Many-Body Dynamics of Spin-Boson Systems

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Outline:

- **Lecture I** Spin-Boson Model with Ohmic Dissipation, Quantum Phase Transitions
- **Lecture II** Dynamics and Non-Perturbative Stochastic Method
- **Lecture III** Application in Mesoscopic RC Circuits and Photons
- **Lecture IV** Application in Cold Atoms

In these Lectures, we focus on the Spin-Boson Model with Ohmic Dissipation.

These lectures notes are based on the following papers:
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Lecture I: 1h30

1 Introduction

Dissipation in quantum mechanics represents an important statistical mechanical problem. A prototype model in this class is the Caldeira-Leggett model describing a quantum particle in a dissipative bath of harmonic oscillators. The spin-boson model can be considered as a variant of the Caldeira-Leggett model where the quantum system is a two-level system. Those dissipative impurity systems are generally interesting because they display both a localized (classical) and delocalized (quantum) phase for the spin. This class of models were intensively investigated to study the quantum-classical transition and the corresponding loss of quantum coherence.

2 Spin-Boson Model with Ohmic Bath

We consider the following Hamiltonian:

$$H = -\frac{\Delta}{2}\sigma_x + \frac{h}{2}\sigma_z + H_{osc} + \frac{1}{2}\sigma_z \sum_n \lambda_n (a_n + a_n^\dagger), \tag{1}$$

where σ_x and σ_z are Pauli matrices and Δ is the tunneling amplitude between the states with $\sigma_z = \pm 1$. H_{osc} is the Hamiltonian of an infinite number of harmonic oscillators with frequencies $\{\omega_n\}$ which couple to the spin degree of freedom via the coupling constants $\{\lambda_n\}$. The heat bath is characterized by its spectral function (Fourier transform of the correlation function in time of the fluctuating operator coupling to σ_z):

$$J(\omega) = \pi \sum_n \lambda_n^2 \delta(\omega_n - \omega) = 2\pi\alpha\omega_c^{1-s}\omega^s e^{-\omega/\omega_c}, \tag{2}$$

where ω_c is a cutoff energy and the dimensionless parameter α measures the strength of the dissipation. Similar models involving a two-level system coupled to a bath of harmonic oscillators are investigated in the context of stimulated photon emission. *For simplicity, hereafter we set the Planck constant $\hbar$ to unity.*

2.1 Simple understanding of quantum phase transitions

First, we consider the regime of a large level asymmetry $\Delta/h \ll 1$ where perturbation theory applies and a correspondence to the well-known $P(E)$ theory for dissipative systems can be formulated.

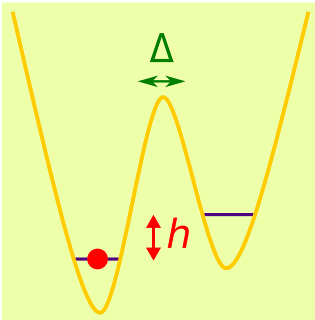

Fig. 1. Simple Picture of a two-level system in the case of a finite detuning (magnetic field).

For $\Delta = 0$, the spin is completely localized in the $\downarrow$ state. Now, for finite Δ, we want to evaluate the probability p_{del} that this spin down electron 'flips'. In the absence of the environment, it is straightforward to obtain that $p_{del} = \Delta^2/h^2 + ...$ Below, we shall include the effect of the environment and see how this quantity gets modified. It should be noted that the latter plays an important role because it is related to $\langle \sigma_z \rangle$, through $\langle \sigma_z \rangle + 1 \sim \mathcal{O}(p_{del})$.

For $\Delta = 0$, one gets two classical states and the Hamiltonian for either of the states depends on the bosons

$$H_{\uparrow,\downarrow} = \sum_n \omega_n a_n^\dagger a_n \pm \sum_n \frac{\lambda_n}{2}(a_n + a_n^\dagger) + cst. \tag{3}$$

We can easily handle this Hamiltonian when $\lambda_n = 0$. The stationary wavefunctions are those with a fixed number of bosons $n_n = a_n^\dagger a_n$ in each mode corresponding to the energy $\sum_n \omega_n n_n$. For $\lambda_n \neq 0$, one may absorb the linear term in the redefinition

$$\begin{aligned} a_{n,\uparrow} &= a_n + \frac{\gamma_n}{2} \\ a_{n,\downarrow} &= a_n - \frac{\gamma_n}{2}. \end{aligned} \tag{4}$$

Since the bosons have been shifted the vacua for the two states are not the same.

More precisely, let us consider the vacuum

$$a_{m,\downarrow}|O_\downarrow\rangle = 0. \tag{5}$$

This is equivalent to

$$a_{m,\uparrow}|O_\downarrow\rangle = \gamma_m|O_\downarrow\rangle, \tag{6}$$

where $\gamma_n = \lambda_n/\omega_n$. We then observe that $|O_\downarrow\rangle$ is not the vaccum for the bosons linked to the state $\uparrow$. This is rather a coherent state wich is defined as an eigenstate of the annihilation operator of a certain mode, and thus

$$|O_\downarrow\rangle = \prod_m \exp\left(-\frac{|\gamma_m|^2}{2}\right) \sum_{n_m=0}^{+\infty} \frac{(\gamma_m)^{n_m}}{\sqrt{n_m!}} |\{n_m\}_\uparrow\rangle. \tag{7}$$

In first order in Δ, this implies that the ground state $|O_\downarrow\rangle$ acquires corrections proportional to all possible states $|\{n_m\}_\uparrow\rangle$:

$$|g\rangle = |O_\downarrow\rangle + \sum_{\{n_m\}} \psi\{n_m\}|\{n_m\}_\uparrow\rangle, \tag{8}$$

with

$$\psi\{n_m\} = \Delta \frac{\langle O_\downarrow|\{n_m\}_\uparrow\rangle}{\epsilon + \sum_m n_m\omega_n}. \tag{9}$$

The delocalization probability of the spin corresponds to

$$p_{del} = \sum_{\{n_m\}} |\psi\{n_m\}|^2 + ... \tag{10}$$

It is relevant to notice that one can make a formal link with the $P(E)$ theory of dissipative tunneling problems through (see Appendix A):

$$p_{del} = \Delta^2 \int_0^{\omega_c} \frac{P(E)}{(h+E)^2} + ... \tag{11}$$

If there is no dissipation, then $P(E) = \delta(E)$, and thus this reduces to the known qubit result $p_{del} = \Delta^2/h^2 + ...$ In the case of Nyquist noise or ohmic dissipation where $P(E) \propto E^{2\alpha-1}$, the delocalization probability rather evolves as

$$p_{del} = \frac{\Delta^2}{h^2}\left(\frac{h}{\omega_c}\right)^{2\alpha}. \tag{12}$$

Note that p_{del} is small at high $h \sim \omega_c$ and one reproduces $p_{del} \sim \Delta^2/h^2$. On the other hand, for $\alpha < 1$, one observes that p_{del} increases for smaller h and eventually reaches the maximum value $p_{del} \to 1$. This shows that the problem becomes highly non-perturbative at small h for $\alpha < 1$; one indeed expects a delocalized phase for $\alpha < 1$ and a localized phase for $\alpha > 1$.

This result is in agreement with the expansion of the ground state energy to second order in Δ (remember that we consider the case where $h > 0$):

$$\mathcal{E}_g = -\frac{h}{2} - \frac{\omega_c}{4}\left(\frac{\Delta}{\omega_c}\right)^2 e^{h/\omega_c}\left(\frac{h}{\omega_c}\right)^{2\alpha-1}\Gamma\left(1-2\alpha, \frac{h}{\omega_c}\right), \tag{13}$$

where Γ is the incomplete gamma function. In the regime of relatively large h (but still small compared to the high-energy cutoff ω_c), one may simplify

$$\Gamma\left(1-2\alpha, \frac{h}{\omega_c}\right) \simeq \Gamma(1-2\alpha) - \frac{(h/\omega_c)^{1-2\alpha}}{1-2\alpha}. \tag{14}$$

This immediately leads to:

$$\mathcal{E}_g = -\frac{h}{2} + \frac{\omega_c}{4}\left(\frac{\Delta}{\omega_c}\right)^2 e^{h/\omega_c} \times \left[\frac{1}{1-2\alpha} - \Gamma(1-2\alpha)\left(\frac{h}{\omega_c}\right)^{2\alpha-1}\right]. \tag{15}$$

Thus, $\langle\sigma_z\rangle = 2\partial\mathcal{E}_g/\partial h$ can be easily derived. At relatively large values of the level asymmetry, one gets:

$$\langle\sigma_z\rangle = -1 + \frac{1}{2}\left(\frac{\Delta}{\omega_c}\right)^2 (1-2\alpha)\Gamma(1-2\alpha)\left(\frac{h}{\omega_c}\right)^{2\alpha-2}. \tag{16}$$

We check that $\langle\sigma_z\rangle + 1 \sim \mathcal{O}(p_{del})$. In a similar way, one can extract $\langle\sigma_x\rangle = -2\mathcal{E}_g/\partial\Delta$, resulting in:

$$\langle\sigma_x\rangle = -\frac{\Delta}{\omega_c}\left[\frac{1}{1-2\alpha} - \Gamma(1-2\alpha)\left(\frac{h}{\omega_c}\right)^{2\alpha-1}\right]. \tag{17}$$

It should be noted that the limit $\alpha \to 1/2$ is always well-defined, since

$$\lim_{x\to 0}\left(\frac{1}{x} - \Gamma(x)y^{-x}\right) = \lim_{x\to 0}\frac{1-y^{-x}}{x} = \ln y. \tag{18}$$

In particular, this leads to:

$$\langle\sigma_x\rangle_{\alpha=1/2} = -\frac{\Delta}{\omega_c}\ln\left(\frac{h}{\omega_c}\right), \qquad \langle\sigma_z\rangle_{\alpha=1/2} = -1 + \frac{\Delta^2}{2\omega_c h}, \tag{19}$$

in agreement with the exact results obtained from the resonant level model.

2.2 *Observables at equilibrium*

For $\alpha \leq 1$, one can use the Bethe Ansatz to compute observables. While general expressions are quite complicated (see for example P. Cedraschi and M. Bütikker, Ann. Phys. N. Y. **289**, 1 (2001)), it is instructive to derive simple *scaling* forms for the observables in the limits $h \ll T_K$ and $h \gg T_K$. As shown in Appendix B, one can derive an exact expression of the Kondo energy scale as a function of the dissipative parameter α:

$$T_K(\alpha) = \Delta\left(\frac{\Delta}{\omega_c}\right)^{\alpha/1-\alpha}, \tag{20}$$

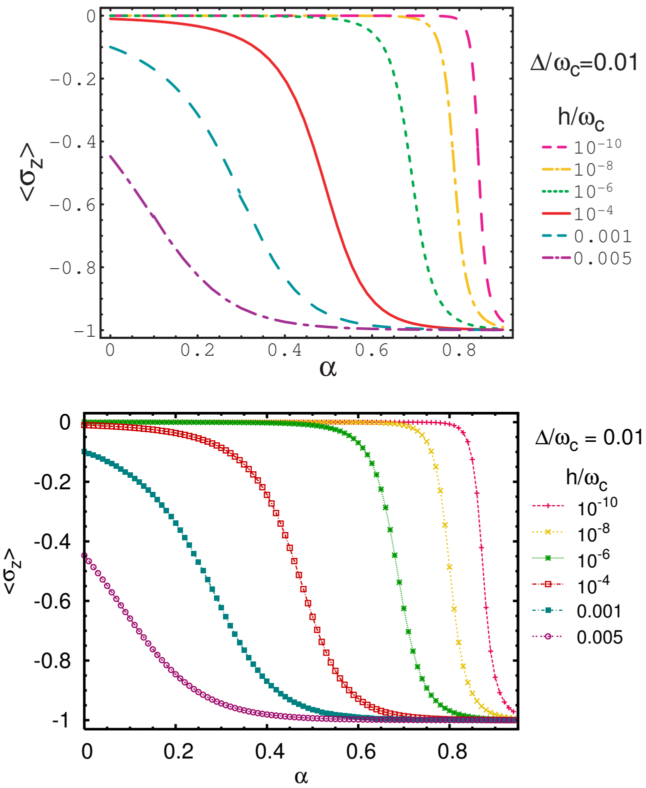

Fig. 2. Longitudinal spin magnetization $\langle \sigma_z \rangle$ as a function of the detuning h from Bethe Ansatz techniques (top) and NRG calculations (bottom).

where ω_c is a high-fequency cutoff for the boson bath. We shall also apply the NRG method which shows an excellent agreement with the Bethe Ansatz results. Technical details are given in Karyn Le Hur, Annals of Physics **323**, 2208-2240 (2008) and are beyond the scope of these lectures.

For $h \ll T_K$, we obtain

$$\lim_{h \ll T_K} \langle \sigma_z \rangle = -\frac{2e^{\frac{b}{2(1-\alpha)}}}{\sqrt{\pi}} \frac{\Gamma[1 + 1/(2-2\alpha)]}{\Gamma[1 + \alpha/(2-2\alpha)]} \left(\frac{h}{T_K} \right), \tag{21}$$

where

$$b = \alpha \ln \alpha + (1-\alpha) \ln(1-\alpha). \tag{22}$$

Note that $\langle \sigma_z \rangle \propto h/T_K$ at small h, in keeping with the Kondo Fermi liquid ground state. The local susceptibility of the spin converges to $1/\Delta$ for $\alpha \to 0$ [1] in accordance with the two-level description and diverges in the vicinity of the KT phase transition as a result of the exponential suppression of the Kondo energy scale. *It is relevant to observe that the longitudinal spin magnetization $\langle \sigma_z \rangle$ only depends on the "fixed point" properties, i.e, this is a universal function of h/T_K in the delocalized phase.*

[1] For $\alpha \to 0$, $\Gamma[1] = 1$, $\Gamma[3/2] = \sqrt{\pi}/2$, $\exp(b/(2(1-\alpha)) = 1$, and $T_K = \Delta$.

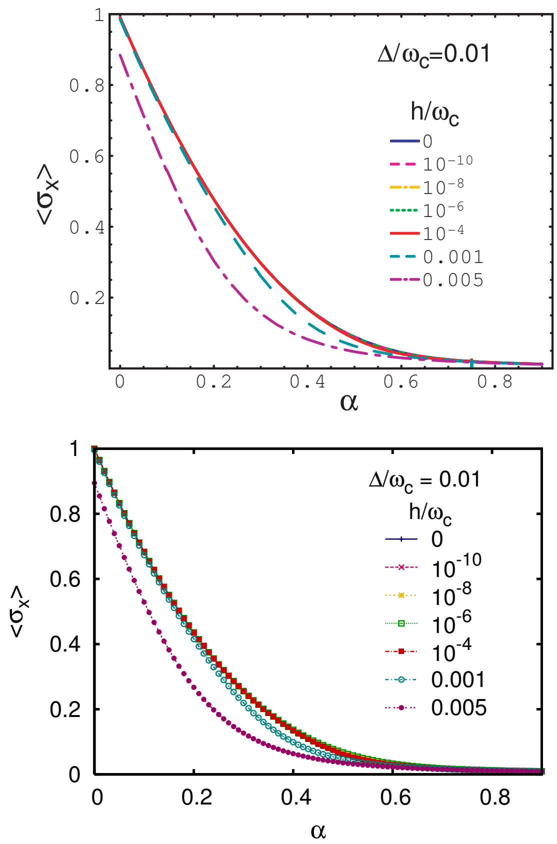

Fig. 3. The transverse spin magnetization $\langle \sigma_x \rangle$ versus α for different level asymmetries h obtained from the Bethe Ansatz calculations (top) and from NRG (bottom).

Exactly at the Kosterlitz-Thouless phase transition, the result is that $\langle \sigma_z \rangle$ jumps by the amount $\sqrt{1/\alpha_c}$ along the quantum critical line where $\alpha_c = 1 + \mathcal{O}(\Delta/\omega_c)$. Remember that in contrast to the universal jump of the superfluid density in two-dimensional XY models, in the spin-boson model, the jump in the longitudinal magnetization is non-universal for finite Δ/ω_c. Finally, far in the localized phase, one rather predicts $\langle \sigma_z \rangle \approx -1 + \mathcal{O}((\Delta/\omega_c)^2)$.

By increasing the detuning h, the abrupt jump in $\langle \sigma_z \rangle$ occurring at the quantum phase transition is progressively replaced by a smooth behavior; at finite h there is a smooth crossover separating the delocalized and the localized regime.

The leading behavior of $\langle \sigma_x \rangle$ in the delocalized phase takes the form:

$$\lim_{h \ll T_K} \langle \sigma_x \rangle = \frac{1}{2\alpha - 1} \frac{\Delta}{\omega_c} + C_1(\alpha) \frac{T_K}{\Delta}, \tag{23}$$

with

$$C_1(\alpha) = \frac{e^{-b/(2-2\alpha)}}{\sqrt{\pi}(1-\alpha)} \frac{\Gamma[1 - 1/(2-2\alpha)]}{\Gamma[1 - \alpha/(2-2\alpha)]}. \tag{24}$$

As $\alpha \to 0$, $T_K \to \Delta$ and $C_1(0) = 1$, so we recover the exact result $\langle \sigma_x \rangle_{\alpha=h=0} =$

1[2]. As we turn on the coupling to the environment, we introduce some uncertainty in the spin direction and $\langle\sigma_x\rangle$ decreases. *It should be noted that $\langle\sigma_x\rangle$ does not only depend on fixed point properties; in the delocalized phase, $\langle\sigma_x\rangle$ still contains a perturbative part in Δ/ω_c!*

For $\alpha < 1/2$, the monotonic decrease of T_K/Δ dominates. In contrast, for $\alpha > 1/2$, the first term in Eq. (23) dominates and we have

$$\langle\sigma_x\rangle_{\alpha>1/2,h\to 0} = \frac{1}{2\alpha-1}\frac{\Delta}{\omega_c}. \tag{25}$$

In fact, this result can also be recovered using the perturbation theory of Sec. 2.1 (consult Eq. (17) for $h \to 0$). This result clearly emphasizes that the observable $\langle\sigma_x\rangle$ is *continuous* and *small* at the KT transition in the ohmic spin-boson model. This is also consistent with the work by Anderson and Yuval which predicts $\langle\sigma_x\rangle \sim \Delta/\omega_c$ exactly at the phase transition.

Lecture II: 1h30

3 Dynamics of two-level systems

3.1 General arguments

Let us study the dynamics of a two-level system subject to dissipation and in an external field $\mathbf{H}_{ext}$ along the $\mathbf{z}$ direction. At a general level, the state of the system at time t can be specified uniquely by giving the expectation values $\langle\sigma_i(t)\rangle$ of the appropriate Pauli operators (spin components) at this time. Specifically omitting the argument t the density matrix reads

$$\hat{\rho} = \begin{pmatrix} \frac{1}{2}\left(1+\langle\sigma_z\rangle\right) & \frac{1}{2}(\langle\sigma_x\rangle + i\langle\sigma_y\rangle) \\ \frac{1}{2}(\langle\sigma_x\rangle - i\langle\sigma_y\rangle) & \frac{1}{2}\left(1-\langle\sigma_z\rangle\right) \end{pmatrix}.$$

A pure state must satisfy $\hat{\rho}^2 = \hat{\rho}$ which implies $\langle\vec{\sigma}\rangle^2 \equiv \langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1$. In this case, the state is an eigenstate of $\mathbf{n}\cdot\sigma$ where $\mathbf{n}$ is some unit vector; the spin points along $\mathbf{n}$. The form of the density matrix implies that, quite generally, a specification of the dynamics of the vector $\langle\vec{\sigma}\rangle(t)$ is a complete description of the dynamics of the system. In traditional NMR problems the (rms) fluctuating environmental field is almost invariably small compared to the dc external field $\mathbf{H}_{ext}$, and this is often the case also in a Quantum Information context (since a system is only likely to be a useful qubit if this condition is well fulfilled). Under those

[2] Exactly at $\alpha = 0$, the (small) first term in Eq. (23) is not present.

conditions it turns out, as we shall see, that the dynamics of $\langle\vec{\sigma}\rangle$ (or equivalently of the spin $\mathbf{S} \equiv \frac{1}{2}\langle\vec{\sigma}\rangle$) is given to a very good approximation by the celebrated Bloch equations

$$\begin{aligned}\frac{dS_z}{dt} =& -\frac{S_z - S_z^{(eq)}}{T_1} \\ \frac{d\mathbf{S}_\perp}{dt} =& \omega_0 \mathbf{z} \times \mathbf{S}_\perp - \mathbf{S}_\perp / T_2,\end{aligned} \tag{26}$$

where ω_0 is the Larmor frequency associated with the external field $\mathbf{H}_{ext} = \omega_0 \mathbf{z}$ (we have taken the gyromagnetic ratio equal to 1) *here, along z direction* and the thermal equilibrium value $S_z^{(eq)}$ in the weak coupling limit is given by

$$S_z^{eq} = \frac{1}{2} \tanh \beta\omega_0/2. \tag{27}$$

The times T_1 and T_2 are called the spin-relaxation time and the transverse relaxation time, respectively. A straightforward theory of T_1 and T_2 can be obtained by writing down the exact equation of motion of $\mathbf{S}$

$$\frac{d\mathbf{S}}{dt} = \omega_0 \mathbf{z} \times \mathbf{S} - \mathbf{S} \times \mathbf{H}_{env}(t), \tag{28}$$

and then average $\mathbf{H}_{env}(t)$ over the unperturbed behavior of the environment.

In the case described above since $\mathbf{H}_{env}$ has only a component along the z direction, the complete behavior of $\mathbf{S}(t)$ is determined by the spectral density $J(\omega)$ defined earlier. The environmental fields are entirely along the z-axis. Let us start at $t = 0$ with the spin polarizes along (say) x-direction. it is convenient to eliminate the effect of Larmor precession by going to a frame rotating with angular velocity ω_0 along the z-axis. Then, if we define the complex quantity $S_x + iS_y \equiv S_\perp e^{i\varphi}$, for any realization of the noise, its magnitude $|S_\perp|$ will remain constant (at 1/2 according to the definitions) will its phase will precess according to

$$\hbar\frac{d\varphi}{dt} = \mathcal{B}(t) \to \varphi(t) = \int_0^t \mathcal{B}(t')dt'/\hbar. \tag{29}$$

Now, we can use the Gaussian statistics of $\mathcal{B}$:

$$\langle \exp i \int_0^t \mathcal{B}(t')dt' \rangle = \exp -\frac{1}{2}\int_0^t dt' \int_0^t dt'' \langle \mathcal{B}(t')\mathcal{B}(t'')\rangle. \tag{30}$$

Using the definition of $\mathcal{B}(t)$ one can show that

$$(S_x + iS_y)(t) \equiv \exp(-t/T_2), \tag{31}$$

where

$$T_2^{-1} = 2\pi \lim_{\omega\to 0} \coth(\beta\omega/2) J(\omega). \tag{32}$$

3.2 *Feynman-Vernon Path Integral of Spin Dynamics: Leggett et al. point of view (1987)*

Now, we come back to the spin-boson model and we want to evaluate the spin reduced density matrix. It is convenient to use the real-time path integral formalism and integrate out the spin degrees of freedom exactly (we have a quadratic action) resulting in:

$$\langle\sigma_f|\rho_S(t)|\sigma'_f\rangle = \int \mathcal{D}\sigma(.) \int \mathcal{D}\sigma'(.) \mathcal{A}(\sigma)\mathcal{A}^*(\sigma')F[\sigma,\sigma'], \tag{33}$$

with σ_f and $\sigma'_f \in (|\uparrow\rangle, |\downarrow\rangle)$. A spin path labelled by $\sigma(s)$ jumps back and forth between the two values $\sigma = \pm 1$. Here, $\mathcal{A}(\sigma)$ and $\mathcal{A}(\sigma')$ denote the amplitude of the spin to follow a path in the absence of the bath. The effect of the environment is entirely caputred in the *influence functional*. The initial condition describes the preparation of the spin and we shall impose $\sigma(t) = \sigma_f$ and $\sigma'(t) = \sigma_f$. The ket point is that the influence functional can be evaluated in a precise way:

$$F[\sigma,\sigma'] = \exp\Big(-\frac{1}{\pi}\int_{t_0}^{t} ds \int_{t_0}^{s} ds' \Big[-iL_1(s-s')\xi(s)\eta(s') + L_2(s-s')\xi(s)\xi(s')\Big]\Big), \tag{34}$$

where we have introduced symmetric and antisymmetric spin paths

$$\eta(s) = \frac{1}{2}[\sigma(s)+\sigma'(s)] \tag{35}$$
$$\xi(s) = \frac{1}{2}[\sigma(s)-\sigma'(s)].$$

It contains the real and imaginary parts of the force autocorrelation function of the environment $\pi\langle X(t)X(0)\rangle_T = L_2(t) - iL_1(t)$ with $X = \sum_n \lambda_n(b_n^\dagger + b_n)$ and

$$L_1(t) = \int_0^\infty d\omega J(\omega)\sin\omega t \tag{36}$$
$$L_2(t) = \int_0^\infty d\omega J(\omega)\cos\omega t \coth\beta\omega/2\,, \tag{37}$$

where $\beta = 1/k_B T$ with temperature T.

Next, one parametrizes a general (double) spin path and inserts it into the functional integral. Since the spin is held fixed at times $t < t_I$, the double spin path is constrained to one of the diagonal (or "sojourn") states $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$. If we are interested in a diagonal element (population) of $\rho_s(t)$, we fix the final state of the spin path to be a "sojourn" state as well. To calculate an off-diagonal element (coherence), we let the spin path end at time t in an off-diagonal (or "blip") state $\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$.

For a path that ends in a sojourn state and makes $2n$ transitions at time $t_I < t_1 <$

$t_2 < \cdots < t_{2n} < t$ along the way, we write the spin paths as

$$\xi(t) = \sum_{j=1}^{2n} \Xi_j \theta(t - t_j) \tag{38}$$

$$\eta(t) = \sum_{j=0}^{2n} \Upsilon_j \theta(t - t_j)\,. \tag{39}$$

The variables $\{\Xi_1, \ldots, \Xi_{2n}\} = \{\xi_1, -\xi_1, \ldots, -\xi_n\}$ with $\xi_j = \pm 1$ describe the n off-diagonal or "blip" parts of the path spent in the states $\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$ during times $t_{2m-1} < t < t_{2m}$ ($m = 1, \ldots, n$), where $\xi(t) = \pm 1$ and $\eta(t) = 0$. The variables $\{\Upsilon_0, \ldots, \Upsilon_{2n}\} = \{\eta_0, -\eta_0, \ldots, \eta_n\}$, on the other hand, characterize the $(n + 1)$ diagonal or "sojourn" parts of the path during times $t_{2m} < t < t_{2m+1}$ ($m = 0, \ldots, n$), where $\eta(t) = \pm 1$ and $\xi(t) = 0$.

The beginning of the initial sojourn is either at $t_0 \to -\infty$ or at $t_0 = t_I$, depending on whether spin and bath are in contact at $t < t_I$. We discuss the influence of this initial preparation on the dynamics in detail later. Formally we have $t_{2n+1} \equiv t$, and the path's boundary conditions specify η_0 and η_n. Altogether, the two-spin path is completely characterized by the variables $\{t_0, t_1, \ldots, t_{2n}; \xi_1, \ldots, \xi_n; \eta_0 = 1, \eta_1, \ldots, \eta_{n-1}, \eta_n\}$. A spin path that ends in a blip state is written in an analogous way.

Using this parametrization of the spin path, we may perform the time integrations in the influence functional, which yields

$$F_n\Big[\{\Xi_j\}, \{\Upsilon_j\}, \{t_j\}\Big] = \mathcal{Q}_1 \mathcal{Q}_2 \tag{40}$$

where

$$\mathcal{Q}_1 = \exp\left[\frac{i}{\pi} \sum_{j>k\geq 0}^{2n} \Xi_j \Upsilon_k Q_1(t_j - t_k)\right] \tag{41}$$

$$\mathcal{Q}_2 = \exp\left[\frac{1}{\pi} \sum_{j>k\geq 1}^{2n} \Xi_j \Xi_k Q_2(t_j - t_k)\right]. \tag{42}$$

The bath functions $Q_{1,2}(t)$ are the second integrals of $L_{1,2}(t)$, *i.e.* $\ddot{Q}_{1,2} = L_{1,2}$. Explicitly, they read for an Ohmic spectral density

$$Q_1(t) = 2\pi\alpha \tan^{-1}(\omega_c t) \tag{43}$$

$$Q_2(t) = \pi\alpha \ln(1 + \omega_c^2 t^2) + 2\pi\alpha \ln\left(\frac{\beta}{\pi t} \sinh \frac{\pi t}{\beta}\right). \tag{44}$$

The influence functional is a product of two terms: $\mathcal{Q}_1$ and $\mathcal{Q}_2$. While $\mathcal{Q}_1$ describes a coupling between the blip and all previous sojourn parts of the path, the term $\mathcal{Q}_2$ contains the interaction between all blips (including a self-interaction).

The environment induces a (long-range) interaction between the spin path at different times. The state of the spin at time t depends on its state at earlier times, which leads to a non-Markovian Heisenberg equation of motion for the spin. The form of the interaction depends on the spectral density $J(\omega)$ and the temperature T. At zero temperature, for example, one finds that

$$L_2(t) = 2\pi\alpha\omega_c^2(1-\omega_c^2 t^2)/(1+\omega_c^2 t^2)^2 \tag{45}$$

only decays algebraically in time. Non-Markovian effects are thus pronounced, especially at long times. At high temperatures, on the other hand, the blip-blip interaction becomes short-ranged. In the white-noise limit at $T > \omega_c$, for example, one derives $L_2(t) = 2\pi\alpha k_B T\delta(t)$ and the dynamics is Markovian.

The path integral of the reduced density matrix also depends on the free spin-path amplitudes $\mathcal{A}[\sigma]$ and $\mathcal{A}^*[\sigma']$. These amplitudes contribute a factor of $i\xi\eta\Delta/2$ for each transition between a sojourn state η and a blip state ξ (as well as a bias-dependent h phase factor) The diagonal element of the density matrix describing the probability

$$p(t) = \langle\uparrow|\,\rho_S(t)\,|\uparrow\rangle \tag{46}$$

to find the system in state $|\uparrow\rangle$ at time t is given by a series in the tunneling coupling Δ^2

$$p(t) = 1 + \sum_{n=1}^{\infty}\left(\frac{i\Delta}{2}\right)^{2n}\int_{t_I}^{t} dt_{2n}\cdots\int_{t_I}^{t_2} dt_1 \sum_{\{\xi_j,\eta_j\}} F_n\,. \tag{47}$$

The sum is only over even exponents of Δ^{2n}, because we are calculating a diagonal element of $\rho_S(t)$. The spin expectation value $\langle\sigma^z(t)\rangle \equiv P(t)$ can be expressed as

$$\langle\sigma^z(t)\rangle \equiv P(t) = 2p(t) - 1\,. \tag{48}$$

In contrast, for an off-diagonal element of $\rho_S(t)$ the path ends in a blip state $\xi_{2n} = \pm 1$ and one finds

$$\begin{aligned}\langle\uparrow|\,\rho_S(t)\,|\downarrow\rangle = \langle\sigma^+(t)\rangle = i\xi_{2n}\sum_{n=1}^{\infty}\left(\frac{i\Delta}{2}\right)^{2n-1}\\ \times\int_{t_I}^{t} dt_{2n-1}\cdots\int_{t_I}^{t_2} dt_1 \sum_{\{\xi_j,\eta_j\}} F_n\,,\end{aligned} \tag{49}$$

where $\xi_{2n} = 1$ for this off-diagonal element and $\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y)$. Note the presence of a boundary term at the final time t, since it now determines the end of the last blip, *i.e.*, $t_{2n} = t$.

What makes these expressions complicated is the fact that the coupling between the spin paths in the influence functional F is long-range in time. Hence, one must consider all terms coupling different blips and sojourns.

Hereafter, we introduce a method that allows us to take all terms in the influence functional exactly into account. In particular, unlike the Non-Interacting Blip Approximation (NIBA) of Leggett *et al.* (1987), we try to consider the long-range interactions between different blips. This is achieved by mapping the problem onto a linear stochastic equation that can be easily solved numerically. We follow the notations of Peter P. Orth, Adilet Imambekov, Karyn Le Hur Phys. Rev. B **87**, 014305 (2013). From mathematical point of view, certain useful steps can also be found in other papers (G. B. Lesovik, A. V. Lebedev, A. Imambekov JETP Lett. **75**, p. 474, (2002); A. Imambekov, V. Gritsev, E. Demler, Phys. Rev. A **77**, 063606 (2008)).

We now present the method to evaluate the full spin reduced density matrix $\rho_S(t)$ in a numerically exact manner. Its element $\langle i| \rho_S(t) |j\rangle$ with $i, j \in \{\uparrow, \downarrow\}$ is calculated by averaging over solutions of a non-perturbative stochastic Schrödinger equation (SSE). We explicitly derive the SSE from the expressions in Eqs. (47) and (49). This method works for all temperatures (and for an arbitrary time-dependent bias field $h(t)$. In contrast to other numerical approaches such as the real-time Monte-Carlo method, we will not directly evaluate the real-time path integral in Eqs. (47) and (49). Instead, we first decouple the terms bilinear in the blip and sojourn variables in the influence functional $F_n = \mathcal{Q}_1 \mathcal{Q}_2$ in Eq. (40) using Hubbard-Stratonovich transformations. We then obtain $\langle i| \rho_S(t) |j\rangle$ as a statistical average over solutions of a stochastic Schrödinger equation. Technical details are provided in Appendix C.

One key point is that one can formally write down

$$\mathcal{Q}_2 = \exp\Big\{-n\alpha\Big[\ln(1+4\omega_c^2 t_{\text{tot}}^2) + G\Big]\Big\} \times \int d\mathcal{S} \exp\Big\{i\sum_{j=1}^{2n} \Xi_j h_s(\tau_j)\Big\}\,. \tag{50}$$

G is defined in Appendix C and the integral over the Hubbard-Stratonovich variables $\{s_m\}$ reads

$$\int d\mathcal{S} = \prod_{m=1}^{m_{\max}} \int_{-\infty}^{\infty} \frac{ds_m}{\sqrt{2\pi}} e^{-s_m^2/2}\,, \tag{51}$$

and we have introduced a real (height) function

$$h_s(\tau) = \sum_{m=1}^{m_{\max}} s_m \sqrt{-\alpha G_m}\,\Psi_m(\tau)\,. \tag{52}$$

All informations about the environment is now contained in this random field.

One can proceed in a similar way with the $\mathcal{Q}_1$ contribution. Fortunately, it takes a particularly simple form for an Ohmic bath and if $\Delta/\omega_c \ll 1$ and $\alpha < 1/2$ (scaling limit), where one may safely approximate

$$Q_1(t) = 2\pi\alpha \tan^{-1}(\omega_c t) \approx \alpha\pi^2\theta(t)\,. \tag{53}$$

We thus find that

$$\mathcal{Q}_1 = \exp\Big[i\pi\alpha \sum_{j>k\geq 0}^{2n} \Xi_j \Upsilon_k\Big] = \exp\Big[i\pi\alpha \sum_{k=0}^{n-1} \xi_{k+1}\eta_k\Big]\,. \tag{54}$$

Let us finally note that we can, in principle, deal with the blip-sojourn interaction term $\mathcal{Q}_1$ in a similar way as with $\mathcal{Q}_2$.

Combining these results, we then obtain:

$$\begin{aligned} p(\tau) = 1 + \int d\mathcal{S} \sum_{n=1}^{\infty} \left(\frac{i\Delta t_{\mathrm{tot}} e^{-(\alpha/2)\left[\ln(1+4\omega_c^2 t_{\mathrm{tot}}^2)+G\right]}}{2} \right)^{2n} \\ \times \int_0^{\tau} d\tau_{2n} \cdots \int_0^{\tau_2} d\tau_1 \sum_{\{\xi_j,\eta_j\}} \exp\Big[i\pi\alpha \sum_{k=0}^{n-1} \eta_k \xi_{k+1}\Big] \\ \times \prod_{j=1}^{2n} \exp\Big[i\,\Xi_j h_s(\tau_j)\Big]\,. \end{aligned} \tag{55}$$

This summation, however, can easily be incorporated into a product of matrices in the vector space of two-spin states $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, that read [3]

$$V = V_0 \begin{pmatrix} 0 & e^{-ih_s(\tau)} & -e^{ih_s(\tau)} & 0 \\ e^{i\pi\alpha}e^{ih_s(\tau)} & 0 & 0 & -e^{-i\pi\alpha}e^{ih_s(\tau)} \\ -e^{-i\pi\alpha}e^{-ih_s(\tau)} & 0 & 0 & e^{i\pi\alpha}e^{-ih_s(\tau)} \\ 0 & -e^{-ih_s(\tau)} & e^{ih_s(\tau)} & 0 \end{pmatrix}, \tag{56}$$

where

$$V_0 = \frac{1}{2}\Delta t_{\mathrm{tot}} \exp\Big\{-(\alpha/2)\Big[\ln(1+4\omega_c^2 t_{\mathrm{tot}}^2) + G\Big]\Big\}\,. \tag{57}$$

Note that $V_0 = \frac{1}{2}\Delta t_{\mathrm{tot}}$ in the scaling limit $\omega_c \to +\infty$. It is worth emphasizing that the two-spin basis states simply correspond to the four elements of the reduced density matrix $\langle i|\,\rho_S\,|j\rangle$. The final two-spin state $|ij\rangle$ with $i,j \in \{\uparrow,\downarrow\}$ of the real-time spin path determines which density matrix element $\langle i|\,\rho_S(t)\,|j\rangle$ is calculated. A product of matrices of the type in Eq. (56) automatically satisfies the requirement that transitions between two-spin states occur via single spin flips.

[3] This step has also been discussed by Lesovik, Lebedev, Imambekov (2002).

We finally arrive at the central result. With Eq. (56) we can express Eq. (55) as a time-ordered exponential

$$p(\tau) = \int d\mathcal{S}\Phi_f T e^{-i\int_0^\tau ds V(s)}\Phi_i\,. \tag{58}$$

Here, T is the usual time-ordering operator. The two-spin states $|\Phi_i\rangle$ and $|\Phi_f\rangle$ are the initial and final states of the spin path. When calculating the diagonal element $p(\tau) = \langle\uparrow|\,\rho_S(\tau)\,|\uparrow\rangle$, we thus have $\Phi_f = |\uparrow\uparrow\rangle$. Since we consider an initial polarization of the spin in state $|\uparrow\rangle$, it follows that $|\Phi_i\rangle = |\uparrow\uparrow\rangle$ as well. We can evaluate the amplitudes on the right-hand side of Eq. (58) by solving the stochastic Schrödinger equation

$$i\frac{\partial}{\partial\tau}\,|\Phi(\tau)\rangle = V(\tau)\,|\Phi(\tau)\rangle \tag{59}$$

with initial and final conditions $\Phi_{i,f} = (1,0,0,0)^T$. The vector $(1,0,0,0)^T$ corresponds to the basis state $|\uparrow\uparrow\rangle$. The integration $\int d\mathcal{S}$ over the Hubbard-Stratonovich variables is performed by averaging the result over N different realizations of the noise variables $\{s_m\}$. One then obtains $p(\tau)$ by averaging over the different results

$$p(\tau) = \frac{1}{N}\sum_{k=1}^{N}\Phi_1^{(k)}(\tau) = \langle\Phi_1(\tau)\rangle_{\mathcal{S}}\,, \tag{60}$$

where $\Phi_1(\tau)$ is the first component of $|\Phi(\tau)\rangle$ and $\langle\cdot\rangle_{\mathcal{S}}$ denotes the average over the Hubbard-Stratonovich random noise variables $\{s_m\}$. We note that $\langle\Phi_1(\tau)\rangle_{\mathcal{S}}$ is purely real as required. The spin expectation value $\langle\sigma^z(t)\rangle$ is given by $\langle\sigma^z(t)\rangle \equiv P(t) = 2p(t) - 1$.

We want to emphasize that our method takes all terms in the influence functional exactly into account. In particular, we fully account for all interactions between different blips. Although this method is quite powerful, it is so far restricted to the case of an Ohmic bath with $0 < \alpha < 1/2$ and a large cutoff frequency $\omega_c \gg \Delta$ (scaling limit). Below, we show results obtained through the SSE approach and compare with the NRG procedure following P. P. Orth, D. Roosen, W. Hofstetter and K. Le Hur, Phys. Rev. B **82**, 144423 (2010). So far, our results work well for α not too close to $\alpha = 1/2$. We are also doing efforts to extend the method to other problems such as the dissipative Rabi problem for example (L. Henriet, Z. Ristivojevic, P. P. Orth, K. Le Hur).

3.4 Weak-Coupling Expansion and NIBA

At a general level, the problem can be reformulated in classical Bloch equations:

$$\frac{d}{d\tau}\mathbf{S}(\tau) = \mathbf{H}(\tau)\times\mathbf{S}(\tau) \tag{61}$$

The effective noisy magnetic field $\mathbf{H}(\tau)$ depends on the random height function $h_s(\tau)$ and lies in the x-y-plane

$$\boldsymbol{H} = H_0\Big(\cos h_s(\tau), \sin h_s(\tau), 0\Big)\,. \tag{62}$$

The amplitude of the magnetic fields reads is proportional to $\Delta\sqrt{\cos\pi\alpha}$. The dissipative dynamics of the *quantum* spin follows from averaging over different random field configurations. The fact that the effective magnetic field lies in the XY plane can be seen by resorting to a polaronic or Schrieffer-Wolff transformation in the original Hamiltonian, then producing an extra 'noisy' phase coupling to the spin ladder (raising and lowering) operators.

We start from the "classical" Bloch equation in the random magnetic field $\boldsymbol{H} = H_0\Big(\cos[h_s(\tau)], \sin[h_s(\tau)], 0\Big)$. The different spin components obey $\dot{S}^x = H_y S^z$, $\dot{S}^y = -H_x S^z$ and $\dot{S}^z = H_x S^y - H_y S^x$. For the z-component we thus obtain

$$\dot{S}^z(t) = -\Delta^2\cos(\pi\alpha)\int_0^t ds\cos\Big[h_s(t) - h_s(s)\Big]S^z(s)\,, \tag{63}$$

It is important to note that there exist correlations between the random height function $h_s(t)$ and the classical spin trajectory $S^z(s)$ such that in general

$$\Big\langle\cos\Big[h_s(t) - h_s(s)\Big]S^z(s)\Big\rangle_{\mathcal{S}} \neq \Big\langle\cos\Big[h_s(t) - h_s(s)\Big]\Big\rangle_{\mathcal{S}}\Big\langle S^z(s)\Big\rangle_{\mathcal{S}}\,. \tag{64}$$

These correlations are absent in the initial state at $t = 0$, but are generated over the course of time, as follows from the differential equation (63). The correlations are thus small at short times t. Note also that since $h_s(t) \sim \sqrt{\alpha}$, the factor $\cos\Big[h_s(t) - h_s(s)\Big] \approx 1$ for small $\alpha \ll 1$. At $\alpha = 0$, both classical and quantum spin undergo undamped Rabi oscillations with frequency Δ. The correlations between $h_s(t)$ and $S^z(t)$ thus become more pronounced for larger values of α. The mean-field decoupling anticipated in Eq. (64) can thus be justified at short times t and/or small dissipation α. Indeed, we now show that one recovers the NIBA from this mean-field decoupling.

Using the mean-field approximation of Eq. (64), we obtain for the equation of motion of the quantum spin

$$\frac{d}{dt}\langle\sigma^z(t)\rangle = \langle\dot{S}^z(t)\rangle_{\mathcal{S}} \approx -\Delta^2\cos(\pi\alpha) \times\int_0^t ds\,\Big\langle\cos\Big[h_s(t) - h_s(s)\Big]\Big\rangle_{\mathcal{S}}\Big\langle S^z(s)\Big\rangle_{\mathcal{S}}\,. \tag{65}$$

Using the statistical properties of the height function $\langle h_s(\tau)\rangle_{\mathcal{S}} = 0$ and $\langle h_s(\tau)h_s(s)\rangle_{\mathcal{S}} =$

$\alpha G_0 - Q_2(\tau - s)/\pi$, one easily computes the expectation value

$$\Big\langle \cos\Big[h_s(t) - h_s(s)\Big]\Big\rangle_{\mathcal{S}} = \exp\Big[-Q_2(t-s)/\pi\Big]\,. \tag{66}$$

The equation of motion thus takes the form

$$\begin{aligned}\frac{d}{dt}\langle\sigma^z(t)\rangle = &-\Delta^2\cos(\pi\alpha) \\ &\times\int_0^t ds\,\exp\Big[-Q_2(t-s)/\pi\Big]\langle\sigma^z(s)\rangle\,,\end{aligned} \tag{67}$$

which we recognize as the NIBA equation of motion.

Solving this equation through Laplace transform, one obtains that the dominant spin dynamics is coherent and corresponds to damped oscillations for $0 < \alpha < 1/2$**. The SSE approach results in a quite similar result. In addition the SSE allows to treat time-dependent problems such as Landau-Zener problems while the NIBA breaks down for these types of problems.**.

Within this approximation, the damping rate and the frequency of oscillations are given by:

$$\begin{aligned}\gamma &= \Delta_{eff}\cos\frac{\pi\alpha}{2(1-\alpha)} \\ \Omega &= \Delta_{eff}\sin\frac{\pi\alpha}{2(1-\alpha)}\end{aligned} \tag{68}$$

and their ratio is independent of the Rabi frequency:

$$\begin{aligned}\Delta_{eff} &= [\Gamma(1-2\alpha)]\cos(\pi\alpha)]^{1/2(1-\alpha)}\,\Delta_r \approx \Delta(1-\alpha\ln(\omega_c/\Delta)) \\ \Delta_r &= T_K = \Delta\left(\frac{\Delta}{\omega_c}\right)^{\alpha/1-\alpha} \approx \Delta[1-\alpha\ln(\omega_c/\Delta)]\end{aligned} \tag{69}$$

and only depends on the dissipative strength α:

$$\frac{\Omega}{\gamma} = \cot\frac{\pi\alpha}{2(1-\alpha)}. \tag{70}$$

we have introduced the renormalized transverse field $\Delta_r = \Delta(\Delta/\omega_c)^{\alpha/1-\alpha}$ which is proportional to the Kondo energy scale T_K. This quality factor also agrees with that obtained via the NIBA of Leggett et al.

These results suggest a coherent-incoherent crossover, corresponding either to the strong suppression of the off-diagonal elements of the spin reduced density matrix $\langle\sigma_x\rangle \sim \Delta/\omega_c \to 0$ or to the complete vanishing of the Rabi quantum oscillations,

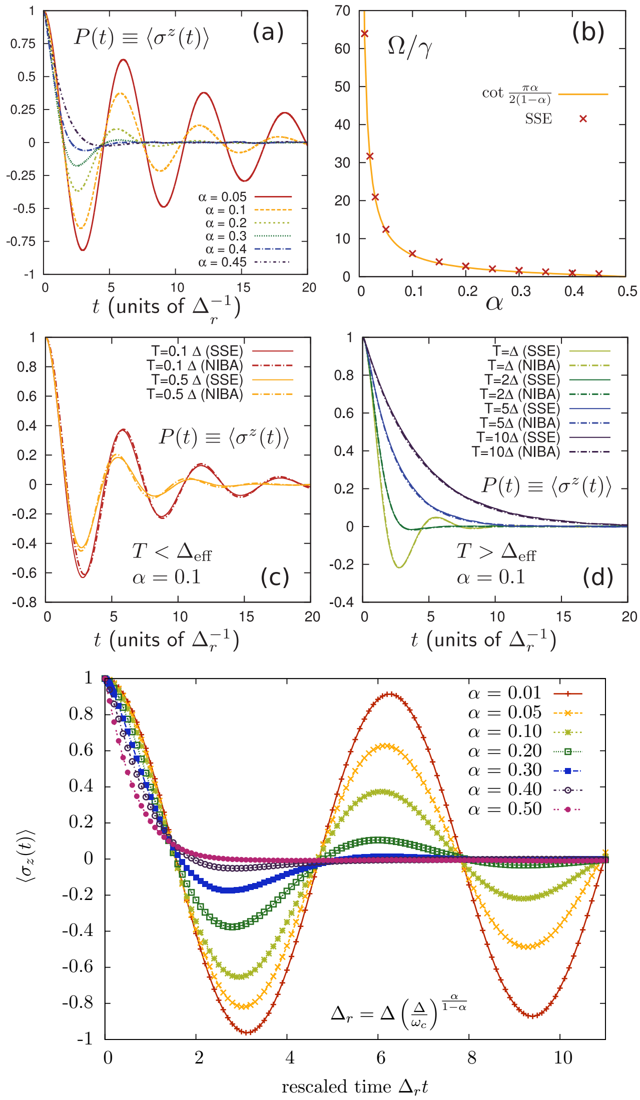

Fig. 4. Spin dynamics $P(t) = \langle \sigma_z(t) \rangle$ obtained from the stochastic approach (upper figures) and time-dependent NRG. Note that Δ_{eff} is an energy scale of the order of Δ_r.

can be identified to the exactly solvable Toulouse limit $\alpha = 1/2$ and not to the quantum phase transition. It is also interesting to underline the correspondence between the prominence of spin-bath entanglement and the emergence of quantum decoherence (see Karyn Le Hur, Annals of Physics 2008). The entanglement entropy S between the spin and the environment displays a plateau at maximal entanglement for $1/2 < \alpha < 1$.

Using the time-dependent NRG, we have also studied the strong coupling regime and in particular the crossover from incoherent decay to localization at the quantum phase transition. In particular, for $1/2 < \alpha < 1$, the spin dynamics remains purely incoherent. The NRG results rather support the form $P(t) = \exp[(-tT_K/2)^{a_\alpha}]$ with the prerequisite that $a_{1/2} = 1$. The exponent a_α evolves linearly with $\alpha - 0.5$. These results are distinct from those by Lesage and Saleur using Conformal Field

Theory.

At low frequency, the spin dynamics is also interesting. For example, as imposed by the formation of a Fermi-liquid fixed point (see also P. Nozières, 1974) one must satisfy the Korringa-Shiba type relation (see Sassetti and Weiss, 1990)

$$\Im m\chi(\omega)_{\omega\to 0} = 2\pi\alpha\omega(\Re e\chi)^2(\omega = 0) \tag{71}$$

The relation, of course, holds for the usual SU(2) Kondo model as a limit of the Anderson model taking $\alpha = 1$ (see also M. Filippone, K. Le Hur, C. Mora 2011). In the present situation, this works assuming that Δ/ω_c is small.

Lecture III: 1h30

4 Mesoscopic circuits: Photons and RC circuits

The spin-boson model can be realized in noisy charge qubits built of mesoscopic quantum dots or Cooper pair boxes (See R. J. Schoelkopf *et al.* et A. Shnirman *et al.*). The gate voltage controls the detuning h and Δ corresponds to the tunneling amplitude between the dot and the lead(s) or the Josephson coupling energy of the junction. If the gate voltage source is placed in series with an external resistor, which can be modelled by a long LC transmission line, this may describe the spin-boson model with ohmic dissipation. **Details are explained in Appendix D**. A one-dimensional Luttinger reservoir could also be used. The spin-boson model can also be derived when coupling a quantum dot to a boson and a fermion bath. These nano-systems may also allow to address new important issues such as the non-equilibrium transport properties at a given quantum phase transition.

Charge measurements provide generally the quantity $\langle\sigma_z\rangle$, which represents the occupation of the dot or island. In a ring geometry, the application of a magnetic flux generates a persistent current which is proportional to $\langle\sigma_x\rangle$ (see P. Cedraschi and M. Buettiker).

4.1 Josephson-Kondo circuit and Transport of Photons

Here, we describe the works (K. Le Hur, PRB 2012) and M. Goldstein, M. Devoret, M. Houzet and L. Glazman PRL 2012) and present a setup realizing the spin-boson model with tunable Ohmic dissipation via photons. The setup is shown in Fig. . This comprises a two-level system corresponding to the 2 charge states of a double-dot qubit and two tunable transmission lines that can be engineered in Josephson junction arrays for example.

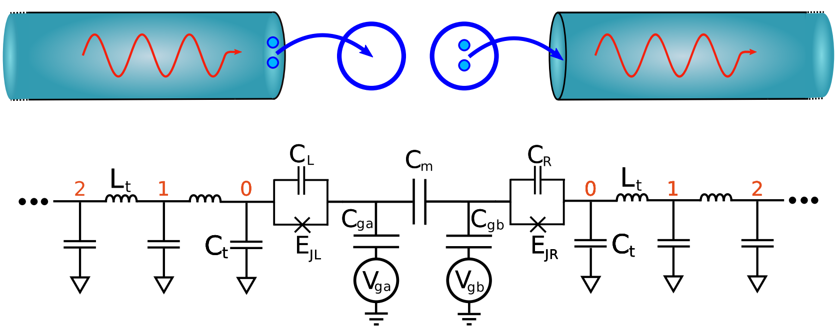

Fig. 5. Principle of the Josephson-Kondo circuit realized with an artificial atom and photon excitations of two long transmission lines.

Quantum excitations in the two long transmission lines are described by collections of harmonic oscillators; b_{lk} and b_{rk} destroy a photon in mode k in the left and right transmission lines, respectively. This produces zero-point fluctuations reminiscent of vacuum fluctuations in free space. We introduce the symmetric b_{sk} and antisymmetric b_{ak} combinations. Using a double Cooper pair box as the spin, the latter only couples to the antisymmetric combination and after unitary transformation the Hamiltonian reads (see):

$$H = \sum_{k>0} \hbar v|k| \left[b_{ak}^{\dagger} b_{ak} + \frac{1}{2} \right] - \frac{\epsilon}{2}\sigma_z - \frac{E_J}{2}\sigma_x - \sum_{k>0} \lambda_k (b_{ak} + b_{ak}^{\dagger}) \frac{\sigma_z}{2}. \tag{72}$$

Each transmission line mimics a physical resistor then producing dissipation in the system. In the present circuit, the spectral function of the environment is defined as $J(\omega) = \pi \sum_{k>0} \lambda_k^2 \delta(\omega - \omega_k) = 2\pi\alpha\omega e^{-\omega/\omega_c}$ where $\omega_c \gg E_J$ represents the high-frequency cutoff of the Ohmic environment and the dissipative parameter α is given by

$$\alpha = \frac{2R}{R_Q}(\gamma_l^2 + \gamma_r^2). \tag{73}$$

Here, $R_Q = h/(2e)^2$ denotes the quantum of resistance where $2e$ is the charge of a Cooper pair, R is the resistance of each transmission line and γ_l and γ_r represent effective dimensionless couplings with the left and right transmission lines. γ_l^2 and γ_r^2 are of the order of unity.

In the coherent regime, similar to the underdamped Rabi oscillations, the spin (qubit) is described by the following dynamical spin susceptibility

$$\chi(\omega) = \frac{\omega_K}{\omega_K^2 - \omega^2 - i\gamma(\omega)}, \tag{74}$$

where $\omega_K = T_K = \Delta_r$. Dissipation from the bosonic environment inevitably results in a prominent broadening $\gamma(\omega) = \omega_K J(\omega)$ of these peaks in accordance with the Korringa-Shiba relation. Interestingly, the spin susceptibility can be measured through the transport of one photon in the circuit leading to a "many-body" resonance in the elastic transmission:

$$t(\omega, P_{in}) = -\frac{2i\gamma_r\gamma_l}{\gamma_l^2+\gamma_r^2} J(\omega)\chi(\omega, P_{in}). \tag{75}$$

P_{in} is the averaged input power. When $P_{in} \to 0$, in the underdamped regime, we find that the scattering matrix is unitary implying that $|r|^2 + |t|^2 = 1$; close to the confinement frequency, we check that $J(\omega_K)\Im m\chi(\omega_K) = 1$ for $P_{in} \to 0$ which shows that the normalized power transmitted to the right transmission line converges to unity assuming that $\gamma_l = \gamma_r$. Increasing the driving power P_{in}, the scattering matrix becomes non-unitary since $J(\omega_K)\Im m\chi(\omega_K, P_{in}) < 1$ (which hides the presence of additional inelastic contributions). Other aspects of the problem have been considered in M. Goldstein *et al.*.

4.2 Quantum RC circuits

Now, we investigate the AC regime, or more specifically the quantum RC circuit consisting of a cavity tunnel-coupled to a metal (essentially, a two-dimensional electron gas or a one-dimensional Luttinger liquid). The study of AC coherent transport was pioneered in a scattering approach by Büttiker, Prêtre and Thomas in 1993 where a universal charge relaxation resistance of $R_q = h/2e^2$ was predicted for a single-mode resistor; the factor $1/2$ is purely of quantum origin and must be distinguished from spin effects. Coulomb blockade effects were ignored and later they have been partially included in an Hartree-Fock theory by Nigg, Lopez and Büttiker (2006). The quantum mesoscopic RC circuit has been successfully implemented in a two-dimensional electron gas and the charge relaxation resistance $R_q = h/2e^2$ was measured by J. Gabelli *et al.* in 2006. This quantized resistance must be thought as the contact resistance between the cavity and the 'reservoir' lead. Our work with C. Mora (Nature Physics 2010) completes the proof of the universal quantized resistance $R_q = h/2e^2$ by including interactions in the cavity non-perturbatively (exactly) and also we show a crossover at finite frequency ω, where the charge relaxation resistance changes from $h/2e^2$ to h/e^2 regardless of the mode transmission.

In the presence of a small time-dependent perturbation of the gate voltage, the charge on the dot $Q = e\langle \hat{N} \rangle$ obeys,

$$Q(\omega) = e^2 K(\omega) V_g(\omega), \tag{76}$$

where the retarded response function, following standard linear response theory $K(t-t') = i\theta(t-t')\langle[\hat{N}(t), \hat{N}(t')]\rangle$, describes charge fluctuations at equilibrium. The quantized resistance $R_q = h/e^2$ can be shown by analogy with the (charge) Kondo effect, following Matveev 1991 and using the identification The formula

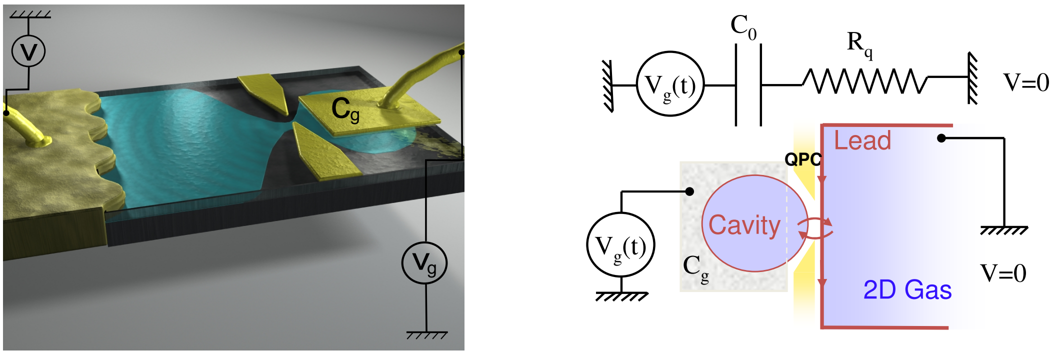

Fig. 6. The principle of the quantum RC circuit built with a mesoscopic capacitor. The quantum RC circuit realized in a two-dimensional electron gas (2D Gas) by J. Gabelli *et al.* and its equivalent circuit.

that gives the charge in the capacitor at low frequency,

$$\frac{Q(\omega)}{V_g(\omega)} = C_0(1 + i\omega C_0 R_q) + \mathcal{O}(\omega^2), \tag{77}$$

with modified values of the capacitance C_0 and the charge relaxation resistance R_q. The crossover from $h/2e^2$ to h/e^2 can be understood from bosonization.

We investigate the regime close to perfect transmission. **Technical Details can be found in C. Mora and K. Le Hur, Nature Physics 6, 697 (2010).** In what follows, we model the complete system by electrons moving along a one-dimensional line, the lead is between $-\infty$ and $-L$ and the cavity between $-L$ and 0. The level spacing on the isolated cavity is still $\Delta = \pi v_F/L$. The Coulomb blockade phenomenon in the cavity, can be treated exactly using the bosonization approach. Integrating all irrelevant modes in an action formalism, at perfect transmission $D = 1$, one finds the action

$$S_0 = \frac{1}{\pi}\sum_n \phi_0(\omega_n)\phi_0(-\omega_n)\left[\frac{|\omega_n|}{1 - e^{-2|\omega_n|L/v_F}} + \frac{E_c}{\pi}\right], \tag{78}$$

where $\omega_n = 2\pi T n$ denote bosonic Matsubara frequencies; the Boltzmann constant k_B is set to unity. Here, the field ϕ_0 is related to the charge on the cavity, $\hat{N} = C_\mu V_g/e + \phi_0/\pi$. From this quadratic action, the response function (76) is straightforwardly calculated. We find:

$$\frac{Q(\omega)}{V_g(\omega)} = \frac{C_g}{1 - \frac{i\omega\pi/E_c}{1-e^{2i\pi\omega/\Delta}}}. \tag{79}$$

Interestingly, the response vanishes each time the frequency ω hits a multiple of Δ corresponding to an eigenstate of the isolated cavity. At low frequency $\omega \ll \Delta$, we extract $C_0 = C_\mu$ - meaning that the Coulomb blockade effect vanishes for perfect transparency $D = 1$ - and $R_q = h/2e^2$ from the comparison of Eq. (79) to the classical RC circuit formula (77). We now discuss the transition to large metallic cavities. Eq. (79) shows an oscillatory behavior for $\omega > \Delta$. We thus average over a finite bandwidth $\delta\omega$, such that $\omega \gg \delta\omega \gg \Delta$, and finally we find:

$$\frac{Q(\omega)}{V_g(\omega)} = \frac{C_g}{1 - i\omega\pi/E_c}. \tag{80}$$

This result is also obtained if one takes $L \to +\infty$ in Eq. (78). It can be checked that this correspondence extends to all correlation functions. Hence, Eq. (78) with $L \to +\infty$ defines the action for the large cavity regime, as shown in the Methods. A comparison of the result (80) with Eq. (77) gives $C_0 = C_\mu = C_g$, indeed corresponding to a vanishing level spacing, and $R_q = h/e^2$. We thus recover the universal resistances $h/2e^2$ and h/e^2 for the small and large cavities, and the fact that the crossover takes place when the frequency ω becomes larger than Δ.

Backscattering at the interface between the cavity and the lead (at $x = -L$) may be incorporated in the model as,

$$S_{BS} = -\frac{v_F r}{\pi a} \int_0^\beta d\tau \cos\left[2\phi_0(\tau) + 2\pi(C_\mu V_g/e)\right], \tag{81}$$

and the total action is now given by $S = S_0 + S_{BS}$ with Eq. (78) and Eq. (81). Here, a is a standard short-distance cutoff and $1/a$ defines the region around k_F where the electron band spectrum can be linearized. r is the dimensionless strength of backscattering and the model only involves the single charge field ϕ_0. For large r, or small transparency D, ϕ_0 gets frozen around the values which minimize Eq. (81). Translated into the charge of the cavity, this gives $\langle \hat{N} \rangle = n \in N$ and we recover charge quantization. The non-linearity of Eq. (81) does not allow a complete analytical approach. We thus consider the case of weak backscattering at the interface, where $r \ll 1$ and the transparency is given by $D = 1 - r^2$.

Let us first discuss the calculation of the charge relaxation resistance R_q. The fluctuations of the phase ϕ_0 are calculated perturbatively to second order in r by expanding the backscattering term S_{BS}. They give access to the number (or charge) fluctuations $K(\omega)$, defined by Eq. (76). The identification with Eq. (77) leads to

$$R_q = \frac{h}{e^2}\frac{B}{A^2}, \tag{82}$$

where the r and r^2 corrections exactly cancel out. The dimensionless coefficients A and B characterize the low frequency expansion of S_0 to first order depending on whether a small or large cavity is considered: $A^{-1} = 1 + \Delta/2E_c$, $B^{-1} = 2(1 + \Delta/2E_c)^2$ and, $A = 1$, $B = 1$, respectively. The universal charge relaxation resistances $h/2e^2$ and h/e^2 for the small and large cavities are finally recovered from Eq. (82).

Other aspects of the problem can also be found in the paper Y. Hamamoto, T. Jonckheere, T. Kato, T. Martin Phys. Rev. B **81** 153305 (2010).

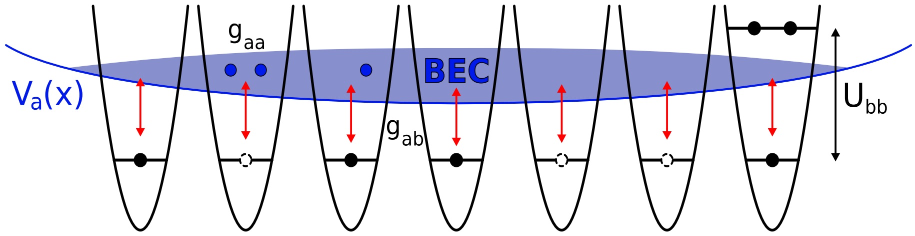

Fig. 7. The setup with 2 different types of atoms

Lecture IV: 1h30

5 Applications in Cold Atoms

We show that a dissipative quantum Ising model can also be implemented in a setup of cold atoms with two bosonic species trapped by different external potentials. Atoms of the first specie, which are in a very deep optical lattice with very well separated and tightly confining potentials, form an array of spins; in each well, the spin-up or spin-down corresponds to occupation by a single or by no atom. The second specie forms a superfluid reservoir. Atoms of different species are coupled via laser transitions and collisional interactions. The phonons in the condensate produce dissipation and a ferromagnetic RKKY (Ising) exchange between the pseudospins whereas the Raman coupling mimics the transverse field of the quantum Ising model. **These notes are based on the paper P. P. Orth, I. Stanic, K. Le Hur (2008) and A. Recati et al. (2005)**.
Hereafter, we reintroduce the Planck constant $\hbar$.

We consider cold bosonic atoms with 2 (hyperfine) ground states a and b. The atoms of the two bosonic species are trapped by two different external potentials. Atoms in state a form a reservoir of atoms in a low-dimensional superfluid phase that can be two-dimensional or one-dimensional, held in a shallow trapping potential $V_a(x)$. The atoms in state b are in a very deep optical lattice with very well *separated* and tightly confining potentials. We consider the collisional blockade limit of large on-site interaction U_{bb}, where only states with occupation $n_b = 0$ and 1 in each well participate to the dynamics, while higher occupations are suppressed by the large collisional shift. Using standard Pauli matrix notations, the occupation operator in each well $\hat{b}_i^\dagger \hat{b}_i$ is replaced by $(1 + \sigma_i^z)/2$ while $\hat{b}_i^\dagger \to \sigma_i^+$ making a close analogy to magnetic systems. We study the dynamics of the b atoms which interact in a collective manner with the host liquid through Raman transitions and collisional interactions. We show that the coupling of each b atom to the density fluctuations in the BEC via collisions, which will be denoted g_{ab} below, can mediate an interaction between the b atoms of Ising type that favors a ferromagnetic ordering in the spin array defined by $\langle \sigma_i^z \rangle \neq 0$. On the other hand, the Raman coupling between a and b atoms suggests a paramagnetic phase where $\langle \sigma_i^z \rangle = 0$; each b atom lies in a superposition of state $|0\rangle$ and $|1\rangle$ and can be eventually entangled to the phonons in the BEC reservoir. By tuning g_{ab}, a second-order transition between the ferromagnetic and the paramagnetic state is expected and we show that this can be understood in terms of a dissipative Ising model with an effective density of states of the heat bath which takes the form $\mathcal{J}(\omega) \propto \omega^d$, where d is the dimensionality of the system.

The BEC can be described by a (quantum) hydrodynamic Hamiltonian

$$H_a = \frac{1}{2} \int d\mathbf{x} \left(\frac{\hbar^2}{m} \rho_a |\nabla \hat{\phi}|^2(\mathbf{x}) + \frac{m v_s^2}{\rho_a} \hat{\Pi}^2(\mathbf{x}) \right), \tag{83}$$

where the density operator has been expressed in terms of the density fluctuation operator $\hat{\Pi}$: $\hat{\rho}_a(\mathbf{x}) = \rho_a + \hat{\Pi}(\mathbf{x})$; we assume the temperature $T \to 0$ such that the superfluid density $\rho_s = \rho_a$. At a general level, the Bose-field operator can be split into magnitude and phase, $\hat{\Psi}_a(\mathbf{x}) \sim \sqrt{\hat{\rho}_a(\mathbf{x})} e^{-i\hat{\phi}(\mathbf{x})}$. The Hamiltonian (1) can be diagonalized in terms of phonon operators:

$$\hat{\phi}(\mathbf{x}) = i \sum_{\mathbf{q}} \left| \frac{m v_s}{2\hbar \mathbf{q} V \rho_a} \right|^{1/2} e^{i\mathbf{q}\mathbf{x}} \left(b_{\mathbf{q}} - b^{\dagger}_{-\mathbf{q}} \right), \tag{84}$$

$$\hat{\Pi}(\mathbf{x}) = \sum_{\mathbf{q}} \left| \frac{\hbar \rho_a \mathbf{q}}{2 v_s V m} \right|^{1/2} e^{i\mathbf{q}\mathbf{x}} \left(b_{\mathbf{q}} + b^{\dagger}_{-\mathbf{q}} \right).$$

V is the volume of the system. For a one-dimensional superfluid, from the Haldane-Luttinger approach, we also identify $\pi/K = m v_s/(\hbar \rho_a)$ and for a weakly-interacting BEC, we get $m v_s^2/\rho_a = g_{aa}$ where g_{aa} embodies the collisional interactions between atoms a. We also infer $v_s = \sqrt{\frac{g_{aa}\rho_a}{m}}$ and thus

$$K = \frac{\pi}{\gamma_{aa}} \text{ with } \gamma_{aa} = \frac{m g_{aa}}{\rho_a \hbar^2}. \tag{85}$$

The coupling Hamiltonian between the two atomic systems can be written as follows:

$$H_c = \sum_i \left(-\frac{\hbar \delta_o}{2} + \frac{g_{ab}}{2} \left(\rho_a + \hat{\Pi}(\hat{\mathbf{x}}_i) \right) \right) \sigma_i^z \tag{86}$$

$$+ \frac{\hbar \Delta}{2} \left(1 - \frac{\hat{\Pi}(\hat{\mathbf{x}}_i)}{2\rho_a} \right) \left(\sigma_i^+ e^{-i\hat{\phi}(\hat{\mathbf{x}}_i)} + \text{H.c.} \right).$$

$\mathbf{x}_i$ denotes the position of a well at site i, Δ is an effective Rabi frequency which embodies the Raman transition from the reservoir atoms to the lowest vibrational state in the Atomic Quantum Dots, g_{ab} is the collisional parameters between atoms a and b (which strongly depends on the scattering length a_{ab}), and δ_o is a mean-field detuning field which can be normalized to include a shift due to the virtual admixture of the double occupied state in the dots. We consider that collisional interactions between b atoms in neighboring wells are negligible since the Wannier wavefunction of a b atom is well centered on each well and we exclude couplings to higher vibrational states assuming that these states are off resonant.

The Rabi term which involves the density fluctuation operator can be ignored in the limit $\hbar\Delta \ll g_{ab}\rho_a$ which will be the limit considered below.

The total Hamiltonian takes the form

$$H = H_a + \sum_i -\frac{\hbar\delta}{2}\sigma_i^z + \left(\frac{g_{ab}}{2}\hat{\Pi}(\hat{\mathbf{x}}_i)\sigma_i^z + \frac{\hbar\Delta}{2}\left(\sigma_i^+ e^{-i\hat{\phi}(\hat{\mathbf{x}}_i)} + \text{H.c.}\right)\right).$$

We consider below that, for a given g_{ab}, the mean-field detuning is adjusted such that $\hbar\delta_o \sim g_{ab}\rho_a$, *i.e.*, so that the charge states $|0\rangle$ and $|1\rangle$ in each well are degenerate: $\delta \to 0$.

5.1 Ferromagnetic Induced Coupling between Spins

First, one can apply a unitary transformation of the form to eliminate the phonons in the Raman coupling which will become a transverse field:

$$U = \exp\left[-i\sum_j \frac{\sigma_j^z}{2}\hat{\phi}(\mathbf{x}_j)\right]. \tag{86}$$

This generates rotations around the σ_i^z-axes in the spin space and thus

$$U^\dagger \sigma_i^+ U = \sigma_i^+ e^{i\hat{\phi}(\mathbf{x}_i)}. \tag{87}$$

On the other hand, H_a does not commute with U and thus

$$\begin{aligned} U^\dagger H_a U = H_a - \sum_i \frac{mv_s^2}{2\rho_a}\hat{\Pi}(x_i)\sigma_i^z \\ + \frac{1}{2}[\mathcal{S}, [\mathcal{S}, H_a]], \end{aligned} \tag{88}$$

where we have defined $\mathcal{S} = i\sum_j \sigma_j^z \hat{\phi}(\mathbf{x}_j)/2$. This results in:

$$\frac{1}{2}[\mathcal{S}, [\mathcal{S}, H_a]] = -\frac{i}{8\rho_a} mv_s^2 \sum_{j,k} \sigma_j^z \sigma_k^z [\hat{\phi}(\mathbf{x}_j), \hat{\Pi}(\mathbf{x}_k)]. \tag{89}$$

In a similar way, we get:

$$[\mathcal{S}, \sum_k \frac{g_{ab}}{2}\hat{\Pi}(\mathbf{x}_k)\sigma_k^z] = \frac{i}{4} g_{ab} \sum_{j,k} \sigma_j^z \sigma_k^z [\hat{\phi}(\mathbf{x}_j), \hat{\Pi}(\mathbf{x}_k)]. \tag{90}$$

Now, we can use the fact that (example in one dimension)

$$
\begin{aligned}
[\hat{\phi}(x), \hat{\Pi}(x')] &= \frac{i}{2L}\sum_{q,q'} e^{iqx}(b_q - b^\dagger_{-q})e^{iq'x}(b_{q'} + b^\dagger_{-q'}) \\
&= \frac{i}{L}\sum_q e^{iq(x-x')} \\
&= \frac{i}{\pi}\int_0^\infty \frac{d\omega}{v_s} e^{-\frac{\omega}{\omega_c}} e^{i\frac{\omega}{v_s}(x-x')} = \frac{i}{\pi}\frac{1}{v_s}\frac{i(x-x')/v_s + 1/\omega_c}{\left(\frac{1}{\omega_c}\right)^2 + \left(\frac{(x-x')}{v_s}\right)^2} \\
&= \frac{i}{\pi}\frac{\omega_c/v_s}{1+\omega_c^2(x-x')^2/v_s^2}.
\end{aligned} \tag{91}
$$

In the limit $\omega_c \to \infty$, this would converge to a δ-function:

$$
[\hat{\phi}(x), \hat{\Pi}(x')] = \frac{i}{\pi}\int_0^\infty \frac{d\omega}{v_s} e^{i\frac{\omega}{v_s}(x-x')} = \frac{1}{\pi}\frac{1}{(x-x')} = i\delta(x-x'). \tag{92}
$$

After the unitary transformation, we get the following Hamiltonian:

$$
\begin{aligned}
\tilde{H} = U^\dagger H U &= H_a + \sum_i \frac{\hbar\Delta}{2}\sigma_i^x \\
&+ \sum_i \left(\frac{g_{ab}}{2} - \frac{m v_s^2}{2\rho_a}\right)\hat{\Pi}(\mathbf{x}_i)\sigma_i^z \\
&- \sum_{\langle i;j\rangle}\left(\frac{g_{ab}}{4}\rho_a - \frac{\rho_a}{8}\frac{m v_s^2}{\rho_a}\right)\sigma_i^z\sigma_j^z.
\end{aligned} \tag{93}
$$

We have used the fact that $\omega_c/v_s = \pi\rho_a$. For simplicity, we only kept the dominant nearest-neighbor interaction in the last term taking into account that the healing length $\xi = v_s/\omega_c = 1/\pi\rho_a$ is not too large compared to the AQD size (each well).

As shown in Appendix F, one can also write the theory as a **dissipative Quantum Ising model:**

$$
\begin{aligned}
S = &\sum_i \int_0^\beta d\tau \hbar\frac{\Delta}{2}\sigma_i^x(\tau) \\
&- \sum_i \frac{1}{8}\int_{-\infty}^{\infty} d\tau' \int_0^{\beta\hbar} d\tau \alpha(\tau-\tau')\sigma_i^z(\tau)\sigma_i^z(\tau') \\
&+ \sum_{\langle i;j\rangle}\int_0^\beta d\tau\, J\sigma_i^z(\tau)\sigma_j^z(\tau).
\end{aligned} \tag{94}
$$

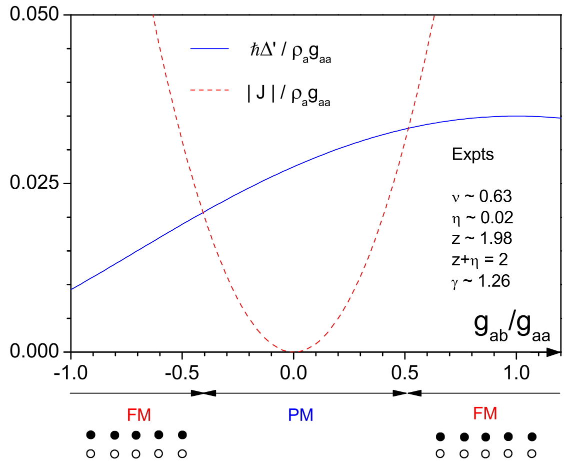

Fig. 8. Two competing energy scales and Phase diagram in 1D

The effective density of states for the dissipative bath takes the form

$$\mathcal{J}(\omega) = \sum_{\mathbf{q}} \lambda_{\mathbf{q}}^2 \delta(\omega - \omega_{\mathbf{q}}) = \alpha \omega^d, \tag{95}$$

where d is the dimensionality and α is the dimensionless dissipative parameter:

$$\alpha = \frac{m v_s}{2(2\pi)^d \rho_a \hbar} \left(\frac{g_{ab}\rho_a}{m v_s^2} - 1 \right)^2 . \tag{96}$$

The fact that the phonon-induced coupling is *ferromagnetic* stems from the fact that unlike fermions the bosons have only a $q = 0$ part in the density operator. This is consistent with the fact that for $g_{ab} = 0$ the original Hamiltonian is invariant under the transformation $\sigma_{iz} \to -\sigma_{iz}$ which ensures $\langle \sigma_{iz} \rangle = 0$.

5.2 Discussion and Conclusion

At $g_{ab} = m v_s^2/\rho_a = g_{aa}$, one gets a *non-dissipative* quantum Ising model. For realistic parameters, the spin array is in a ferromagnetic ground state (for one dimension and two dimensions). One can detect the ferromagnetic ground state by following the population of the b atoms which should obey $n_{bi} = \hat{b}_i^\dagger \hat{b}_i = 0$ or 1 at zero detuning. **The effect of Dissipation on the quantum phase transition has been discussed in the paper P. P. Orth, I. Stanic, K. Le Hur, 2008.** A coupling between spins has been recently observed in M. R. Delbecq et al. Nature Communications **4**, 1400 (2013).

A Spin-boson model and P(E) theory

Here, we formulate a precise correspondence between the perturbative calculation at small $\Delta/h \ll 1$ and the well-known $P(E)$ theory of dissipative tunneling problems. By definition $P(E)$ obeys:

$$P(E) = \prod_m e^{-|\gamma_m|^2} \sum_{n_m} \frac{|\gamma_m|^{2n_m}}{n_m!} \delta\left(E - n_m \hbar\omega_m\right). \tag{.1}$$

We have explicitly taken into account that in $P(E)$ all the modes m contribute. This expression can be rewritten as

$$\begin{aligned} P(E) &= \int \frac{dt}{2\pi} \prod_m \sum_{n_m} \frac{|\gamma_m|^{2n_m}}{n_m!} e^{-|\gamma_m|^2} e^{i(E - n_m\omega_m)t} \\ &= \int \frac{dt}{2\pi} e^{iEt} \prod_m \sum_{n_m} \frac{|\gamma_m|^{2n_m}}{n_m!} e^{-in_m\omega_m t - |\gamma_m|^2} \\ &= \int \frac{dt}{2\pi} e^{iEt} \prod_m e^{|\gamma_m|^2 (e^{-i\omega_m t} - 1)} \\ &= \int \frac{dt}{2\pi} e^{iEt} e^{\sum_m |\gamma_m|^2 (e^{-i\omega_m t} - 1)} \\ &= \int \frac{dt}{2\pi} e^{iEt} e^{K(t)}, \end{aligned} \tag{.2}$$

where we have defined $K(t) = \tilde{K}(t) - \tilde{K}(0)$ and

$$\begin{aligned} \tilde{K}(t) &= \int \frac{d\omega}{\pi} \sum_m \frac{\lambda_m^2}{\omega^2} \delta(\omega - \omega_m) e^{-i\omega t} \\ &= 2 \int_0^{\omega_c} d\omega \frac{\alpha}{\omega} e^{-i\omega t}. \end{aligned} \tag{.3}$$

At long imaginary times τ, in the ohmic case, this gives rise to the usual function:

$$\begin{aligned} K(\tau) &= -2\alpha \int_0^{+\infty} \frac{d\omega}{\omega} (1 - e^{-\omega|\tau|}) e^{-\omega/\omega_c} \\ &= -2\alpha \ln(1 + \omega_c|\tau|). \end{aligned} \tag{.4}$$

The logarithm appears due to the summation over the infinite number of modes. At low energy E, this results in $P(E) \propto E^{2\alpha-1}$; there is an orthogonality catastrophe for $\alpha < 1$ whereas for $\alpha > 1$, $P(E) \to 0$ at $E \to 0$. $P(E)$ *can be interpreted as the probability to "emit" the energy E to the bath when flipping the spin*. At finite temperature, one rather gets:

$$K(t) = \alpha \int_{-\infty}^{+\infty} d\omega \frac{J(\omega)}{\omega^2} (n(|\omega|) + \theta(\omega))(e^{-i\omega t} - 1). \tag{.5}$$

The step-function $\theta(\omega)$ corresponds to the zero-point fluctuations and $n(\omega)$ is the Bose-Einstein distribution function. If the temperature is large, $K(t)$ is controlled by thermal excitations of the bath oscillators. The real part of $K(t)$ thus becomes $\sim -\alpha T t$ in the long-time limit. The bath function $P(E)$ becomes a Lorentzian and for very small α the result converges to that of an uncoupled two-level system.

B Exact Mapping Onto the Kondo Model

Here, we discuss the relation between the spin-boson model with Ohmic dissipation $(J(\omega) \propto \omega)$ and the Kondo model following the review by A. J. Leggett *et al.*. See also the papers by M. Blume, V. J. Emery and A. Luther Phys. Rev. Lett. **25**, 450 (1970) and F. Guina, V. Hakim and A. Muramatsu Phys. Rev. B **32**, 4410 (1985).

Let us remind that the Kondo model takes the general form

$$H_K + \sum_k \epsilon(\mathbf{k}) c_k^\dagger c_k + J\mathbf{S} \cdot s(0), \tag{.1}$$

where the operators $c_k^\dagger$ create conduction electrons of wavevector $\mathbf{k}$ and spin $\sigma =\uparrow,\downarrow$ or $\sigma = \pm$. The impurity spin is $\mathbf{S}$ whereas $\mathbf{s}(0)$ denotes the effective spin due to the conduction electrons at $\mathbf{r} = 0$. To show the correspondence between the spin-boson model and the Kondo model one can proceed as follows.

First, it is important to observe that when the exchange J is pointlike only s-wave scattering occurs. One can expand the plane-wave electron states $\mathbf{k}$ in spherical waves around the impurity and the only electrons affected are those with angular momentum quantum numbers $l = m = 0$. Therefore, we may characterize the relevant states simply by the magnitude $|\mathbf{k}|$ of the wavevector, which reduces the problem to an essentially **one-dimensional** problem.

The next step is to consider that for low-temperature compared to T_F the dominant excitations are those in the immediate vicinity of the Fermi surface. We may thus linearize the dispersion relation $\epsilon(\mathbf{k})$ around the Fermi energy ϵ_F in the form

$$\epsilon(\mathbf{k}) = \epsilon_F + \hbar v_F(|\mathbf{k}| - k_F). \tag{.2}$$

The kinetic term for the free fermions becomes

$$H_{Kin} = \hbar v_F \sum_{p\sigma} p c_{p\sigma}^\dagger c_{p\sigma}, \tag{.3}$$

where $c_{p\sigma}^\dagger$ creates an electron with spin σ and momentum $|\mathbf{k}| = p + k_F$ and $l = m = 0$. We introduce Wannier operators for an electrons spin σ at the origin by

$$c_\sigma^\dagger = L^{-1/2} \sum_p c_{p\sigma}^\dagger, \tag{.4}$$

where L is the length of a normalization box, such that the wavevector $\mathbf{p}$ have values $p = 2\pi/L.n$, $n = 0, \pm 1, \pm 2, ...$, and the limit $L \to \infty$ is taken according to

$L^{-1}\sum_p \to \int dp/2\pi$. We redefine $\mathbf{S} = \frac{1}{2}\sigma$ and

$$\mathbf{s}(0) = \frac{1}{2}\sum_{\sigma\sigma'} c^\dagger \sigma_{\sigma\sigma'} c_{\sigma'}, \tag{.5}$$

where σ is the vector of the 2×2 Pauli matrices. The Kondo Hamiltonian can be rewritten as

$$\begin{aligned} H_K = &\hbar v_F \sum_{p\sigma} p c^\dagger_{p\sigma} c_{p\sigma} + \frac{J_z}{4}\sigma_z \sum_\sigma \sigma c^\dagger_\sigma c_\sigma \\ &+ \frac{J_\perp}{2}\left(\sigma_+ c^\dagger_\downarrow c_\uparrow + H.c.\right), \end{aligned} \tag{.6}$$

and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. Note that J_z and $J_\perp$ now have dimensions of energy times length, and therefore the relevant dimensionless coupling parameters are ρJ_z and $\rho J_\perp$ with

$$\rho = (2\pi\hbar v_F)^{-1} \tag{.7}$$

as the single spin density of states at the Fermi surface.

To derive the equivalence between this Kondo model and the spin-boson model the idea is to follow the ideas due to Schotte (1970) and to make a link with the bosonization technique which will be discussed more deeply in the next chapter. Let us introduce the **bosonic** charge- and spin-density operators for the fermions by

$$\begin{aligned} \rho(k) &= \sum_{p\sigma} c^\dagger_{p+k\sigma} c_{p\sigma},\ \rho(-k) = \rho^\dagger(k), \\ \sigma(k) &= \sum_{p\sigma} \sigma c^\dagger c^\dagger_{p+k\sigma} c_{p\sigma},\ \sigma(-k) = \sigma^\dagger(k), \end{aligned} \tag{.8}$$

with $k > 0$. Then, for a semi-infinite band with all the states $p < 0$ filled, it is straightforward to show that the operators

$$b_k = \left[\frac{\pi}{kL}\right]^{1/2} \rho(-k) \text{ and } a_k = \left[\frac{\pi}{kL}\right]^{1/2} \sigma(-k), \tag{.9}$$

obey commutation relations $[a_k, a^\dagger_{k'}] = [b_k, b^\dagger_{k'}] = \delta_{kk'}$. In one dimension, it is easy to write the kinetic term of fermions using the bosonized form

$$H_{Kin} = \hbar v_F \sum_{k>0} k(a^\dagger_k a_k + b^\dagger_k b_k). \tag{.10}$$

The non-interacting gas in one dimension can be replaced by Bose-like excitations of charge and spin type which are decoupled.

Now, we can use $\sum_k \sigma(k) = L\sum_\sigma \sigma c^\dagger_\sigma c_\sigma$ and therefore the z part of the Kondo Hamiltonian can be exactly rewritten as

$$\frac{J_z}{4}\sigma_z \sum_{k>0}\left[\frac{k}{\pi L}\right]^{1/2}(a_k + a_k^\dagger). \tag{.11}$$

By contrast, the mixed products $c^\dagger_\uparrow c_\downarrow$ that occur in the spin-flip scattering term cannot be linearly related to $\sigma(k)$ and $\rho(k)$. Thus, the nontrivial task is to find a nonlinear representation of these combinations. **The central idea of the bosonization is to realize that the exponential of a particular combination of Bose operators can be made into an anticommuting Fermi field.** Let us define

$$j_\sigma(x) = \sum_{k>0} e^{-ak/2}\left[\frac{2\pi}{kL}\right]^{1/2}(b_{k\sigma}e^{i\sigma kx} - b^\dagger_{k\sigma}e^{-i\sigma k\sigma}) = -j^\dagger_\sigma(x), \tag{.12}$$

with Bose operators

$$b_{k\sigma} = \frac{b_k + \sigma a_k}{\sqrt{2}}, \tag{.13}$$

and an exponential cutoff that eliminates k values large compared to the inverse microscopic length a^{-1}. The coefficients have been chosen such that (for $a \to 0$)

$$[j_\sigma(x), j_\sigma(y)] = -i\pi\sigma \text{sign}(x-y). \tag{.14}$$

Using

$$e^A e^B = e^B e^A e^{[A,B]}, \tag{.15}$$

we deduce that $\exp j_\sigma(x)$ behaves like a Fermi field $\Psi_\sigma(x)$. In general, we can define fermion operators in one dimension as

$$\Psi(x) = (2\pi a)^{-1/2}\exp j_\sigma(x). \tag{.16}$$

One can then rewrite the Kondo Hamiltonian in terms of the spin-density wave component $\sigma(k)$

$$\begin{aligned} H_K^B =& \hbar v_F \sum_k a_k^\dagger a_k \\ &+ \frac{J_z}{4}\sigma_z \sum_{k>0} e^{-ak/2}\left[\frac{k}{\pi L}\right]^{1/2}(a_k + a_k^\dagger) \\ &+ \frac{J_\perp}{4\pi a}\left[\sigma_+ \exp(\xi) + \sigma_- \exp(-\xi)\right], \end{aligned} \tag{.17}$$

with

$$\xi = \sum_k e^{-ak^2}\left[\frac{4\pi}{kL}\right]^{1/2}(a_k - a_k^\dagger). \tag{.18}$$

The idea is now to perform a Schrieffer-Wolff transformation $S^\dagger H_K^B S$ with $S = \exp(\frac{1}{2}\sigma_z\xi)$ to obtain a spin boson model!

$$S^\dagger H_K^B S = \frac{J_\perp}{4\pi a}\sigma_x + \left[\frac{J_z}{4\pi} - \hbar v_F\right]\sigma_z \sum_{k>0} e^{-ak/2}\left[\frac{\pi k}{L}\right]^{1/2}(a_k + a_k^\dagger) \qquad (.19)$$
$$+ \hbar v_F \sum_{k>0} k a_k^\dagger a_k.$$

Choosing the following convention for the spin-boson model

$$H_{SB} = -\frac{\Delta}{2}\sigma_x + \frac{h}{2}\sigma_z + H_{osc} \qquad (.20)$$
$$+ \frac{\sqrt{2\alpha}}{2} v_F \sigma_z \sum_q \sqrt{\frac{2\pi q}{L}}(a_q + a_q^\dagger)e^{-aq/2},$$

where α is the dimensionless dissipative parameter.

We identify the important connections:

$$\frac{\hbar\Delta}{2} = -\frac{J_\perp}{4\pi a} \qquad (.21)$$
$$-\sqrt{\alpha} = \frac{J_z}{4\pi\hbar v_F} - 1.$$

The level asymmetry in the spin-boson model can be also generated in the Kondo model by applying a local field. The transverse field in the spin-boson model can be identified to the transverse coupling of the Kondo model whereas the dissipative coupling with the bosons is embodied by the Ising part of the Kondo model. For small couplings with the bosons $\alpha \ll 1$, we deduce that the ground state of the spin-boson model is equivalent to this of the Kondo model with $J_z > 0$. This produces a Fermi liquid type ground state. In particular, one predicts $\langle\sigma_z\rangle \to 0$ for $\epsilon \to 0$. One can derive an exact expression of the Kondo energy scale as a function of α

$$T_K(\alpha) = \Delta\left(\frac{\Delta}{\omega_c}\right)^{\alpha/1-\alpha}, \qquad (.22)$$

where ω_c is a high-fequency cutoff for the boson bath. For $\alpha \to 1$, the Kondo energy scale takes a similar form as in the SU(2) Kondo problem discussed earlier with $J_\perp \sim J_z$. For $\alpha > 1$, the ground state of the spin-boson model becomes equivalent to the ferromagnetic regime of the Kondo model; the impurity remains unscreened at zero temperature. **At $\alpha_c = 1$, there is a quantum phase phase transition induced by the zero-point fluctuations of the environment.** There is a finite jump in $\langle\sigma_z\rangle$ which is typical of a Kosterlitz-Thouless transition.

C Steps to Treat Blip and Sojourn Contributions

These technical details can be found in the paper P. P. Orth, A. Imambekov, K. Le Hur Phys. Rev. B (2013). We first analyze the $\mathcal{Q}_2$ part of the influence functional

F_n in Eq. (40), that describes the interactions between blips. Before we can apply a Hubbard-Stratonovich transformation, we diagonalize the kernel $Q_2(t)$ and write it in a factorized form as following A. Imambekov, V. Gritsev and E. Demler:

$$Q_2(t_j - t_k) = \pi\alpha\Big[G_0 + \sum_{m=1}^{m_{\max}} G_m \Psi_m(t_j)\Psi_m(t_k)\Big]\,. \tag{.1}$$

We truncate the sum and keep $m_{\max}$ terms, but always check that the final result is independent of $m_{\max}$.

To achieve this, we expand $Q_2(t)$ in a Fourier series. To obtain only negative Fourier coefficients, we rather expand $\tilde{Q}_2(\tau) = Q_2(\tau) - Q_2(2)$ and write

$$Q_2(\tau) = Q_2(2) + \pi\alpha\Big[g_0 + \sum_{m=1}^{m_{\max/2}} g_m \cos\frac{m\pi\tau}{2}\Big]\,, \tag{.2}$$

on the interval $\tau \in (-2, 2)$. Here, $\tau = (t - t_I)/t_{\text{tot}}$ is a rescaled time which depends on the total length of our numerical simulation t_{tot}. The time $\tau = 0$ corresponds to the initial time t_I, when the large-bias constraint on the spin is turned-off. The time $\tau = 1$ corresponds to the final time $t_{\max} = t_{\text{tot}} + t_I$ of our numerical simulation. Note that $Q_2(2)$ is a constant that depends on the length of the simulation t_{tot}. At $T = 0$, it reads for example $Q_2(2) = \pi\alpha \ln[1 + 4\omega_c^2 t_{\text{tot}}^2]$. Since we obtain the Fourier coefficients g_m numerically, this approach is quite general and can be used for various forms of the bath correlation function $Q_2(t)$, as arise for different spectral densities $J(\omega)$. From the Fourier expansion, we identify the (trigonometric) eigenfunctions as $\Psi_{2k-1}(\tau) = \cos\frac{k\pi\tau}{2}$ as well as $\Psi_{2k}(\tau) = \sin\frac{k\pi\tau}{2}$. The coefficients in Eq. (.1) read

$$G_0 = g_0 + \frac{1}{\pi\alpha} Q_2(2) \tag{.3}$$

$$G_{2k-1} = G_{2k} = g_k < 0\,. \tag{.4}$$

We can thus write $\mathcal{Q}_2$ in factorized form as

$$\mathcal{Q}_2 = \exp\Big\{-n\alpha\Big[\ln(1 + 4\omega_c^2 t_{\text{tot}}^2) + G\Big]\Big\} \times \prod_{m=1}^{m_{\max}} \exp\Big\{\frac{1}{2}\Big[\sqrt{\alpha G_m}\sum_{j=1}^{2n} \Xi_j \Psi_m(t_j)\Big]^2\Big\}\,, \tag{.5}$$

with constant $G = \sum_{m=0}^{m_{\max}/2} g_m$. Note that $\lim_{m_{\max}\to\infty} G = -\ln(1 + 4\omega_c^2 t_{\text{tot}}^2)$, so the prefactor in Eq. (.5) approaches unity in this limit. To derive Eq. (.5), we have used that $\sum_{j>k\geq 1}^{2n} \Xi_j \Xi_k = -n$ and $\Xi_j^2 = 1$. Since $G_m < 0$ for $m \geq 1$ it is more appropriate to write $\sqrt{\alpha G_m} = i\sqrt{-\alpha G_m}$. Next, we decouple the blip variables $\{\Xi_j\}$ in the exponent in Eq. (.5) using a total of $m_{\max}$ Hubbard-Stratonovich transformations, resulting in

$$\mathcal{Q}_2 = \exp\Big\{-n\alpha\Big[\ln(1+4\omega_c^2 t_{\rm tot}^2)+G\Big]\Big\} \\ \times\int d\mathcal{S}\exp\Big\{i\sum_{j=1}^{2n}\Xi_j h_s(\tau_j)\Big\}\,. \tag{.6}$$

The integral over the Hubbard-Stratonovich variables $\{s_m\}$ reads

$$\int d\mathcal{S} = \prod_{m=1}^{m_{\rm max}}\int_{-\infty}^{\infty}\frac{ds_m}{\sqrt{2\pi}}e^{-s_m^2/2}\,, \tag{.7}$$

and we have introduced a real (height) function

$$h_s(\tau) = \sum_{m=1}^{m_{\rm max}} s_m\sqrt{-\alpha G_m}\Psi_m(\tau)\,. \tag{.8}$$

The function $h_s(\tau)$ contains information about the environment via the eigenfunctions and eigenvalues of the bath correlation function $Q_2(t)$. It also depends on the Hubbard-Stratonovich variables $\{s_m\}$, which can be interpreted as Gaussian distributed random (noise) variables. One thus finds that $\langle h_s(t)\rangle_{\mathcal{S}} = 0$ and

$$\langle h_s(t)h_s(s)\rangle_{\mathcal{S}} = \alpha G_0 - Q_2(t-s)/\pi, \tag{.9}$$

while all higher moments vanish.

Let us now turn to the $\mathcal{Q}_1$ part of the influence functional $F_n = \mathcal{Q}_1\mathcal{Q}_2$, that couples the blip and sojourn part of the spin path. It is important to distinguish between the first sojourn, which occurs during initial time $t_0 \le t \le t_I$ when the spin is polarized, and all other sojourns. We thus separate

$$\mathcal{Q}_1 = \mathcal{Q}_1^{(0)}\mathcal{Q}_1^{(1)}. \tag{.10}$$

The contribution of the first sojourn $\mathcal{Q}_1^{(0)}$ encodes the initial preparation of the system. It is given by the terms where $k = 0,1$ in Eq. (41) and reads $\mathcal{Q}_1^{(0)} = \exp[\frac{i}{\pi}\sum_{j=1}^{2n}\Xi_j\{Q_1(t_j-t_0)-Q_1(t_j-t_1)\}]$.

The contribution of all later sojourns $\mathcal{Q}_1^{(1)}$ is given by the terms with $k \ge 2$ in Eq. (41) and reads $\mathcal{Q}_1^{(1)} = \exp[\frac{i}{\pi}\sum_{j>k\ge 2}^{2n}\Xi_j\Upsilon_k Q_1(t_j-t_k)]$. Fortunately, it takes a particularly simple form for an Ohmic bath and if $\Delta/\omega_c \ll 1$ and $\alpha < 1/2$ (scaling limit), where one may safely approximate

$$Q_1(t) = 2\pi\alpha\tan^{-1}(\omega_c t) \approx \alpha\pi^2\theta(t)\,. \tag{.11}$$

We thus find that

$$\mathcal{Q}_1^{(1)} = \exp\Big[i\pi\alpha \sum_{j>k\geq 2}^{2n} \Xi_j \Upsilon_k\Big] = \exp\Big[i\pi\alpha \sum_{k=1}^{n-1} \xi_{k+1}\eta_k\Big]. \tag{.12}$$

This reflects the fact that the main contribution to the path integral stems from paths with spin-flip separations larger than ω_c^{-1}.

If we use the scaling form $Q_1(t) = \alpha\pi^2\theta(t)$ for the first sojourn as well, this corresponds to the spin-bath preparation where $t_0 = t_I$. In this case, the complete blip-sojourn interaction term $\mathcal{Q}_1 = \mathcal{Q}_1^{(0)}\mathcal{Q}_1^{(1)}$ is given by

$$\mathcal{Q}_1 = \exp\Big[i\pi\alpha \sum_{j>k\geq 0}^{2n} \Xi_j \Upsilon_k\Big] = \exp\Big[i\pi\alpha \sum_{k=0}^{n-1} \xi_{k+1}\eta_k\Big]. \tag{.13}$$

Let us finally note that we can, in principle, deal with the blip-sojourn interaction term $\mathcal{Q}_1$ in a similar way as with $\mathcal{Q}_2$. In this case, we must first separate the bath correlation function $Q_1(t)$ in the exponent into a symmetric part $Q_1(|t|)$ and an anti-symmetric part $Q_1(t)$, in order to extend the sum over the blip and sojourn variables to $j \leq k$. Then, we can diagonalize the kernels, complete the square in the exponent and linearize it using Hubbard-Stratonovich transformations. The resulting expression for the height function $h_s(\tau)$, however, is no longer purely real, but also contains an imaginary component.

D Resistance as a Transmission Line and Spin-Boson Model

First, we build the Hamiltonian of a one-dimensional transmission line. The system is a collection of harmonic oscillators (normal modes) and therefore can be readily quantized. Each capacitor possesses an energy $Q_n^2/(2C_t)$ with Q_n being the charge on each capacitor plate. Each inductance generates the energy

$$W = \frac{(\varphi_{n+1} - \varphi_n)^2}{2L_t}. \tag{.1}$$

Here, φ embodies the magnetic flux. We expand the Josephson energies to second order in $\varphi_{n+1} - \varphi_n$ (since φ_n evolves smoothly along the transmission line), resulting in the inductance L_t. Then, we get the following Hamiltonian

$$H_0 = \sum_{n=0}^{\infty} \left[\frac{Q_n^2}{2C_t} + \frac{(\varphi_{n+1} - \varphi_n)^2}{2L_t}\right]. \tag{.2}$$

Then, it is convenient to introduce the capacitance $c = C_t/a$ and inductance $l = L_t/a$ per unit length such that:

$$H_0 = \int_0^{\mathcal{L}} dx \left(\frac{1}{2c} q(x)^2 + \frac{1}{2l}\left(\partial_x\varphi\right)^2\right). \tag{.3}$$

We have substituted $x = na$. The commutator $[\varphi_m, Q_n] = i\delta_{m,n}$ then turns into

$$[\varphi(x), q(y)] = i\delta(x - y). \tag{.4}$$

$q(x) = Q_n/a$ represents the charge density and $\varphi_n = \varphi(x)$ the flux variable. This "continuum" description is appropriate as long as $a/\mathcal{L} \ll 1$ where $\mathcal{L} \to +\infty$ is the length of each transmission line and a the size of a unit cell. Introducing the variable $\theta(x,t)$ such that $\partial_x\theta(x) = q(x)$, then leads to the usual Euler-Lagrange (or wave) equation for the transmission line. For open boundaries, the spatial solution of the modes can be expressed in terms of the wavevectors $k \approx m\pi/(2\mathcal{L})$ where m is odd for symmetric modes and even for antisymmetric modes and the transmission line can be easily diagonalized introducing the bosonic creation and annihilation operators

$$[b_k, b_{k'}^\dagger] = \delta_{k,k'}. \tag{.5}$$

Then, photon excitations in a given transmission line are described through the Hamiltonian:

$$H_0 = \sum_{k>0} \hbar v|k| \left[b_k^\dagger b_k + \frac{1}{2} \right]. \tag{.6}$$

The Hamiltonian H_0 for the transmission line is justified for frequencies smaller than $\omega_c = v/a$.

E Josephson-Kondo circuit

Close to a charge degeneracy line, we employ the pseudospin representation for the charge states $(0,1)$ and $(1,0)$ reinterpreting them as spin-up and spin-down eigenstates of the operator σ_z. The effective detuning $\epsilon = (E_{10} - E_{01}) \to 0$, where E_{10} (E_{01}) corresponds to the energy of the spin-down (spin-up) eigenstate, can be adjusted through the gate voltages.

Transfer of Cooper pairs between islands and leads is described through the terms E_{JL} and E_{JR} in Fig. of the main text. In the limit of weak Josephson tunneling $(E_{JL}, E_{JR}) \ll \min(E_{11} - E_{10}, E_{00} - E_{10})$ one can perform a standard perturbation theory and cotunneling of Cooper pairs then results in:

$$E_J = \frac{E_{JL}E_{JR}}{4} \sum_{j=0,1} \left[\frac{1}{E_{jj} - E_{01}} + \frac{1}{E_{jj} - E_{10}} \right], \tag{.1}$$

where E_{11} (E_{00}) corresponds to the energy to add (remove) one extra Cooper on the double-island.

The Josephson Hamiltonian then takes the form $-(E_J/2)\sigma^+ \exp[i(\phi_l - \phi_r)(x = 0)] + h.c.$ where, in one dimension, the Josephson phases $\phi_l(x = 0)$ and $\phi_r(x = 0)$ are defined in Sec. II. Hereafter, we introduce the parameter

$$\alpha_k = \frac{1}{\sqrt{c\mathcal{L}}} \sqrt{\hbar\omega_k}. \tag{.2}$$

To rewrite the Josephson term as a transverse field $H_J = -(E_J/2)\sigma_x$ one can perform a spin rotation $U = \exp(A_l - A_r)$ where $A_i = \sum_{k>0} \frac{\alpha_k}{\hbar\omega_k}(b^\dagger_{ik} - b_{ik})\sigma_z/2$.

Since the Hamiltonian H_0 of the transmission lines does not commute with A_i this results in an extra term in the Hamiltonian $H^J_{Int} = [H_0, A_l - A_r]$:

$$H^J_{Int} = \sum_{k>0} \left[\alpha_k(b_{lk} + b^\dagger_{lk}) - \alpha_k(b_{rk} + b^\dagger_{rk})\right] \frac{\sigma_z}{2}. \qquad (.3)$$

Here, we discuss details of the charging energy for the double Cooper-pair box system and we closely follow the notations in the main text. The energetics of this double-island system then can be obtained from

$$\begin{aligned} Q_a &= C_L(V_a - V_l) + C_{ga}(V_a - V_{ga}) + C_m(V_a - V_b) \\ Q_b &= C_R(V_b - V_r) + C_{gb}(V_b - V_{gb}) + C_m(V_b - V_a), \end{aligned} \qquad (.4)$$

where Q_a and Q_b represent the charges on the two islands, V_a and V_b are the related potentials, C_m is the capacitance between islands, C_L and C_R describe the capacitive couplings with the transmission lines and V_l and V_r can be identified to the electric potentials at the position $x = 0$ of the two transmission lines.

Then,

$$\begin{pmatrix} Q_a + C_L V_l + C_{ga} V_{ga} \\ Q_b + C_R V_r + C_{gb} V_{gb} \end{pmatrix} = \begin{pmatrix} C_{\Sigma a} & -C_m \\ -C_m & C_{\Sigma_b} \end{pmatrix} \begin{pmatrix} V_a \\ V_b \end{pmatrix} \qquad (.5)$$

where:

$$\begin{aligned} C_{\Sigma a} &= C_L + C_{ga} + C_m \\ C_{\Sigma_b} &= C_R + C_{gb} + C_m. \end{aligned} \qquad (.6)$$

By inverting the matrix, the electrical potentials on the two islands satisfy:

$$\begin{pmatrix} V_a \\ V_b \end{pmatrix} = \frac{1}{C_{\Sigma a} C_{\Sigma b} - C_m^2} \begin{pmatrix} C_{\Sigma b} & C_m \\ C_m & C_{\Sigma_a} \end{pmatrix} \begin{pmatrix} Q_a + C_L V_l + C_{ga} V_{ga} \\ Q_b + C_R V_r + C_{gb} V_{gb} \end{pmatrix}. \qquad (.7)$$

The electrostatic energy of the double island then is given by

$$U(Q_a, Q_b) = \frac{1}{2}(Q_a + C_{ga} V_{ga}) V_a + \frac{1}{2}(Q_b + C_{gb} V_{gb}) V_b, \qquad (.8)$$

where V_a and V_b are given above. The effect of the finite (but small) capacitive couplings H^C_{Int} with the transmission lines takes the form

$$\frac{1}{2} \frac{C_{\Sigma b}}{C_{\Sigma a} C_{\Sigma b} - C_m^2} C_L (Q_a V_l), \qquad (.9)$$

and similarly

$$\frac{1}{2}\frac{C_{\Sigma a}}{C_{\Sigma a}C_{\Sigma b}-C_m^2}C_R(Q_bV_r) \tag{.10}$$

where:

$$\begin{aligned} C_{\Sigma a} &= C_L + C_{ga} + C_m \\ C_{\Sigma_b} &= C_R + C_{gb} + C_m. \end{aligned} \tag{.11}$$

There are additional contributions in Q_aV_r and Q_bV_l due to the interdot capacitive coupling C_m:

$$\begin{aligned} &\frac{1}{2}\frac{C_m}{C_{\Sigma a}C_{\Sigma b}-C_m^2}C_R(Q_aV_r) \\ &\frac{1}{2}\frac{C_m}{C_{\Sigma a}C_{\Sigma b}-C_m^2}C_L(Q_bV_l). \end{aligned} \tag{.12}$$

Using the qubit representation $Q_b = \frac{1}{2}(1+\sigma_z)$ and $Q_a = \frac{1}{2}(1-\sigma_z)$ then the total Hamiltonian $H = H_0 + H_J + H_{Int}^J + H_{Int}^C$ can be summarized as:

$$\begin{aligned} H =& \sum_{i=l,r}\sum_{k>0}\hbar v|k|\left[b_{ik}^\dagger b_{ik} + \frac{1}{2}\right] - \frac{\epsilon}{2}\sigma_z - \frac{E_J}{2}\sigma_x \\ &+ \sum_{k>0}\alpha_k\left(-\gamma_l(b_{lk}+b_{lk}^\dagger) + \gamma_r(b_{rk}+b_{rk}^\dagger)\right)\frac{\sigma_z}{2}, \end{aligned} \tag{.13}$$

where the detuning satisfy $\epsilon \to 0$ and

$$\begin{aligned} \gamma_r &= -1 + \frac{C_R}{2}\left(\frac{C_{\Sigma a}}{C_{\Sigma a}C_{\Sigma b}-C_m^2} - \frac{C_m}{C_{\Sigma a}C_{\Sigma b}-C_m^2}\right) \approx -1 \\ \gamma_l &= -1 + \frac{C_L}{2}\left(\frac{C_{\Sigma b}}{C_{\Sigma a}C_{\Sigma b}-C_m^2} - \frac{C_m}{C_{\Sigma a}C_{\Sigma b}-C_m^2}\right) \approx -1. \end{aligned} \tag{.14}$$

Remember that the charge $2e$ has been normalized to unity. The form (.13) of the Hamiltonian is used for studying photon transport. The analogy with the single-channel Kondo model becomes apparent when rewriting the Hamiltonian in terms of:

$$\begin{aligned} b_{sk} &= \cos\theta b_{lk} + \sin\theta b_{rk} \\ b_{ak} &= \sin\theta b_{lk} - \cos\theta b_{rk}. \end{aligned} \tag{.15}$$

Choosing $\cos\theta = \gamma_r/\sqrt{\gamma_l^2+\gamma_r^2}$ and $\sin\theta = \gamma_l/\sqrt{\gamma_r^2+\gamma_l^2}$, we note that the boson operator b_{ak} only couples to the effective spin-1/2 object resulting in

$$\lambda_k = \alpha_k \sqrt{\gamma_l^2 + \gamma_r^2} = \sqrt{\frac{\gamma_l^2 + \gamma_r^2}{c\mathcal{L}}} \sqrt{\hbar\omega_k}. \tag{.16}$$

Applying a unitary transformation (similar to U^{-1}), the Hamiltonian then can be rewritten as in the main text:

$$\tilde{H} = -\frac{\epsilon}{2}\sigma_z - \frac{E_J}{2}\sigma^+ e^{i(\Phi_l - \Phi_r)} + h.c. + \sum_{i=l,r}\sum_{k>0} \hbar v|k| \left[b_{ik}^\dagger b_{ik} + \frac{1}{2}\right], \tag{.17}$$

where the phases $\Phi_l = -\gamma_l \phi_l(x=0)$ and $\Phi_r = -\gamma_r \phi_r(x=0)$ contain Josephson physics as well as (weak) charging effects.

F Quantum Ising Model

The Hamiltonian can be rewritten with the proper counterterm to complete the square:

$$\begin{aligned} \tilde{H} = &\sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} \left(b_{\mathbf{q}} + \sum_i \frac{\lambda_{\mathbf{q}}}{\omega_{\mathbf{q}}} e^{-i\mathbf{q}\mathbf{x}_i} \frac{\sigma_i^z}{2}\right)^\dagger \\ &\qquad\qquad \times \left(b_{\mathbf{q}} + \sum_i \frac{\lambda_{\mathbf{q}}}{\omega_{\mathbf{q}}} e^{-i\mathbf{q}\mathbf{x}_i} \frac{\sigma_i^z}{2}\right) \\ &+ \sum_i \frac{\hbar\Delta}{2}\sigma_i^x + \sum_{\langle i;j\rangle} J\sigma_i^z\sigma_j^z, \end{aligned} \tag{.1}$$

where

$$\begin{aligned} \lambda_{\mathbf{q}} = &\left(\frac{g_{ab}\rho_a}{mv_s^2} - 1\right) \left|\frac{mv_s^3 \mathbf{q}}{2V\rho_a\hbar}\right|^{1/2} \\ J = &-\frac{g_{ab}^2\rho_a^2}{8mv_s^2}. \end{aligned} \tag{.2}$$

We have used the fact that

$$\begin{aligned} \sum_{\mathbf{q}} \frac{\hbar\lambda_{\mathbf{q}}^2}{4\omega_{\mathbf{q}}} e^{i\mathbf{q}(\mathbf{x}_i - \mathbf{x}_j)} = &\frac{mv_s^2}{4V\rho_a}\left(\frac{g_{ab}\rho_a}{mv_s^2} - 1\right)^2 \sum_{\mathbf{q}} e^{i\mathbf{q}(\mathbf{x}_i - \mathbf{x}_j)} \\ = &\frac{mv_s^2}{8}\left(\frac{g_{ab}\rho_a}{mv_s^2} - 1\right)^2 \frac{1}{1 + \omega_c^2(\mathbf{x}_i - \mathbf{x}_j)^2/v_s^2}. \end{aligned} \tag{.3}$$

This results in:

$$\sum_{\mathbf{q},i,j} \frac{\hbar\lambda_{\mathbf{q}}^2}{4\omega_{\mathbf{q}}} e^{i\mathbf{q}(\mathbf{x}_i-\mathbf{x}_j)} \sigma_i^z \sigma_j^z \approx \sum_{\langle i;j\rangle} \frac{mv_s^2}{8} \left(\frac{g_{ab}\rho_a}{mv_s^2} - 1 \right)^2 \sigma_i^z \sigma_j^z, \tag{.4}$$

and thus in

$$J = -\left(\frac{g_{ab}}{4}\rho_a - \frac{\rho_a}{8}\frac{mv_s^2}{\rho_a} \right) - \frac{1}{8} mv_s^2 \left(\frac{g_{ab}\rho_a}{mv_s^2} - 1 \right)^2 = -\frac{g_{ab}^2\rho_a^2}{8mv_s^2}. \tag{.5}$$

We integrate out the phonons as in the Caldeira-Leggett model or similar to the Section on Spin Dynamics. The main effect is site dissipation resulting in a **dissipative Quantum Ising model.**