Quantum Information
Entanglement
Entropy

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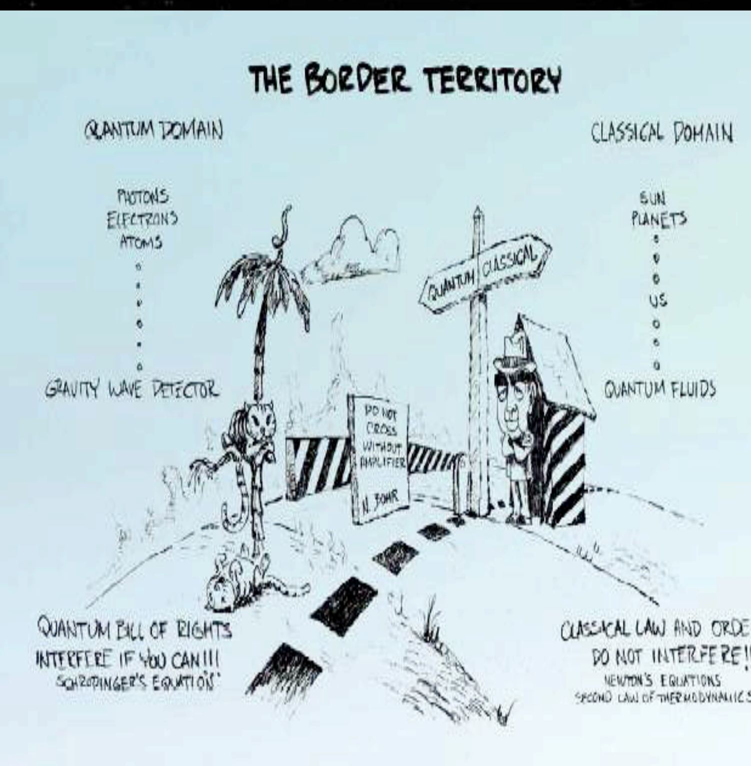
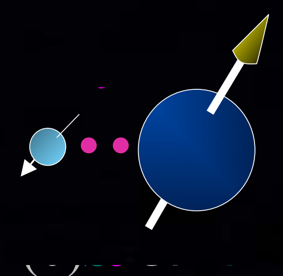
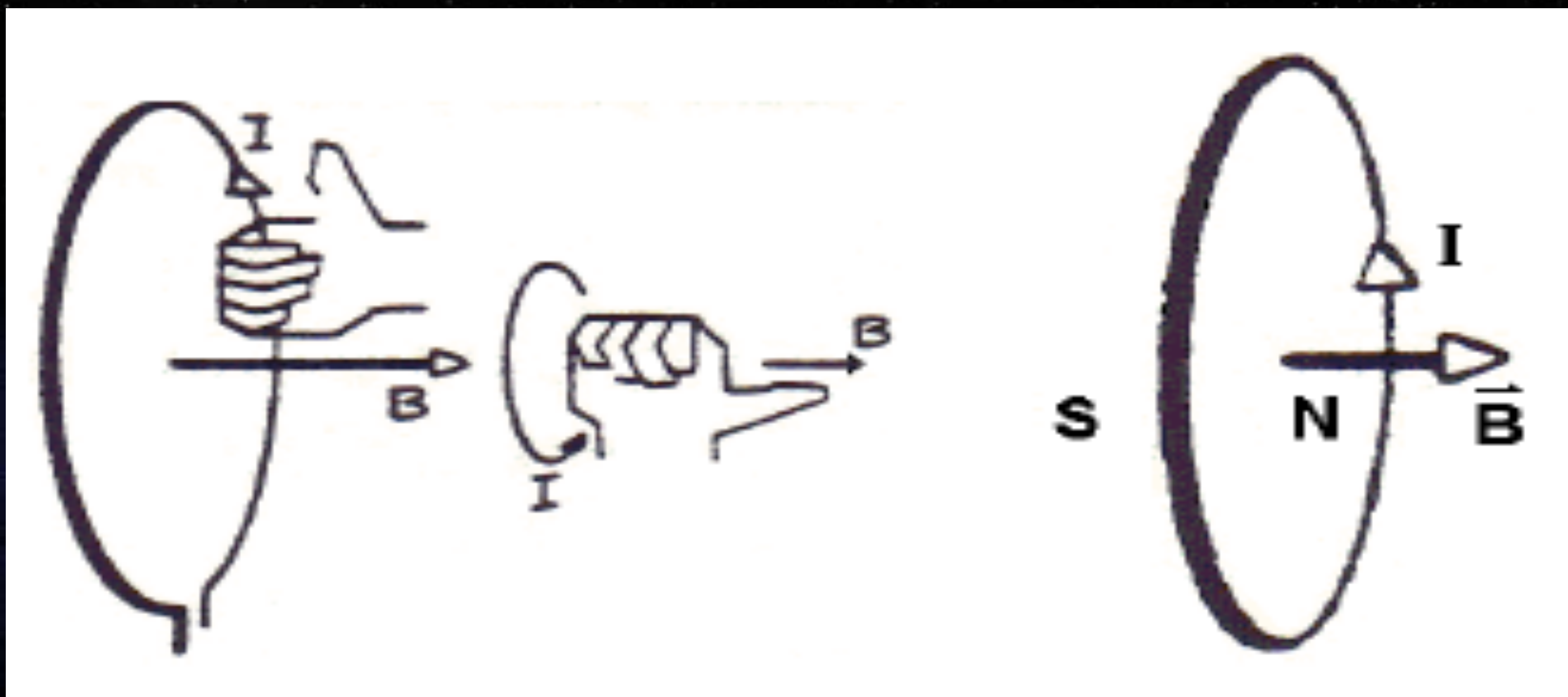


COLLOQUIUM:
May 11th 2012



1 aspect of my interests

Angular momentum and Magnetism



XIX century
Ampere, Biot, Savart…

XX: Scientists play with "toupies"

Spin?
Neutrino?

Quantum Weirdness

- We move down to the tiny scale of atoms...

Spin

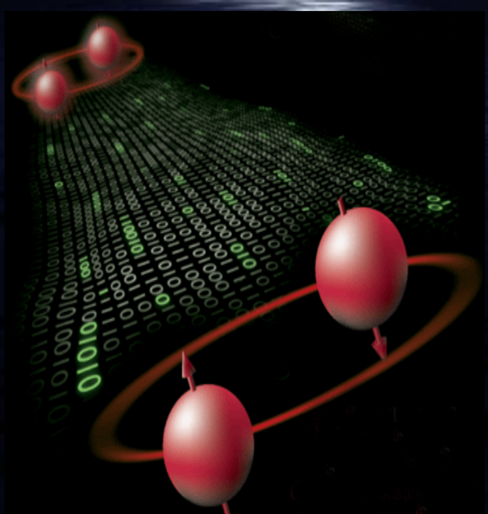
Uhlenbeck-Goudsmit, 1925
Pauli, 1927
Dirac, 1928

- Quantum effects?

particle-wave duality $p=\hbar k$

photo-electric effect $E=\hbar\omega$

quantum superposition,...



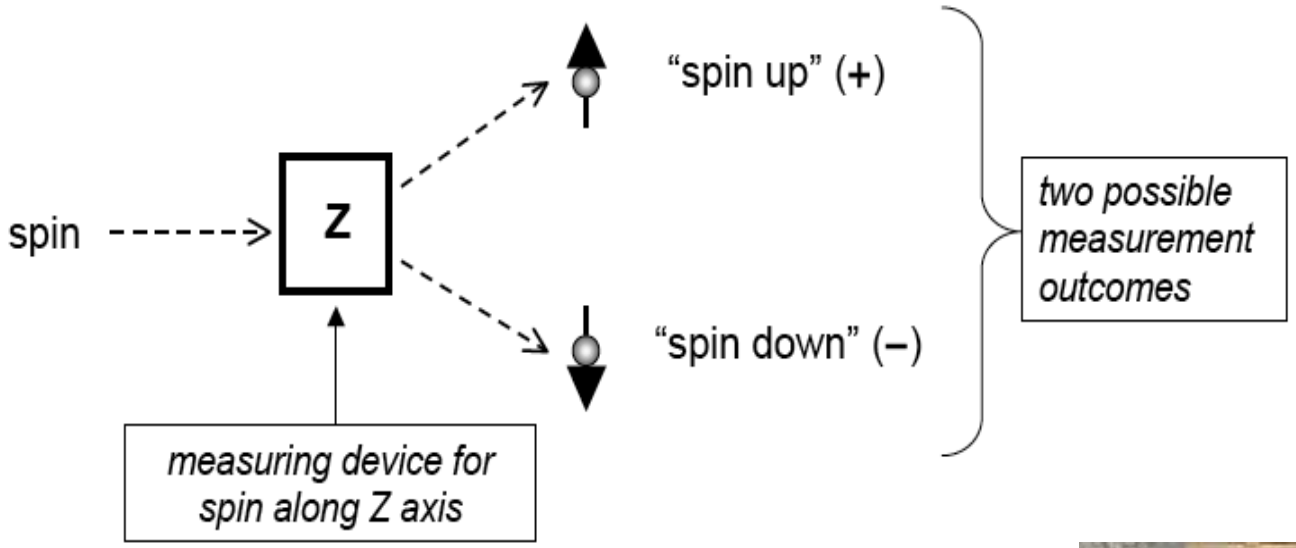
Quantum computer

- qubits $|a\rangle, |b\rangle \hat{=} \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$
- physical implementation:
  quantum 2-level-system: $|\uparrow\rangle \equiv |0\rangle, \quad |\downarrow\rangle \equiv |1\rangle$

Schrödinger Cat: Character of the quantum

Measurement

Copenhagen interpretation: wave-function collapse
system is quantum and measuring apparatus is <u>classical</u>



When we measure: **cat** is alive or dead

Sample two-state systems

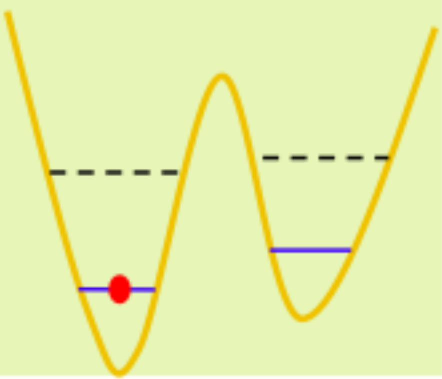
- Intrinsic two-state

Nuclei spin S=1/2

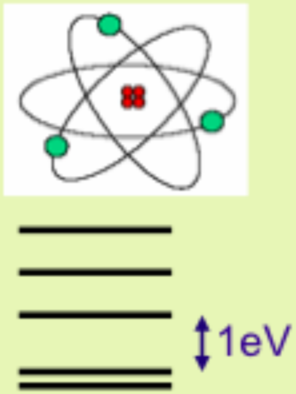
Polarization of photon (electromagnetic cavity)

- Truncated two-state

Particle in double well



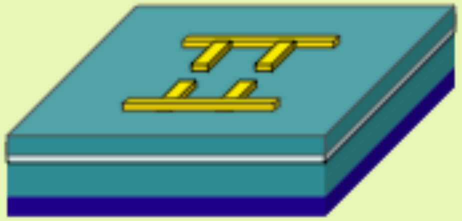
Atom



Condensed matter systems

- rf SQUID

- Quantum dot

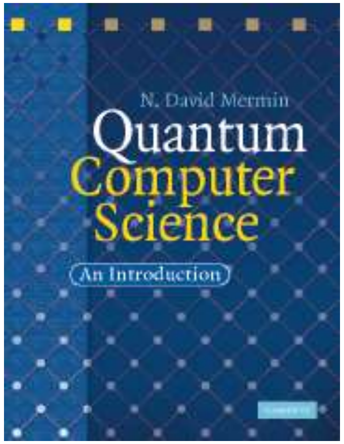


Cbits and Qbits:

A classical computer operates on string of zeros and ones such as 1100101…, Converting them into other such strings.

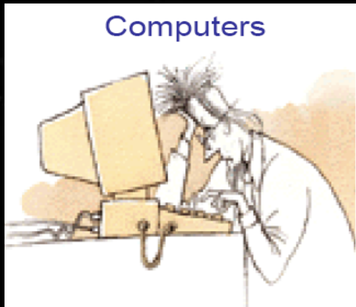
Each position is called a <u>bit</u>: either a 0 or 1

<u>Two-state realization</u>: a switch that can be open or shut, or a magnet whose magnetization is up or down



Qbits correspond to 2-level quantum systems or spins

http://people.ccmr.cornell.edu/~mermin



Quantum Computation: 2-level system as a « Qubit »

Qubit investigation

“ ħ/2 = 1”

2 level system

$$S^{\pm} = S_x \pm i S_y$$

$$S_x = \frac{\hbar}{2}\sigma_x$$
$$S_y = \frac{\hbar}{2}\sigma_y$$
$$S_z = \frac{\hbar}{2}\sigma_z$$

Pauli matrices

$[S_x,S_y]=i\hbar S_z$

$$S_{\pm}|s,m\rangle = \hbar\sqrt{s(s+1)-m(m\pm 1)}|s,m\pm 1\rangle$$

$$S_z|s,m\rangle = \hbar m|s,m\rangle$$

$|\uparrow\rangle = |1/2, 1\rangle$ and $|\downarrow\rangle = |1/2, -1\rangle$

Computers

Quantum Information Basis: 1970s

<u>Classical bits</u>: the only non-trivial operator is the **NOT** operation (flips the state of the Cbit)

$$|0\rangle \longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix}, \quad |1\rangle \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

$$\mathbf{x} \longleftrightarrow \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

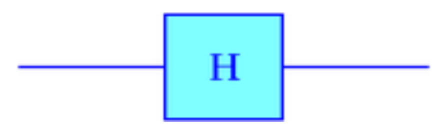
X^2 =XX=1

X^3= XXX=X

…

Quantum computer

- qubits $|a\rangle, |b\rangle \mathrel{\hat{=}} \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$
- physical implementation:
  quantum 2-level-system: $|\uparrow\rangle = |0\rangle, \quad |\downarrow\rangle = |1\rangle$
- 'quantum gate': unitary transformation
  (is reversible!)



$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$\mathrm{H}|0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad \mathrm{H}|1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

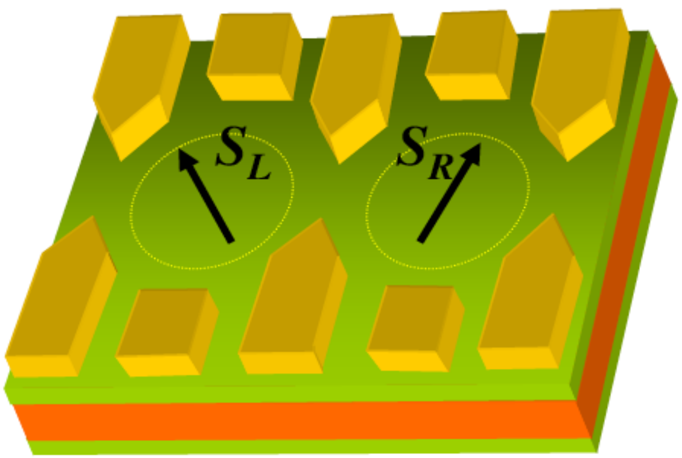
Hadamard Gate!

H1 in Hadamard transform

What are the questions and challenges

Either find a way to minimize decoherence at sufficiently relevant temperatures:
certainly one needs to understand the quantum matter in a better way
Error correction codes

Or Find another type of qubits: topological aspects, **Majorana Fermions**



Spin Qubits
Loss & DiVincenzo

Qubit candidates (?)

- -NMR
- -quantum dots
- -Superconducting circuits
- -Ion traps
- -Polar molecules
- -Semiconductor impurities,…

Majorana fermion is its own antiparticle:
Advantage, delocalized object occurring in exotic condensed matter systems
topological protection? IMPORTANT point

XXI, Detect the Majorana in topological SCs: L. Kouwenhoven Delft, 2012

See F. Wilczek, Majorana returns, Nat. Physics 2009

They appeared first
In spin chains: via Jordan-Wigner
transformation (1928)

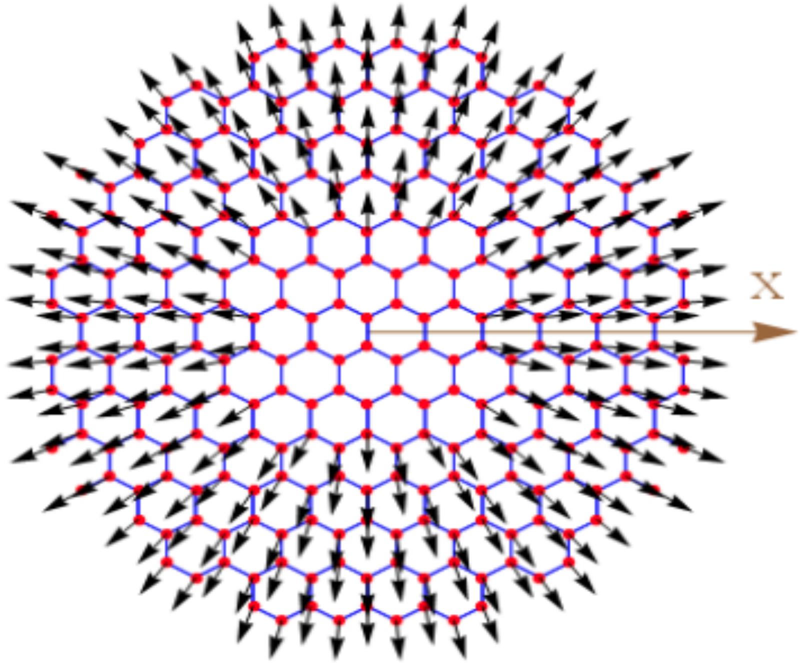
Challenge taking
into account that the
Man who discovered
the Majorana
disappeared 1938

Proposals:

Alexei Kitaev
Nick Read
Leonid Levitov
Hans Mooij
Liang Fu
Charles Kane
…

Note: Majorana fermions in graphene
But 4 of them…
P. Ghaemi & F. Wilczek, 2007 (arXiv); published in 2012
D. Bergman & K. Le Hur, 2009

Superconducting
Graphene



Topological defect
In quantum matter

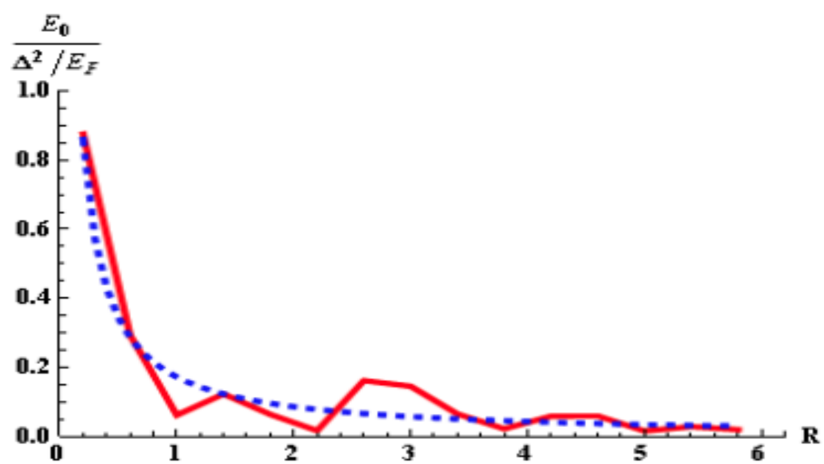
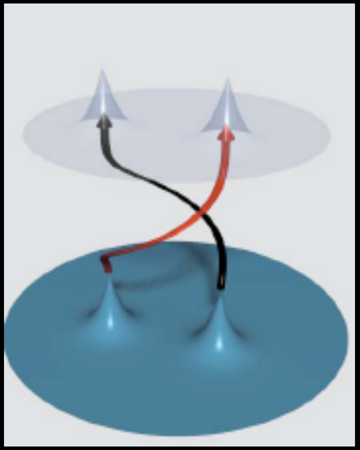


FIG. 9: Ratio of lowest energy scale to de Gennes energy scale, with varying vortex core size. The vortex core sizes range from 0.2 to 6.0 in increments of 0.4, and the lattice patch is circular with a radius of 6.0. The nearest neighbor distance is $1/\sqrt{3}$. The raw data is denoted by the continuous (red) curve, and the fit is denoted by a dashed (blue) curve.

Braiding and Strange Gates

The Majorana fermion states must be occupied in pairs, since the entire physical system can only occupy real fermion states.
So only combinations of Majorana fermions can be occupied



This occupied state is inherently delocalized – it has weight in two spatially separated vortex cores.

$$\hat{c}^{\dagger}|\Psi_0\rangle = (\hat{\gamma}_1 + i\hat{\gamma}_2)\,|\Psi_0\rangle$$

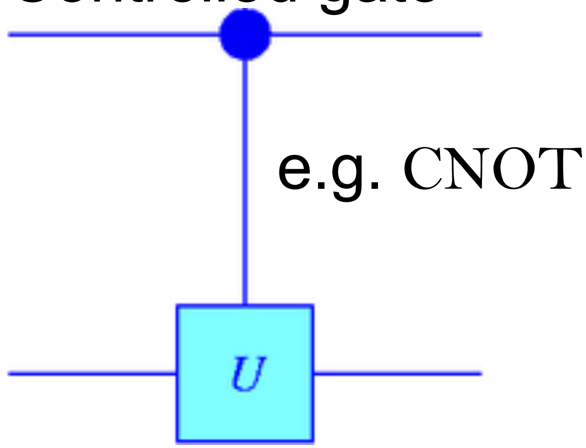
Exchange of 1 and 2 $\quad \gamma_1 \to \gamma_2$

$\gamma_2 \to -\gamma_1$

$$(\hat{\gamma}_2 + i\hat{\gamma}_1)\,|\Psi_0\rangle = i\,(\hat{\gamma}_1 - i\hat{\gamma}_2)\,|\Psi_0\rangle = i\hat{c}|\Psi_0\rangle$$

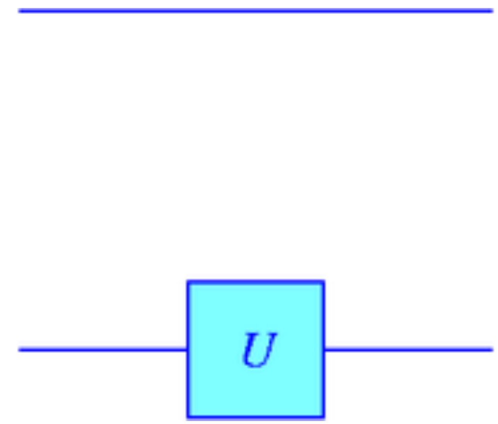
Different final state! – Non-Abelian statistics.

Controlled gate



$$|00\rangle \mapsto |00\rangle$$
$$|01\rangle \mapsto |01\rangle$$
$$|10\rangle \mapsto |1\rangle U|0\rangle = |1\rangle\left(x_{00}|0\rangle + x_{10}|1\rangle\right)$$
$$|11\rangle \mapsto |1\rangle U|1\rangle = |1\rangle\left(x_{01}|0\rangle + x_{11}|1\rangle\right)$$

Uncontrolled gate



$$|00\rangle \mapsto |0\rangle U|0\rangle$$
$$|01\rangle \mapsto |0\rangle U|1\rangle$$
$$|10\rangle \mapsto |1\rangle U|0\rangle$$
$$|11\rangle \mapsto |1\rangle U|1\rangle$$

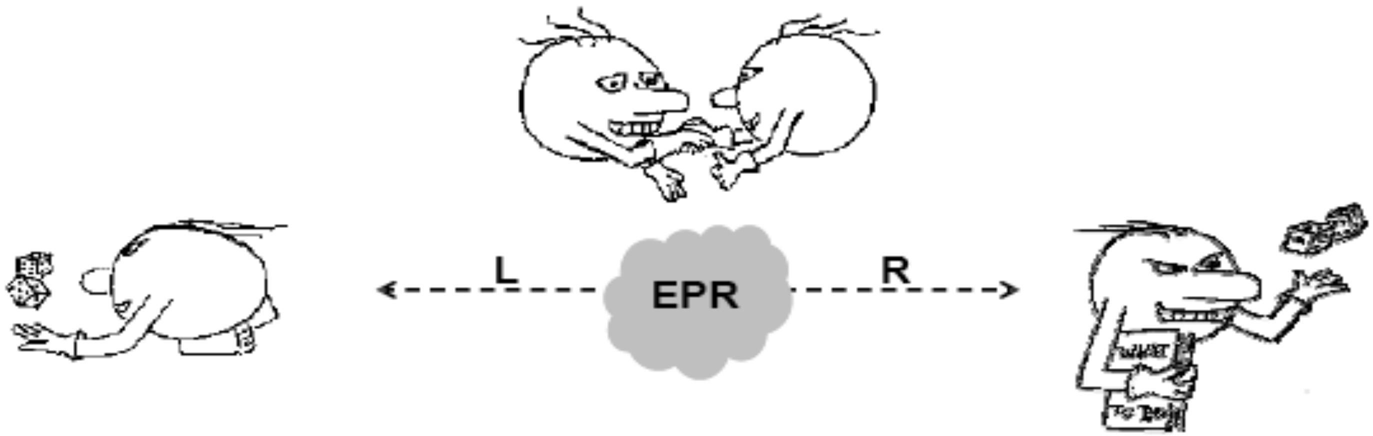
Quantum gates based on exchange interaction:

$$\boxed{H(t) = J(t)\, S_L \cdot S_R}$$

$$U_{SW} : \uparrow\downarrow \;\Rightarrow\; \downarrow\uparrow$$
$$U_{SW}^{1/2} : \uparrow\downarrow \;\Rightarrow\; \boxed{\uparrow\downarrow + e^{i\alpha}\downarrow\uparrow}$$

(2-qubit gates are the basic ingredients even for n>2 qubits)

What is Entanglement?
Spooky action at Distance (Einstein)



Simple example: 2 Qbits forming a singlet pair

$$|\Psi_S\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\rangle|\downarrow_B\rangle - |\downarrow_A\rangle|\uparrow_B\rangle)$$

Wave function is NOT factorizable into individual wave functions…
Quantum states of 2 (or more) particles are linked together

Detection (for photons) lies on violation of Bell's inequalities

(see for example experiment by A. Aspect, P. Grangier, G. Roger 1981)

Question: Why should we care about entanglement?

ANSWER: BECAUSE Nature provides "intrication"…

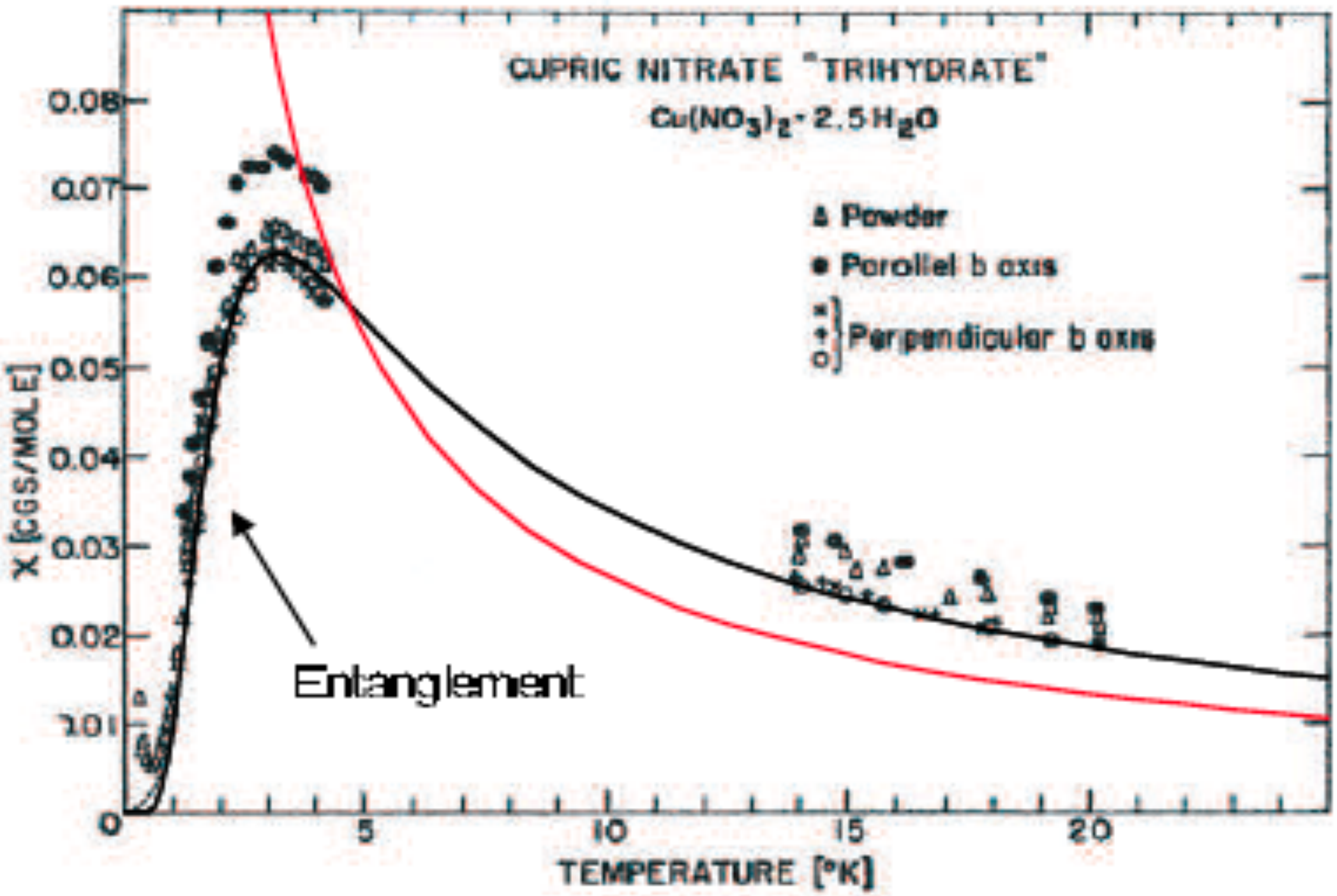
Sculptor Henry Moore, 1930
Inspired by French descriptive Geometry
Fabre de Lagrange
Gaspard Monge

See Nature Physics, May 2012

Also, perhaps our description of many-body systems in the quantum limit is not complete: **based on thermodynamical quantities**, choice of correlation functions
How does this tell us about "quantum-ness" ?

How do we know that quantum mechanics exists at large length scales?

Quantum Entanglement in Bulk Properties of Solids:
Quantum spin chain different from Ising classical chain? How?

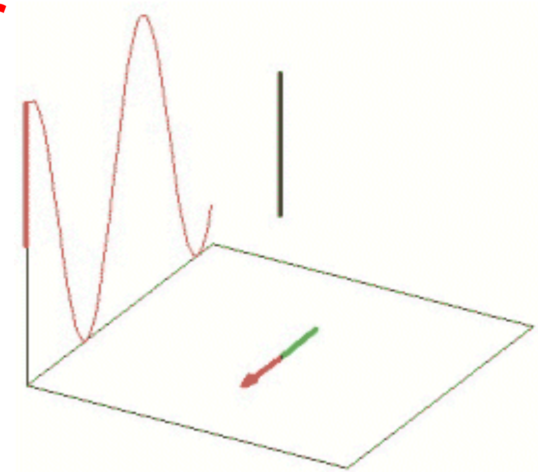


Brukner,
Vedral,
Zeilinger, 2004

$$\chi \geq \frac{g^2 \mu_B^2 N}{kT} \frac{1}{6}$$

implies separable state

Entanglement can also be bad for quantum computing: so let's care



The red segment visualizes the longitudinal component of the spin $\langle \mathbf{S}_z \rangle$

Apply a magnetic field in the plane (red/green arrow): Rabi oscillations…

Nobel price, 1944

Important question of interest:
Entanglement with environment produces decoherence
Environment is Big; challenge to describe quantitatively this problem

Important contributions: R. Feynman & A. Leggett

$$\boxed{\langle \dot{S}_z \rangle + \int_{-\infty}^{t} F(t-t') \langle S_z(t') \rangle dt' = 0}$$

F(t) captures the effect of the environment

No universality here:

The derivation of the function F depends on the type of environment (universe)

A. Leggett et al. Rev. Mod. Phys. **59**, 1 (1987)
U. Weiss, Quantum Dissipative Systems, 2nd edition
K. Le Hur, chapter in the book « Understanding Quantum Phase Transitions », ed. L. D. Carr 2010

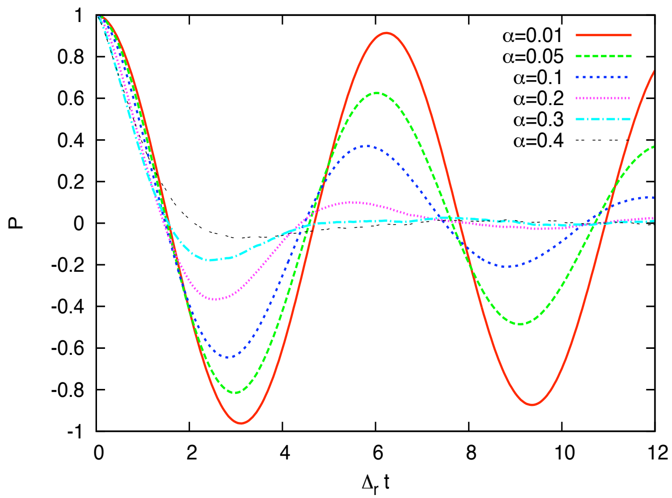
Develop analytical Approaches & “tricky” numerics

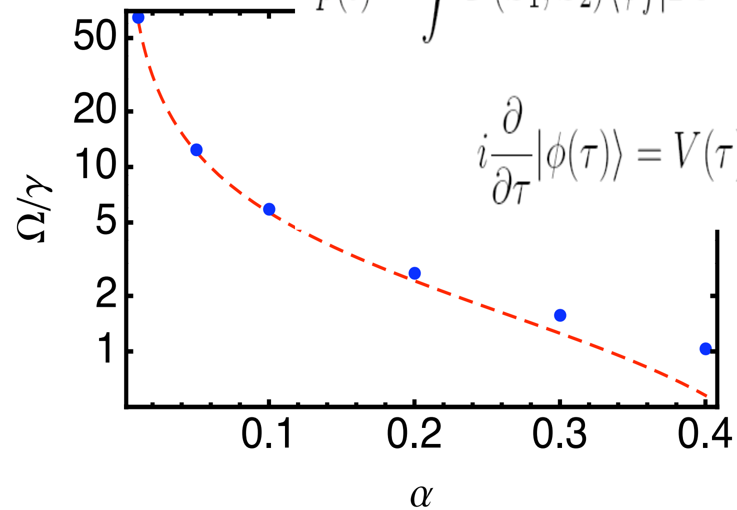
P. Orth, A. Imambekov, K. Le Hur, stochastic approach

D. Roosen, K. Le Hur, W. Hofstetter: time-dependent Wilson NRG

$$p(t) = \int \mathcal{D}(S_1, S_2) \langle \phi_f | T e^{-i \int_0^t dt' V(t') dt'} | \phi_i \rangle$$

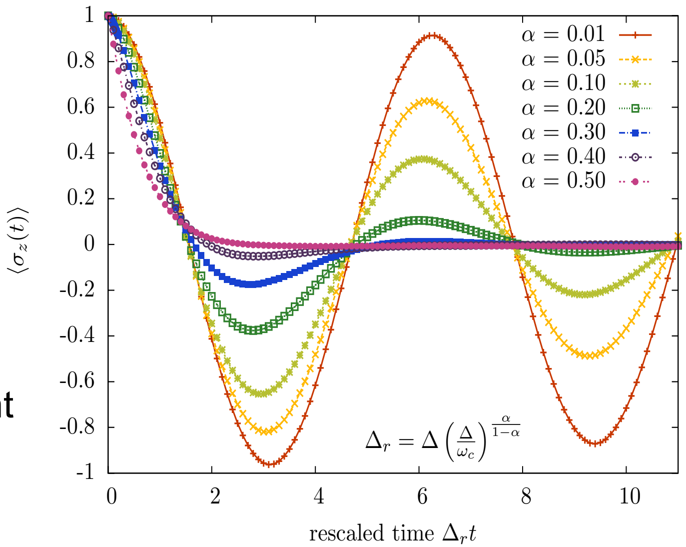
Stochastc Approach

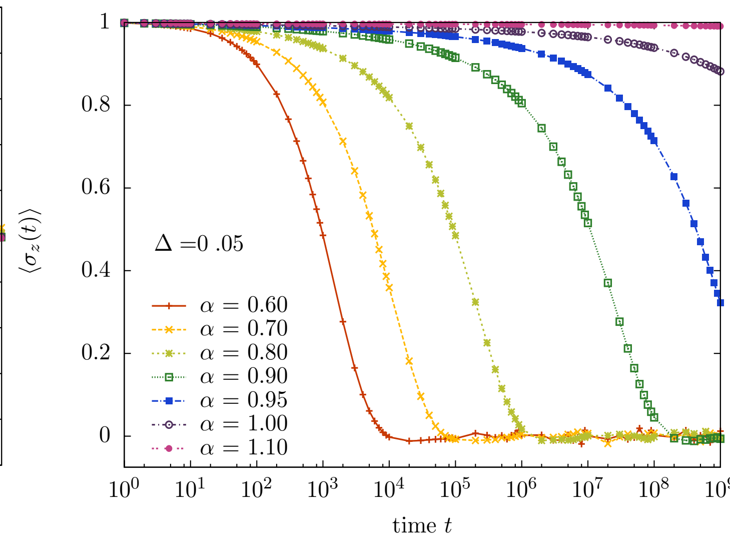




$$i\frac{\partial}{\partial \tau}|\phi(\tau)\rangle = V(\tau)|\phi(\tau)\rangle$$

Time-dependent NRG





Entanglement is GOOD or BAD….

Good: Allows to entangle 2 spins and perform more operations for example **faster computer and teleportation**

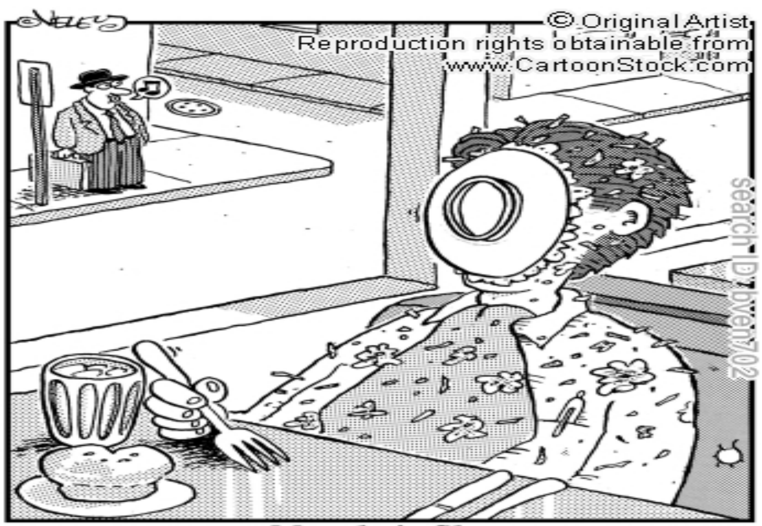
Bad: Entanglement with the environment inevitably produces decoherence

Entanglement is also a source of entropy

But what do we mean by Entropy?

1 possible answer

"Cleaning goes against entropy and the natural order."



Murphy's Slaw.

With many particles, disorder (complexity) seems a Natural State (many possible configurations leading to solid, gas, liquid Etc…)

Certainly, **temperature** helps realizing **entropy** or disorder (complexity)

Relation between **entropy** and **heat**…

From Sadi Carnot to …

Thermodynamics
2nd principle, **1824**
Carnot cycle

1818: death of **Gaspard Monge**

Sadi Carnot (1796-**1832**)
died at 36 years old

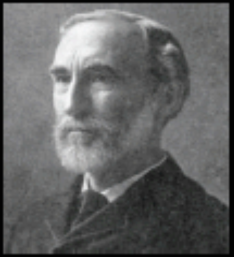
Entropy: « energy » & « tropos »

The word entropy was introduced by ***Clausius***, 1864

Thermodynamics Foundations:

Change of entropy is obtained by adding the small portions of heat received by the system divided by T during the heat process

Measure of uncertainty by ***Boltzmann & Gibbs:***



$$S_G = -k_B \int \rho(X) \,|\log \rho(X)|\, dX$$

Here, $\rho(X)\,dX$ is the *ensemble* density

Operator & Quantum mechanical perspective by ***von Neumann***, 1927

$$S_G = -\,\mathrm{Tr}\,\rho \log \rho$$

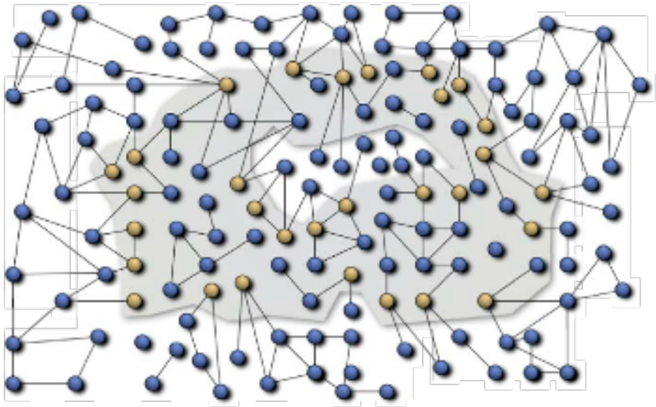
Entanglement Entropy

quantum mechanics:
entropy > 0 without an objective lack of information

non-degenerate
pure ground state

$$\rho_0 = |\psi\rangle \langle\psi|$$

$$\Rightarrow S(\rho_0) = 0$$



(Eisert *et al.*, RMP, 2010)

shaded region A
remainder B

$$\rho_A = \mathrm{tr}_B\,(\rho)$$

$$\Rightarrow S(\rho_A) \neq 0$$

Entanglement
Entropy

von Neumann entropy

$$S(\rho) = -\mathrm{tr}\,(\rho \log_2 \rho)$$

L. Amico et al. Rev Mod Phys (2008)

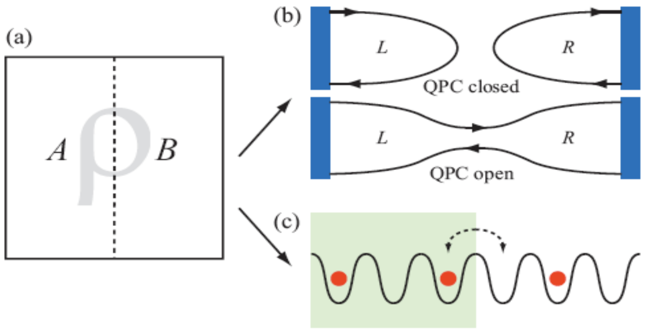
Is this Entropy Important?

Perhaps: Beckenstein-Hawking black hole entropy scales as the area and not volume; entanglement origin ? (Bombelli et al 1986) non-gravitational view of black holes…

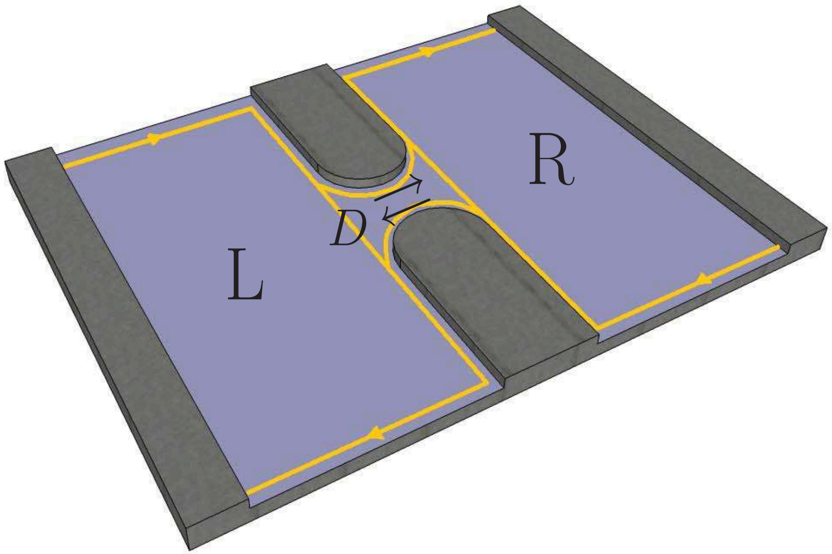
But, other possible interpretations (holographic principle)

Simple Understanding: Take free electrons (quasiparticles)

Mixing entropy at zero temperatures



Role of Nanotechnology & Math



Bipartite Fluctuations as a Probe of Many-Body Entanglement

H. **Francis Song**, Stephan Rachel, Christian Flindt, Israel Klich, Nicolas Laflorencie, KLH, arXiv:1109.1001, 30 pages+25 pages Supplementary Material
Phys. Rev. B **85**, 035409 (2012) [27 pages] (Editors' Suggestion) + long Suppl. Material

Entanglement entropy of free fermions

$$\mathcal{S} = \lim_{K\to\infty} \sum_{n=1}^{K+1} \alpha_n(K) C_n,$$

where

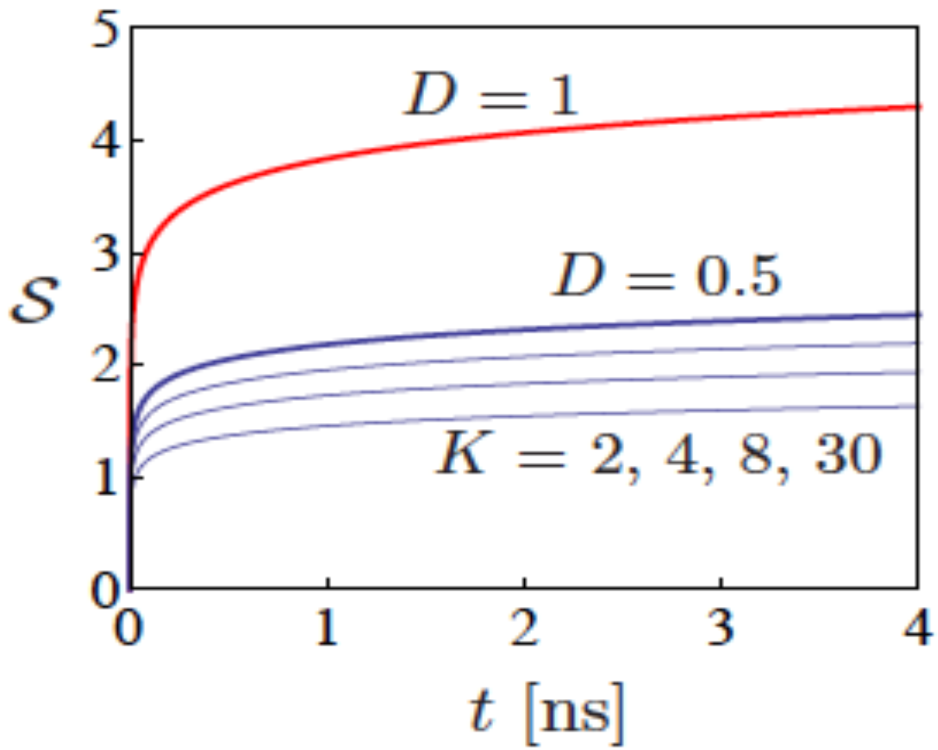
$$\alpha_n(K) = \begin{cases} 2\sum_{k=n-1}^{K} \frac{S_1(k,n-1)}{k!k} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

Here $S_1(n,m)$ are **unsigned Stirling numbers of the first kind**. Practically, K is the number of available cumulants and should be taken to be even.

Nano-Entangler

Generating Function known: Levitov, Lee and Lesovik (1996)
We are then able to compute all the cumulants and measure S…

GigaHertz limit:

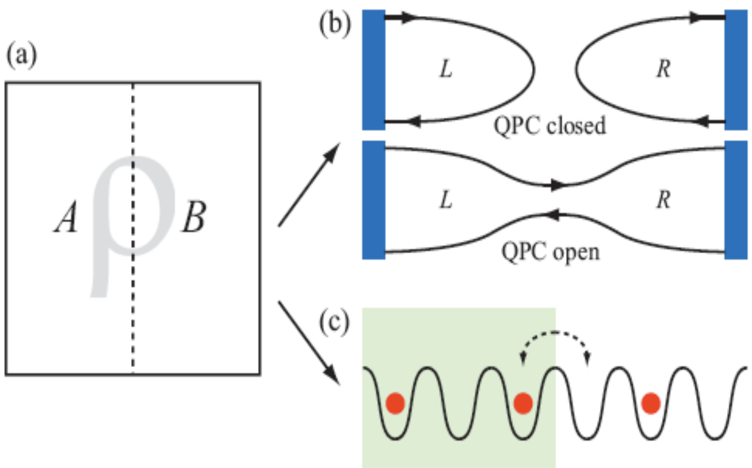


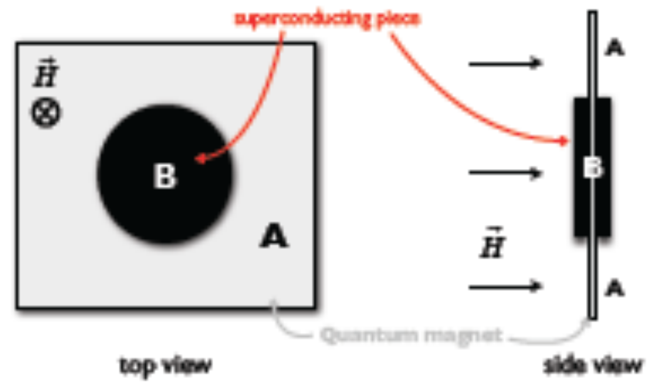
Gaussian equilibrium
Quantum noise

D=1: Results from CFT (Klich-Levitov): entropy grows logarithmically with time
D=0.5: Higher cumulants matter, but the entropy maintains its logarithmic growth
With K=2: $S=5/(2\pi)^2 * \ln(t/\tau)$: **Lower bound on the full entanglement entropy**

Note that quantifying entanglement in many-body systems is not unique:
Here, we have chosen cases where naturally we can divide the system into two pieces; still this gives new ways to think about “fluctuations”

Interacting cases: see recent paper by J. Cardy (PRL 2011); P. Calabrese etc…

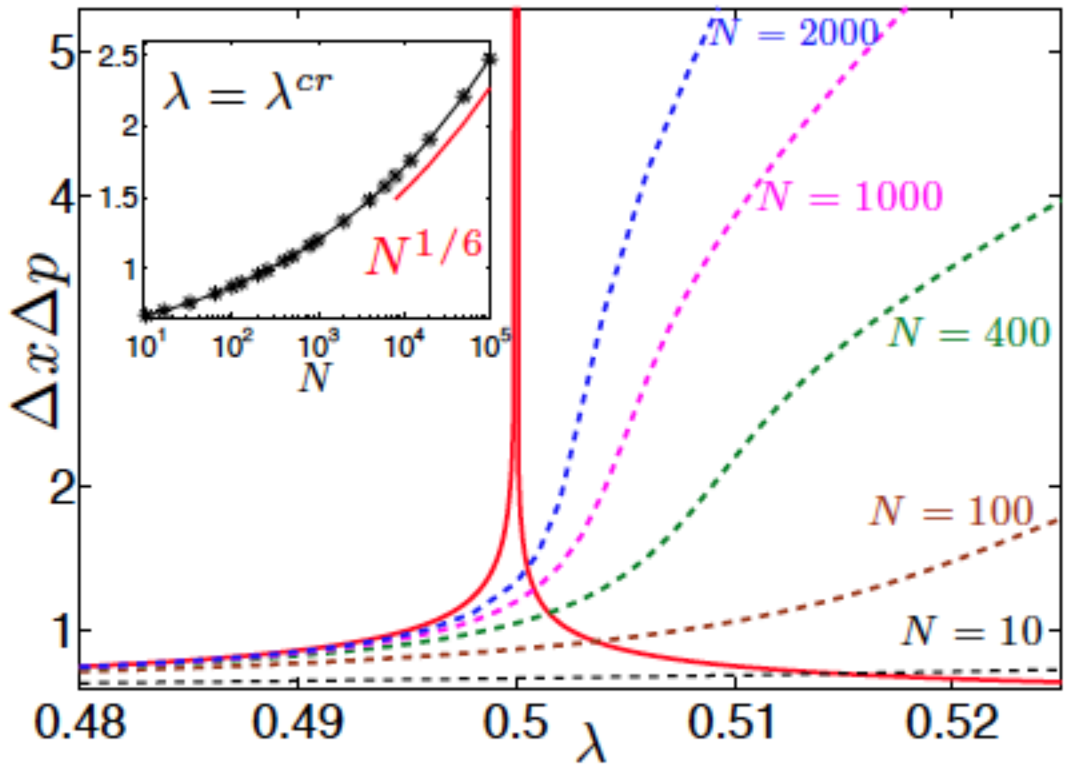




Also, experimental progress in cold atom systems
- Measure of local spin susceptibility, T. Esslinger ETH Zuerich 2012
- Parity number correlation functions (Harvard, Muenich)

Entanglement between light and matter

Heisenberg principle



Example
Dicke model

$$S \sim \log_2\{\Delta x \Delta p\} \quad \text{if} \quad \Delta x \Delta p \gg 1$$

Pierre Nataf, Mehmet Dogan and Karyn Le Hur, arXiv: 1204.3065

A little View on Q-Matter and Schrödinger-Cat

From Q-Superposition towards:

- Quantum information
- Condensed Matter
- NanoScience
- Quantum optics

Thank you for your attention!
Thanks to Students, Postdocs & Collaborators!!