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4 classes Saclay Lectures Series: 1h30 each

Institut d'Optique Graduate School

- Class I: Quantum Geometry, Information and Topological Physics from Bloch Sphere (June 9)
- Class II: Application Topological Lattice Models and Quantum Matter (June 16)
- Class III: Applications in Transport and Light-Matter Interaction (June 23)
- Class IV: Entangled WaveFunction and Fractional Topology (June 30)

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Topological lattice model from a simple matrix

2D: Graphene, Honeycomb lattice

Quantum Hall Effect

Haldane Model

Quantum Spin Hall effect

Interactions

Relations with class 1 ?

Geometry

One-dimensional Model : Su-Schrieffer-Heeger model

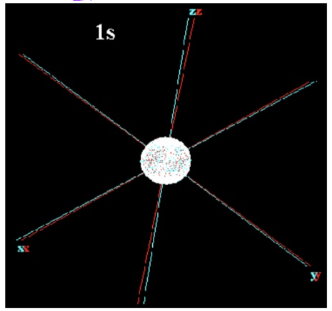
Superconductor and Majorana fermions

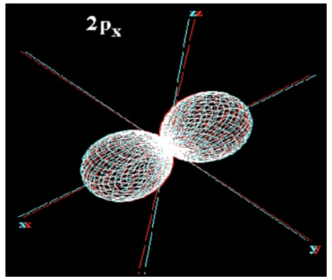
Applications

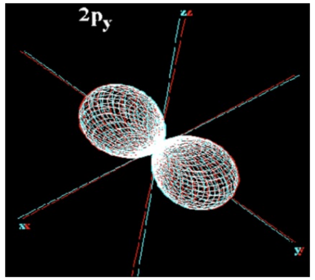
sp2 hybridization in graphene

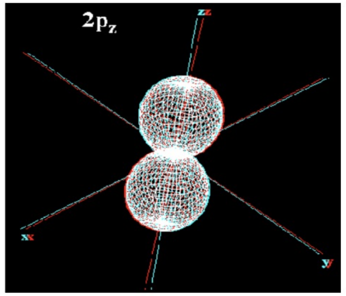
6
electrons
C

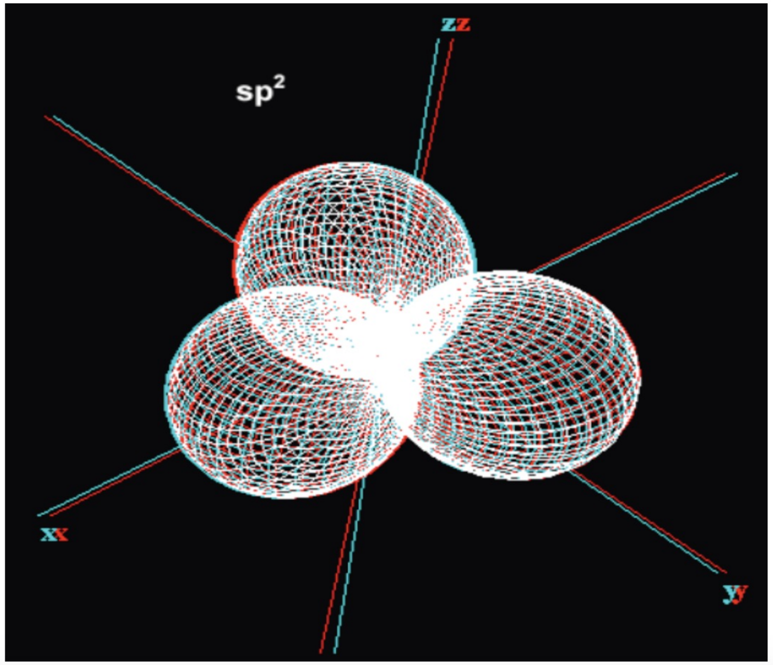
2s





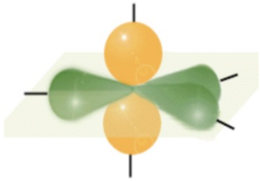






Within graphite, 2s and 2p orbitals undergo a sp2 hybridization

The geometry of the hybridized orbital is trigonal planar:
3 nearest neighbors

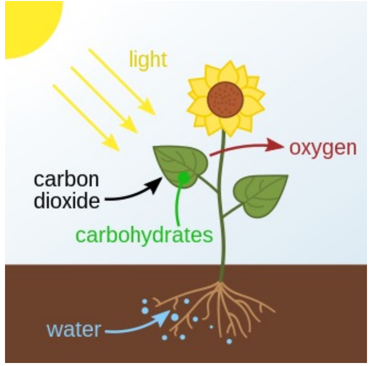


The last p_z-orbital forms the π-orbital

1 electron

Diamond : sp3 hybridization

Useful
For pencils
For tennis Rackets
For bicycles…
For photo-synthesis



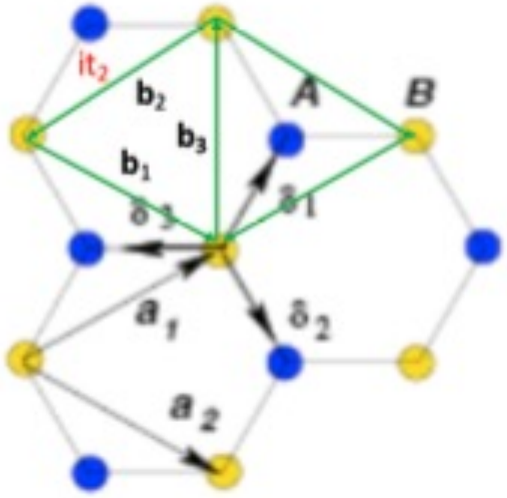
Yet be careful
with "gas" emission
For the planet

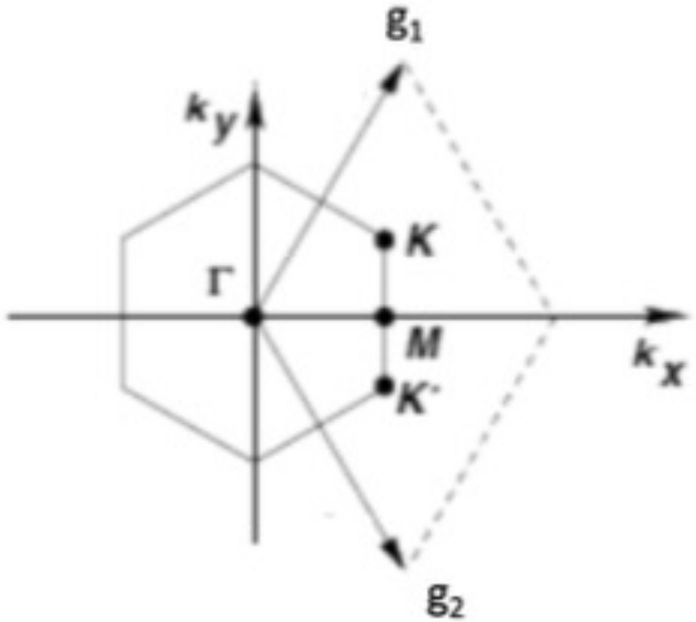
Picture from
Ph. Kim

Special about honeycomb lattice

1 plane of graphene (3D graphite; present research 2 planes and Moire magic angles…)

$$\mathbf{b}_1 = \frac{a}{2}(3, -\sqrt{3})$$
$$\mathbf{b}_2 = -\frac{a}{2}(3, \sqrt{3})$$
$$\mathbf{b}_3 = (0, \sqrt{3}a)$$





$$\boldsymbol{\delta}_1 - \boldsymbol{\delta}_3 = -\mathbf{b}_2$$
$$\boldsymbol{\delta}_2 - \boldsymbol{\delta}_3 = \mathbf{b}_1$$

$$\vec{g}_i \cdot \vec{a}_j = 2\pi\, \delta_{ij}$$

2 Triangular lattices from Translation Operators in two dimensions
defined through the Bravais lattice vectors $\mathbf{a}_i$ and equivalently $\mathbf{b}_i$: `2 sublattices'

Within these definitions: $\vec{\delta}_1 = \frac{a}{2}(1, \sqrt{3})$, $\vec{\delta}_2 = \frac{a}{2}(1, -\sqrt{3})$, $\vec{\delta}_3 = (-a, 0)$

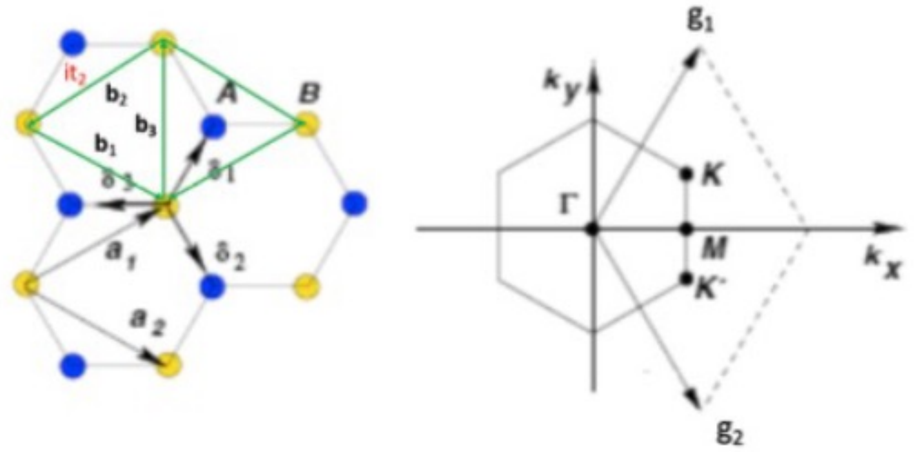
Special honeycomb lattice: 2*2 Matrix Model

$$\psi_{j\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_{\mathbf{R}_m} e^{i\mathbf{k}\cdot\mathbf{R}_m} \Phi_j(\mathbf{r} - \mathbf{R}_m)$$

The restricted Bloch wave is

$$\psi_{j\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_{\mathbf{R}_m} e^{i\mathbf{k}\cdot\mathbf{R}_m} \Phi_j(\mathbf{r} - \mathbf{R}_m).$$

From Bloch theorem, $\psi(\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}} u(\mathbf{r})$ with $u(\mathbf{r}) = \sum_j c_j \Phi_j(\mathbf{r})$. The functions Φ are centered at a site $\mathbf{R}_m$ — meaning that $\Phi_j(\mathbf{r} - \mathbf{R}_m)$ refers to an electronic wave-function around the site $\mathbf{R}_m$ — and periodic if we apply the translation operator of vector $\mathbf{b}_j$. From symmetries, the honeycomb lattice can be viewed as formed with two triangular lattices made of A and B sites respectively. In this description, N represents the number of A or B sites and a particle has equal probabilities to occupy a site such that $c_j = 1/\sqrt{N}$.



$$\sum_{i=A or B sites} e^{i(\mathbf{k}-\mathbf{k}')\mathbf{R}_i} = N\delta(\mathbf{k} - \mathbf{k}').$$

From the definitions, the Hamiltonian takes the form[1]

$$H = -t \sum_{\mathbf{R}_m} \sum_{\boldsymbol{\delta}_j} |\Phi(\mathbf{R}_m)\rangle\langle\Phi(\mathbf{R}_m + \boldsymbol{\delta}_j)| + h.c.$$

After Fourier transform, the Hamiltonian takes the form $H = \sum_{\mathbf{k}} H(\mathbf{k})$ with

$$H_{\mathbf{k}} = -t \sum_{\boldsymbol{\delta}_j} e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_j} |\psi_{A\mathbf{k}}\rangle\langle\psi_{B\mathbf{k}}| + h.c.$$

$$H(\mathbf{k}) = \begin{pmatrix} 0 & -t\sum_{\boldsymbol{\delta}_j} e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_j} \\ -t\sum_{\boldsymbol{\delta}_j} e^{i\mathbf{k}\cdot\boldsymbol{\delta}_j} & 0 \end{pmatrix}$$

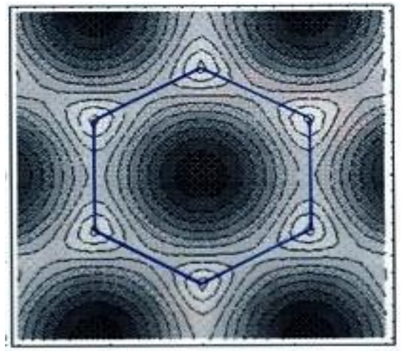
Close to the Dirac points

$$E^2 = t^2 \left(\sum_{\delta_j} e^{i\mathbf{k}\cdot\delta_j} \right) \cdot \left(\sum_{\delta_j} e^{-i\mathbf{k}\cdot\delta_j} \right)$$

$$k_x = K_x + p_x = \frac{2\pi}{3a} + p_x$$

$$k_y = K_y + p_y = \frac{2\pi}{3\sqrt{3}a} + p_y .$$

Wallace, 1947



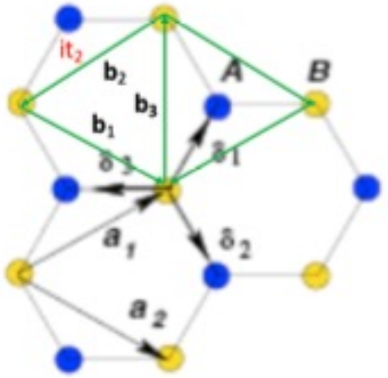
$$\boxed{E^2 \approx \frac{9}{4}(ta)^2(p_x^2 + p_y^2)}$$

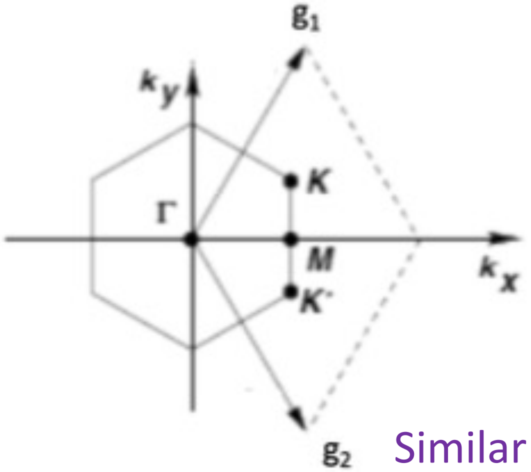
implying

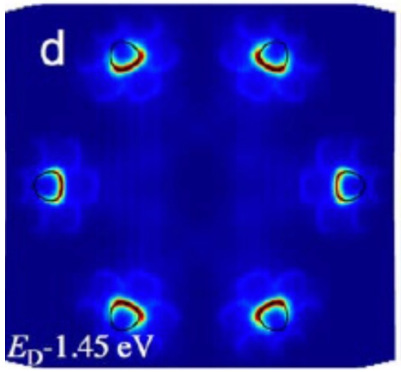
$$\boxed{E(p) = \pm \hbar v_F |\mathbf{p}|}$$

$$v_F = \frac{1}{\hbar}\frac{\partial E}{\partial |\mathbf{p}|} = \frac{3}{2\hbar} ta$$

$\simeq 10^6 \, m/s \ll c$







A. Bostwick et al.
Nature Physics **3** 36 (2007)
Photoemission

Similar Dirac points in high-Tc Superconductors

Linear energy dispersion

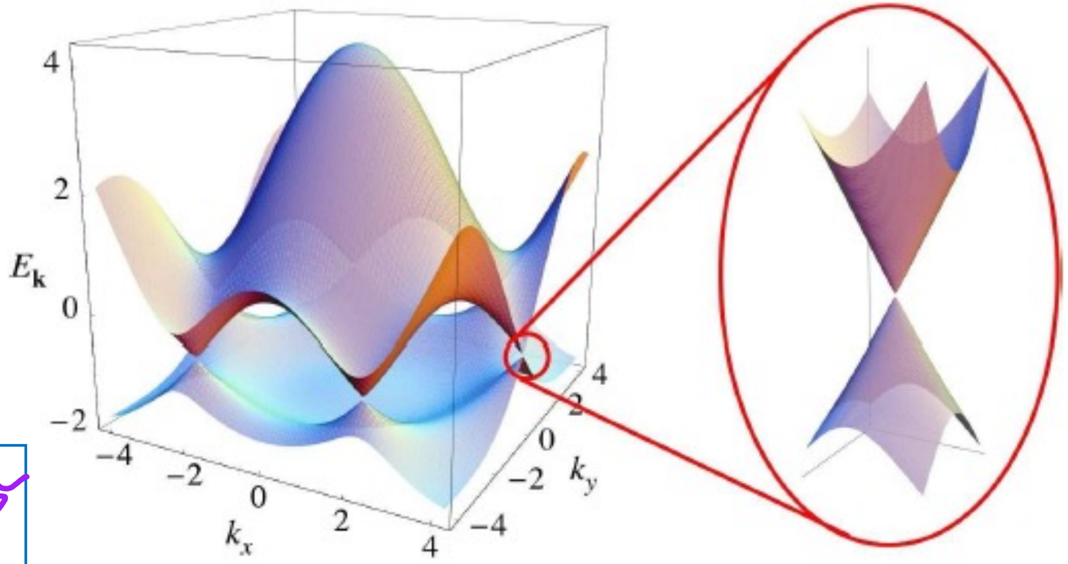
$$H = \hbar v_F \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix}$$

$$H(\mathbf{p})|\psi(\mathbf{p})\rangle = \pm\hbar v_F|\mathbf{p}||\psi(\mathbf{p})\rangle$$

$$\Pi = (p_x + i p_y) = |\vec{p}| e^{i\tilde{\varphi}}$$

$$H = v_F (p_x \sigma_x + \sigma_y p_y)$$

$\vec{\sigma}$: Pauli matrices



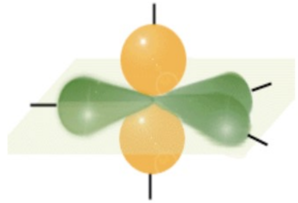
Relation with Dirac Equation in 2D:

$$\boxed{-iv_F\hbar\nabla\cdot\boldsymbol{\sigma}|\Phi(\mathbf{r})\rangle = E|\Phi(\mathbf{r})\rangle.}$$

The <u>isospin</u> **σ** (helicity) acts on each branch (sublattice)

Useful Review:
Rev. Mod. Phys. **81**, 109 (2009).

6 electrons in carbon:
2 in s1, 3 in sp2, 1 in p_z



Question:
Where is E_F for graphene?

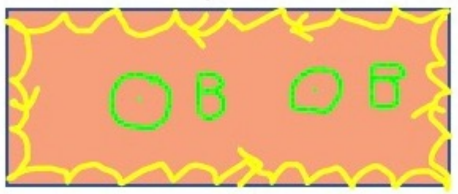
Answer:
$E_F=0$
Graphene is a semimetal

K. Von Klitzing, G. Dorda, M. Pepper

Quantum Hall Effect

McClure 1954

$\vec{F} = q(\vec{\dot{r}} \times \vec{B})$



$$\boxed{-i\hbar\nabla \rightarrow -i\hbar\nabla - qA_x}$$

$A_x = -By$
$A_y = 0$

$$H = \begin{pmatrix} 0 & -i\hbar v_F\partial_x + v_F qBy - \zeta\hbar v_F\partial_y \\ -i\hbar v_F\partial_x + v_F qBy + \zeta\hbar v_F\partial_y & 0 \end{pmatrix}$$

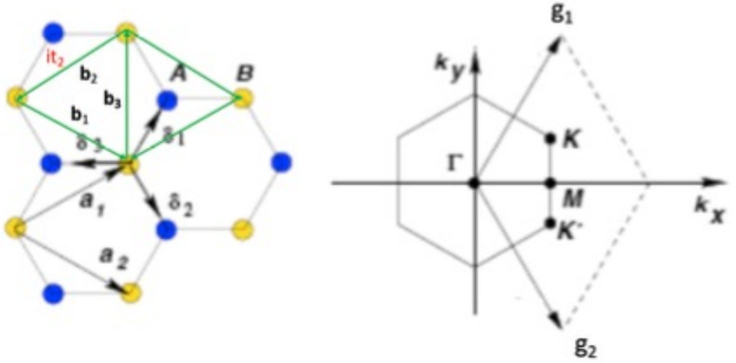
$\begin{cases} \xi = +1, & K \\ \xi = -1, & K' \end{cases}$

The solutions take the form

$$\Phi(\mathbf{r}) = e^{ikx}\Phi(y),$$

where $\Phi(y)$ associated to the spinor $|\Phi_A(y), \Phi_B(y)\rangle$. Therefore,

$$H = \begin{pmatrix} 0 & \hbar v_F k + v_F qBy - \zeta\hbar v_F\partial_y \\ \hbar v_F k + v_F qBy + \zeta\hbar v_F\partial_y & 0 \end{pmatrix}.$$



$l_B = \sqrt{\frac{\hbar}{|q|B}}$

$\omega_c = \frac{v_F}{l_B}$

$$H = \hbar\omega_c \begin{pmatrix} 0 & -l_B\zeta\partial_y + \left(kl_B - \frac{y}{l_B}\right) \\ l_B\zeta\partial_y + \left(kl_B - \frac{y}{l_B}\right) & 0 \end{pmatrix}$$

$\hat{r} = -\frac{y}{l_B} + kl_B$ $\qquad$ $-i\hbar\partial_r = l_B(i\hbar\partial_y)$ such that $[\hat{r}, -i\hbar\partial_r] = i\hbar$

$$\mathcal{O} = \frac{1}{\sqrt{2}}(\hat{r} + \partial_r) = \mathcal{O}_K = \mathcal{O}^\dagger_{K'}$$
$$\mathcal{O}^\dagger = \frac{1}{\sqrt{2}}(\hat{r} - \partial_r) = \mathcal{O}^\dagger_K = \mathcal{O}_{K'}$$

$\boxed{[\mathcal{O}, \mathcal{O}^\dagger] = 1}$

$$H = \hbar\omega_c^* \begin{pmatrix} 0 & \mathcal{O}^\dagger \\ \mathcal{O} & 0 \end{pmatrix} = \hbar\omega_c^* \left(\mathcal{O}^\dagger\sigma^+ + \mathcal{O}\sigma^-\right).$$

It is also useful to introduce $\hat{N} = \mathcal{O}^\dagger\mathcal{O}$

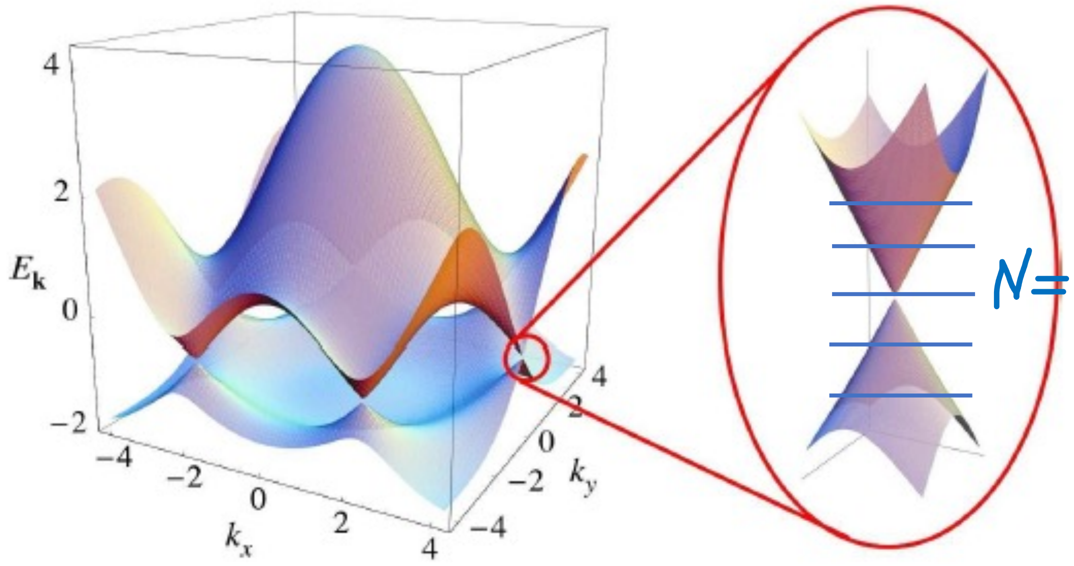
$\omega_c^* = \sqrt{2}\,\omega_c$

$\Phi_A(N)$

$\Phi_B(N-1)$

$$\boxed{\begin{aligned} \hbar\omega_c^* \mathcal{O}^\dagger \mathcal{O} \Phi_A(r) &= E \mathcal{O}^\dagger \Phi_B(r) = \frac{E^2}{\hbar\omega_c^*}\Phi_A(r) \\ \hbar\omega_c^* \mathcal{O}\mathcal{O}^\dagger \Phi_B &= E_\pm(N)\mathcal{O}\Phi_A(N) = \frac{E_\pm^2}{\hbar\omega_c^*}\Phi_B \end{aligned}}$$

$$E_\pm(N) = \pm\hbar\omega_c^*\sqrt{N}.$$



$N=0$ Special !

$$\Phi_\pm(K) = \pm\Phi_B(N-1)\otimes B + \underline{\Phi_A(N)\otimes A}.$$

$$\Phi_A(y) = c.e^{-\frac{(y-k)^2}{2l_B^2}}$$

$$\Phi_\pm(K') = \underline{\pm\Phi_B(N)}\otimes B + \Phi_A(N-1)\otimes A.$$

Analogy charged particle in GaAs and drift velocity

$$\mathcal{O}^\dagger\mathcal{O} = \frac{1}{2}(\hat{r}^2 - \partial_r^2 + [\hat{r}, \partial_r]) = \frac{1}{2}(\hat{r}^2 - \partial_r^2 - 1).$$

$$\frac{\hbar\omega_c^*}{2}\left(\hat{r}^2 - \partial_r^2 + [\hat{r}, \partial_r]\right)\Phi_A(y) = \frac{\hbar\omega_c^*}{2}\left(\hat{r}^2 - \partial_r^2 - 1\right)\Phi_A(y) = \frac{E^2}{\hbar\omega_c^*}\Phi_A(y)$$

$$\hat{H}_{eff}\Phi_A(r) = \left(\frac{E^2}{\hbar\omega_c^*} + \frac{\hbar\omega_c^*}{2}\right)\Phi_A(r) = \hbar\omega_c^*\left(N + \frac{1}{2}\right)\Phi_A(r)$$

$$-eV(y) = eEy$$

$$m = \frac{\hbar}{\omega_c^* l_B^2}$$

$$\hat{H}_{eff} = \frac{p_y^2}{2m} + eEy + \frac{1}{2}m\omega_c^*\left(y - l_B^2 k\right)^2$$

$$k \to k - \frac{eE}{m\omega_c^{*2} l_B^2}$$

$$\langle v_x \rangle = \frac{\hbar k}{m} = -\frac{E}{B}.$$

This velocity can be justified from physical arguments. If we include both a Coulomb and Lorentz force for a charge q, $q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$ then this is equivalent to modify the electric field along y direction such that $E_y \to E_y - v_x B$.

(Karyn Le Hur, Review ArXiv:2209.15381)

Simple estimation of quantum Hall conductivity

$$\boxed{j_x = n e |\langle v_x \rangle| = \frac{n e E}{B} = \frac{N_e e E}{B A}}$$

$N_e = nA$ with $A = L_x L_y$

We assume (2N+1) filled energy Landau levels

$$j_\perp = j_x = \frac{2(2N+1)\mathcal{N}}{A}\frac{E}{B}e$$

$$\boxed{\sigma_{xy} = 2(2N+1)\frac{e^2}{h}}$$

$\mathcal{N}$: degeneracy cyclotron orbits

cyclotron orbits coordinate $y_0 = k l_B^2$.

$$|k|_{max} = \frac{L_y}{l_B^2}$$

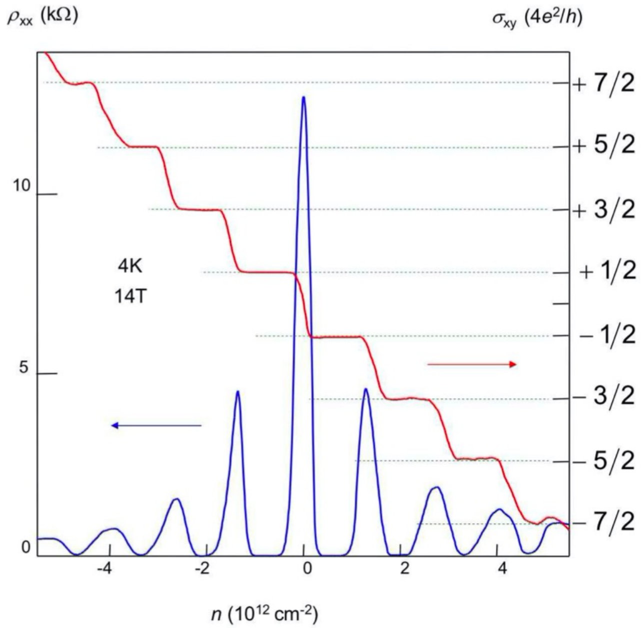
$$\Phi_0 = \frac{h}{e}$$

$$\mathcal{N} = L_x \int_0^{|k|_{max}} \frac{d|k|}{2\pi}$$

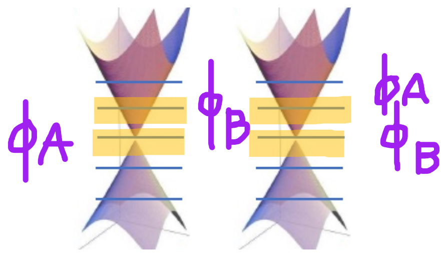
$$\mathcal{N} = \frac{ABe}{2\pi\hbar} = \frac{\Phi}{\Phi_0}$$

$$j_\perp = j_x = \frac{2(2N+1)\mathcal{N}}{A}\frac{E}{B}e = \frac{2(2N+1)e^2}{h}E,$$

K. Novoselov, A. K. Geim
Ph. Kim



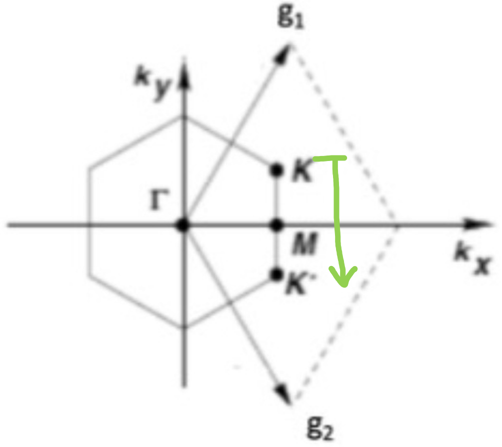
Agrees with lattice calculations and Hofstadter model (Rev. Mod. Phys. **81**, 109 (2009))



Correspondence Bloch sphere $(k_y,k_x) \rightarrow (\theta,\varphi)$

class I

$$\boxed{-d(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta) = (v_F|\mathbf{p}|\cos\tilde{\varphi}, v_F|\mathbf{p}|\sin(\zeta\tilde{\varphi}), -\zeta m).}$$

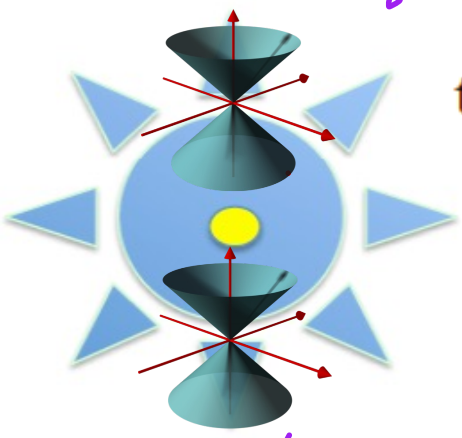


$$\mathcal{H}(\mathbf{k}) = -\mathbf{d}\cdot\boldsymbol{\sigma} = |\mathbf{d}|\begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}$$

$$|\psi_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}} \end{pmatrix}, \quad |\psi_-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \\ \cos\frac{\theta}{2}e^{i\frac{\varphi}{2}} \end{pmatrix}$$

$K \quad d_z = +m$

$A_\varphi = -i\langle\psi|\partial_\varphi|\psi\rangle$



$$\tan\theta = \frac{v_F|\mathbf{p}|}{m}$$
$$\tilde{\varphi} = \varphi \pm \pi$$

Eigenstates $\psi_\pm$ associated to energy -/+ |**d**|

$K' \quad d_z = -m$

Topology from Electromagnetism on the Sphere & quantum physics

Related to quest of Dirac monopoles and Skyrmions (P. Curie, 1894 ; P. Dirac 1931)

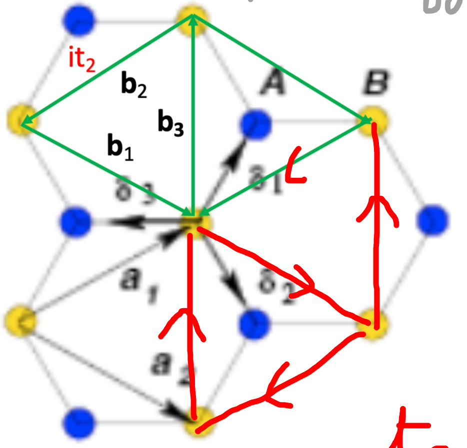
Relation with physics of planets

$C = A_\varphi(\pi) - A_\varphi(0)$

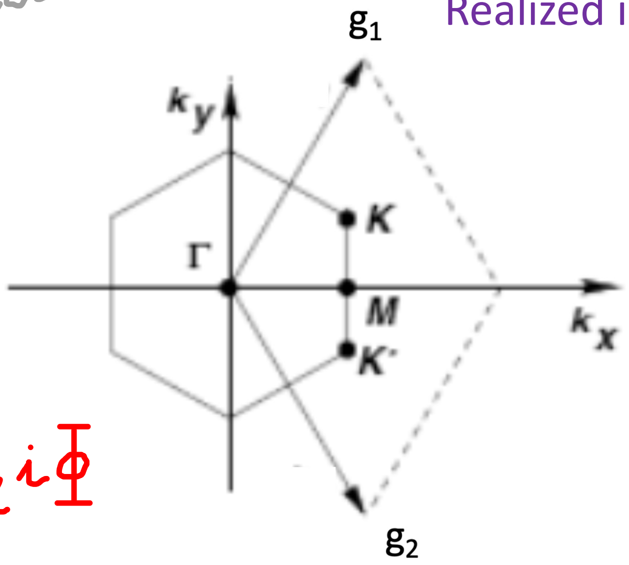
“zero net flux” quantum anomalous Hall effect

Haldane Model 1988

Realized in quantum materials, graphene, ultra-cold atoms, light systems



$t_2 e^{i\Phi}$



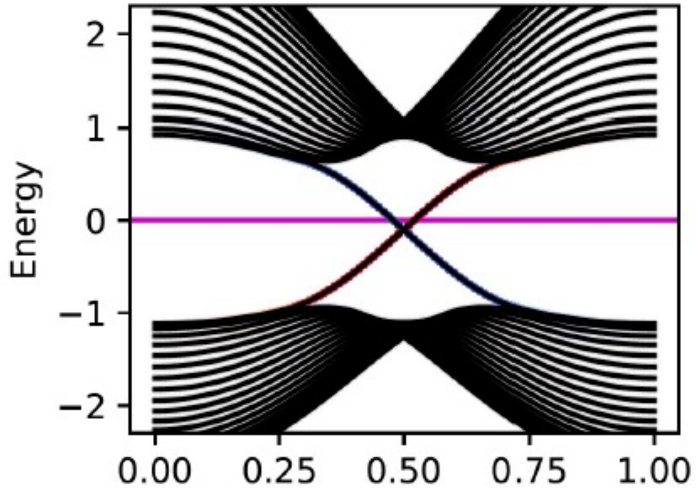
$$\mathcal{H}(\mathbf{k}) = -\mathbf{d}\cdot\boldsymbol{\sigma}$$

$\Phi = \frac{\pi}{2}$

$$\mathbf{d} = \Big(t\sum_{\boldsymbol{\delta}_j}\cos(\mathbf{k}\cdot\boldsymbol{\delta}_j),\, t\sum_{\boldsymbol{\delta}_j}\sin(\mathbf{k}\cdot\boldsymbol{\delta}_j),\, +t_2\sum_{\mathbf{b}_j}\sin(\mathbf{k}\cdot\mathbf{b}_j)\Big).$$

$$+d_z(\mathbf{K}) = 2t_2\sum_{\mathbf{b}_j}\sin(\mathbf{K}\cdot\mathbf{b_j}) = 3\sqrt{3}t_2 = m$$

$$+d_z(\mathbf{K}') = 2t_2\sum_{\mathbf{b}_j}\sin(\mathbf{K}'\cdot\mathbf{b_j}) = -3\sqrt{3}t_2 = -m.$$



The non-trivial topology of the Bloch bands translates to a nonzero winding number of the map $\hat{\mathbf{d}} = \mathbf{d}/|\mathbf{d}|$ from the torus (the first Brillouin zone) to the unit sphere.

http://www.physics.rutgers.edu/pythtb/

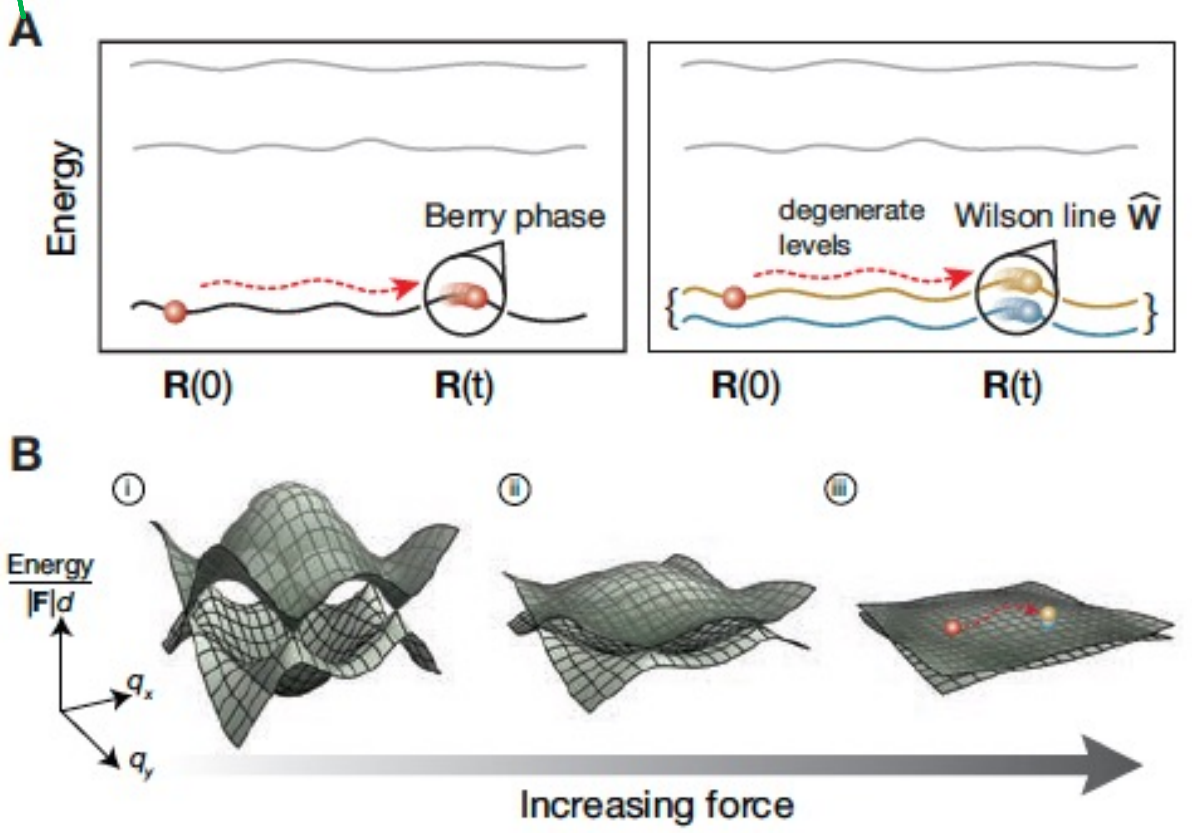
F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988)

Class III
Realization with light in graphene

Driving in cold atoms

Munich's group

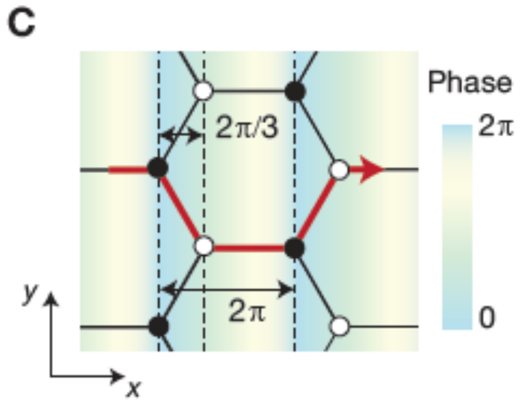
T. Li et al. Science 2016, arXiv:1509.02185



$$\mathbf{q}(t) = \mathbf{q}(0) + \mathbf{F}t/\hbar$$

$$W^{mn}_{\mathbf{Q}\to\mathbf{q}} = \langle \Phi^m_{\mathbf{q}} | e^{i(\mathbf{q}-\mathbf{Q})\cdot\hat{\mathbf{r}}} | \Phi^n_{\mathbf{Q}} \rangle = \langle u^m_{\mathbf{q}} | u^n_{\mathbf{Q}} \rangle.$$

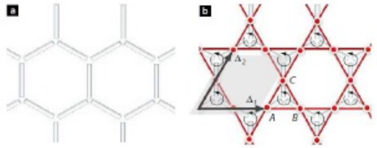
$$|u^1_{\mathbf{q}}\rangle = \cos\frac{\theta_{\mathbf{q}}}{2}|1\rangle + \sin\frac{\theta_{\mathbf{q}}}{2} e^{i\phi_{\mathbf{q}}}|2\rangle.$$



Haldane model of Light

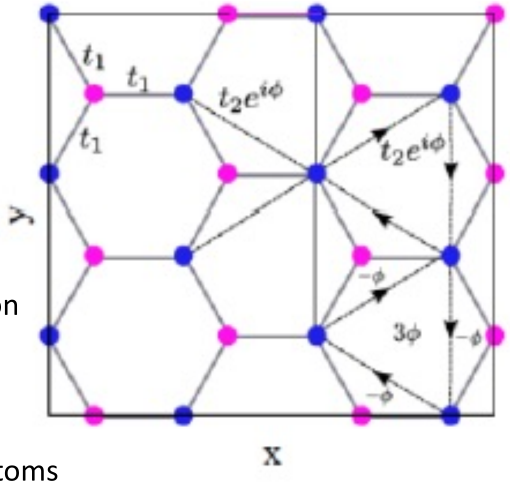
Other reviews
I. Carusotto & C. Ciuti
Lu, Johannpoulos, Soljacic
T. Ozawa, H. Price, A. Amo
et al. 2018

F. D. M. Haldane 1988

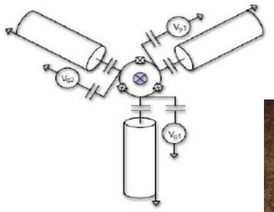


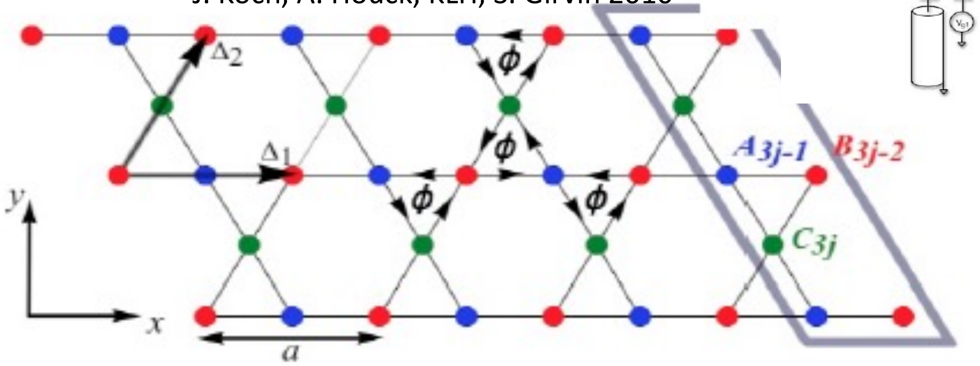
Wallace
Graphene
And then
McLure, theory

A. Geim
K. Novoselov
Scotch realization
Ph. Kim

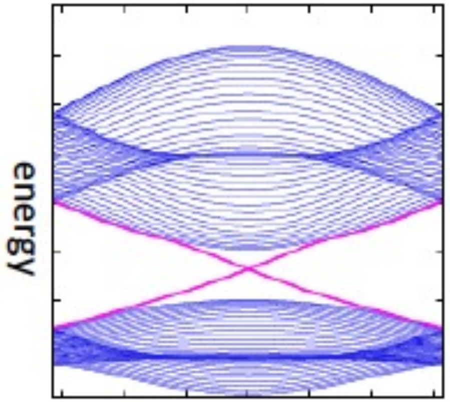


A. Petrescu, A. A. Houck and KLH, 2012
J. Koch, A. Houck, KLH, S. Girvin 2010

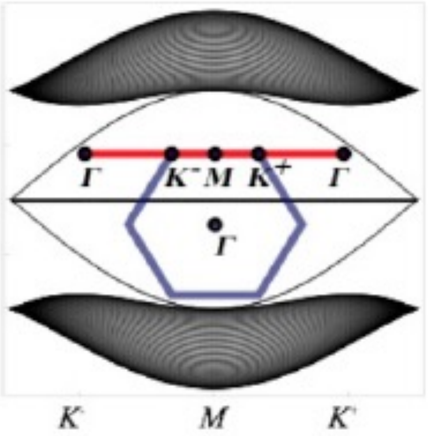


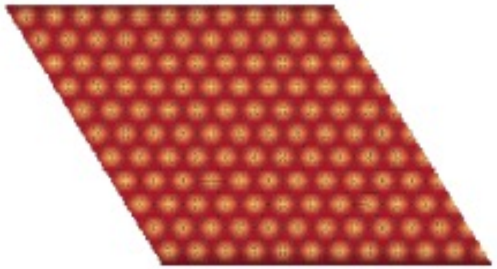


Realized with atoms
Jotzu et al, Zurich
Materials & light



Flat bands observed in polaritons (A. Amo , J. Bloch); also, work by J. Meyer & G. Refael





Localization in
Hexagon rings

Figure from KLH, Henriet, Petrescu, Plekhanov, Roux, Schiro Académie of Sciences 2016

Karyn Le Hur,
Review ArXiv:2209.15381

Interaction Effects : Simple Understanding

Mean-Field Approach

$$H_V = V\sum_{i,p} \hat{n}_i \hat{n}_{i+p}$$

$$H_V = V\sum_{i,p}[-(\phi_0+\phi_z)c^\dagger_{i+p}c_{i+p} - (\phi_0-\phi_z)c^\dagger_i c_i + c^\dagger_i c_{i+p}(\phi_x - i\phi_y) + c^\dagger_{i+p}c_i(\phi_x+i\phi_y) - (\phi_0^2-\phi_z^2-\phi_x^2-\phi_y^2)].$$

$$\boxed{\phi_r = -\frac{1}{2}\langle \Psi^\dagger_i \boldsymbol{\sigma} \Psi_i\rangle}$$

$$\cos\theta(\mathbf{p}) = \frac{1}{\epsilon(\mathbf{p})}(\zeta d_z(\mathbf{p}) - 3V\phi_z)$$

$$\sin\theta(\mathbf{p}) = \frac{1}{\epsilon(\mathbf{p})}(\hbar v_F - \frac{3}{2}Va(\phi_x+i\phi_y))|\mathbf{p}|,$$

$$H(\boldsymbol{k}) = \begin{pmatrix} \gamma(\boldsymbol{k}) & -g(\boldsymbol{k}) \\ -g^*(\boldsymbol{k}) & -\gamma(\boldsymbol{k}) \end{pmatrix}$$

$\Psi_i = (c_i, c_{i+p})$

$$\gamma(\mathbf{k}) = 3V\phi_z - 2t_2\sum \sin(\mathbf{k}\cdot\mathbf{b}_p)$$

$$g(\mathbf{k}) = (t_1 - V(\phi_x+i\phi_y))\cdot\left(\sum_{\mathbf{p}}\cos(\mathbf{k}\cdot\boldsymbol{\delta}_p) - i\sin(\mathbf{k}\cdot\boldsymbol{\delta}_p)\right)$$

$\epsilon = \sqrt{\gamma^2 + g^2}$

Hellmann-Feynman : $E_{gs} = -\sum_{\mathbf{k}}\epsilon(\mathbf{k})$

$$-2\phi_z = \langle c^\dagger_i c_i - c^\dagger_{i+p}c_{i+p}\rangle = \frac{1}{6N}\sum_{\mathbf{k}}\frac{\partial E_{gs}}{\partial(V\phi_z)}$$

$$\boxed{\frac{4}{3V} = \frac{1}{N}\sum_{\mathbf{k}}\frac{1}{\epsilon(\mathbf{k})}}$$

$V_c = \frac{4}{3}t_1$

$t_2 \to 0$

Path-Integral Approach

$$\begin{aligned}\mathcal{H}_V &= V\sum_{i,p}\left(n_i-\frac{1}{2}\right)\left(n_{i+p}-\frac{1}{2}\right)\\ &= V\sum_{i,p,r}\eta_r\left(c_i^\dagger\sigma^r_{i,i+p}c_{i+p}\right)^2\\ &-\frac{V}{2}\sum_{i,p}\left(c_i^\dagger c_i+c_{i+p}^\dagger c_{i+p}-\frac{1}{2}\right),\end{aligned}$$

Generalization of H. J. Schulz, Phys. Rev. Lett. 65, 2462 (1990)

Choice of parameters η_r matter when applying a variational approach

$$\boxed{-\eta_0=\eta_x=\eta_y=\eta_z=-\frac{1}{8}.}$$

SU(2) invariance

$$\mathcal{Z}=\int \mathrm{D}(\Psi,\Psi^\dagger,\phi^0,\phi^x,\phi^y,\phi^z)e^{-\mathcal{S}},$$

$$\mathcal{S}=\int_0^\beta \mathrm{d}\tau\sum_{\boldsymbol{k}}\Psi^\dagger_{\boldsymbol{k}}\left(\partial_\tau+h_0(\boldsymbol{k})\cdot\boldsymbol{\sigma}\right)\Psi_{\boldsymbol{k}}+\sum_{\boldsymbol{k},\boldsymbol{q},p}\Psi^\dagger_{\boldsymbol{q}}h_V(\boldsymbol{k},\boldsymbol{q},p)\Psi_{\boldsymbol{k}}+\sum_{\boldsymbol{k},r}6V\,\phi^r_{\boldsymbol{k}}\phi^r_{-\boldsymbol{k}},$$

where the interaction density matrix reads

$\vec{k}-\vec{q}=0$

$\omega\to 0$

$$h_V(\boldsymbol{k},\boldsymbol{q},p)=V\begin{pmatrix} e^{-\frac{i}{2}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{a}_p}\left(i\phi^0_{\boldsymbol{k}-\boldsymbol{q}}+\phi^z_{\boldsymbol{k}-\boldsymbol{q}}\right)-\frac{1}{2} & e^{\frac{i}{2}(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{a}_p}\left(\phi^x_{\boldsymbol{k}-\boldsymbol{q}}-i\phi^y_{\boldsymbol{k}-\boldsymbol{q}}\right)\\ e^{-\frac{i}{2}(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{a}_p}\left(\phi^x_{\boldsymbol{k}-\boldsymbol{q}}+i\phi^y_{\boldsymbol{k}-\boldsymbol{q}}\right) & e^{\frac{i}{2}(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{a}_p}\left(i\phi^0_{\boldsymbol{k}-\boldsymbol{q}}-\phi^z_{\boldsymbol{k}-\boldsymbol{q}}\right)-\frac{1}{2}\end{pmatrix}.$$

Variational approach ground state

S. Capponi, ED

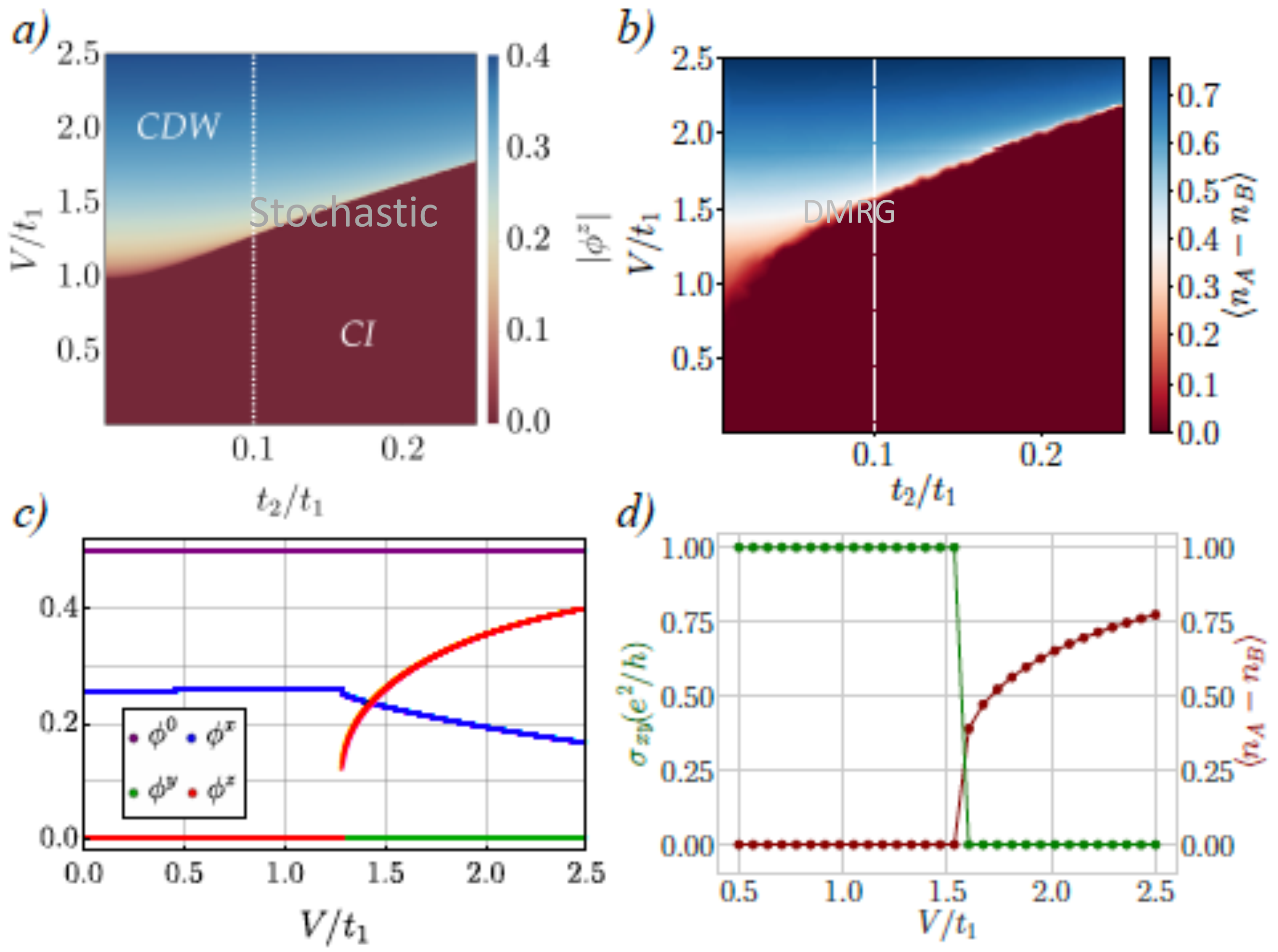
$V_c \sim 1.38 t_1$

$t_2 \to 0$

$\frac{4}{3} = 1.33\ldots$

Stochastic View of Interactions

Haldane model $+ V n_A n_B$



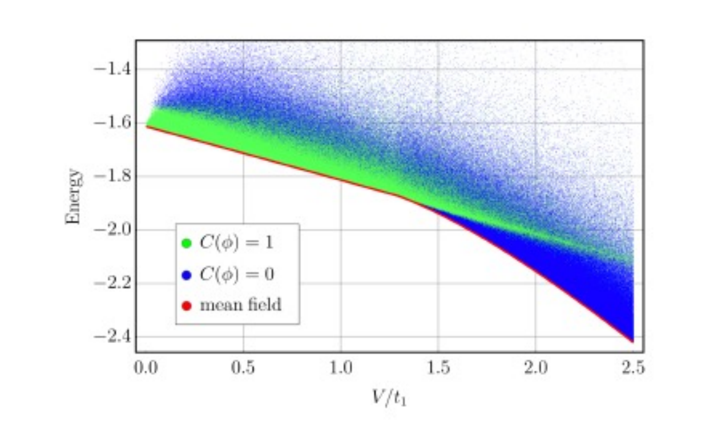
Philipp Klein, Adolfo Grushin, Karyn Le Hur, Phys. Rev. B 2021

$$\mathcal{H}_{\rm mf}(\boldsymbol{k}) = \begin{pmatrix} \gamma(\boldsymbol{k}) - 3V(\phi^0 + \frac{1}{2}) & -g(\boldsymbol{k}) \\ -g^*(\boldsymbol{k}) & -\gamma(\boldsymbol{k}) - 3V(\phi^0 + \frac{1}{2}) \end{pmatrix},$$

$$\gamma(\boldsymbol{k}) = 3V\phi^z - 2t_2 \sum_p \sin(\boldsymbol{k}\cdot\boldsymbol{b}_p),$$

$$g(\boldsymbol{k}) = [t_1 - V(\phi^x + i\phi^y)] \sum_p (\cos(\boldsymbol{k}\cdot\boldsymbol{a}_p) - i\sin(\boldsymbol{k}\cdot\boldsymbol{a}_p)).$$

$$\mathcal{F}(\phi^z) = \mathcal{F}_0 + \alpha(\phi^z)^2 + \beta(\phi^z)^4 + \gamma(\phi^z)^6,$$



Agree with ED calculations at Maryland (V. Galitski and collaborators, 2010)

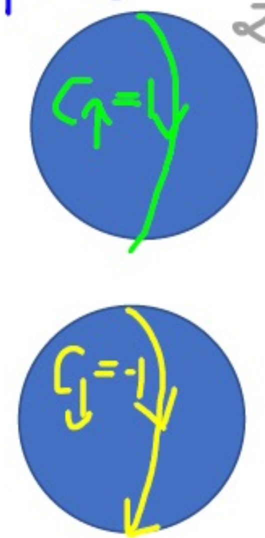
Topological Insulators (TI) & Quantum Spin Hall Effect (QSH)

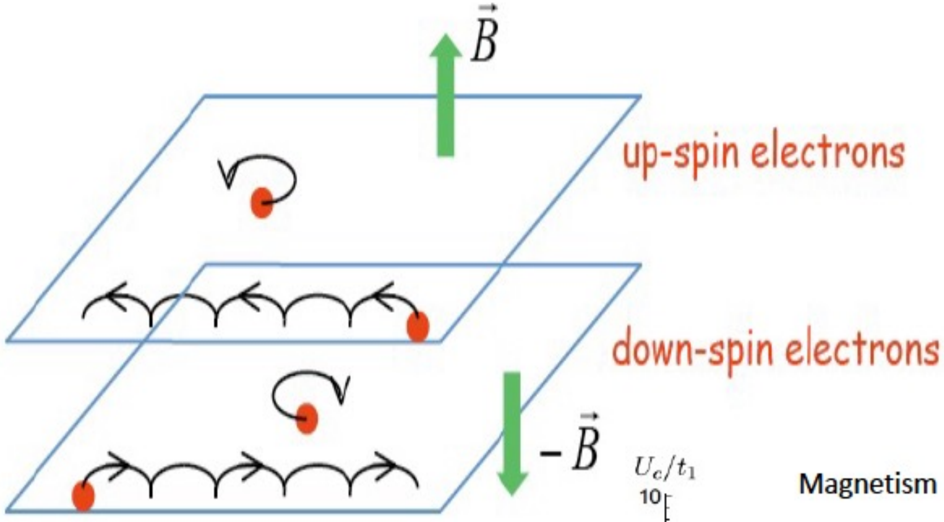
- Time-reversal invariant band insulator
- Strong spin-orbit interaction $\lambda \vec{L} \cdot \vec{\sigma}$
- Gapless helical edge mode (Kramers pair)

Mercury, Wurzburg
Bismuth, Princeton

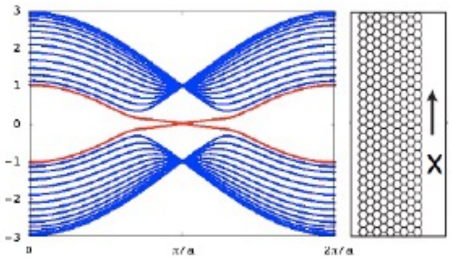
$\lambda L_z \sigma_z$
$L_z \to -L_z, \sigma_z \to -\sigma_z$

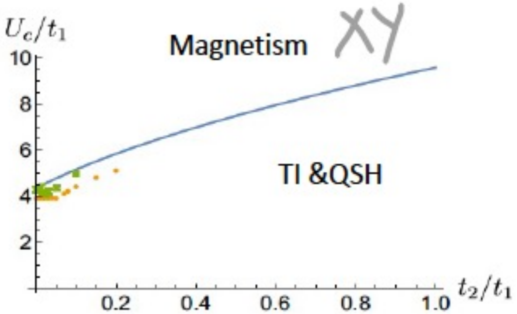
$C_\uparrow - C_\downarrow = 2$ $\mathbb{Z}_2$





Kane-Mele Model 2005, 2006
B. A. Bernevig & T. Hughes, S.C. Zhang





— Variational [***] (CPHT)
· CDMFT [*] (Yale)
· QMC [**] Wurzburg

Measurable with light from Dirac points
Relation with zeros of Kane-Mele Pfaffian
K. Le Hur, arXiv:2106.15665

class III

Interaction Effects + Mott : S. Rachel and K. Le Hur (2010) [*]; W. Wu et al. (*, CDMFT, 2012); F. Assaad et al. (**. 2010, QMC)
Analytical Solution of Mott Transition [***] J. Hutchinson, Ph. Klein, K. Le Hur, Phys. Rev. B 104, 075120 (2021) (fluctuations)

Proximity Effects with Graphene

$C_1 + C_2 = 0$ $\quad$ $C_1 - C_2 = 2$

$\mathbb{Z}_2$ state with asymmetric masses

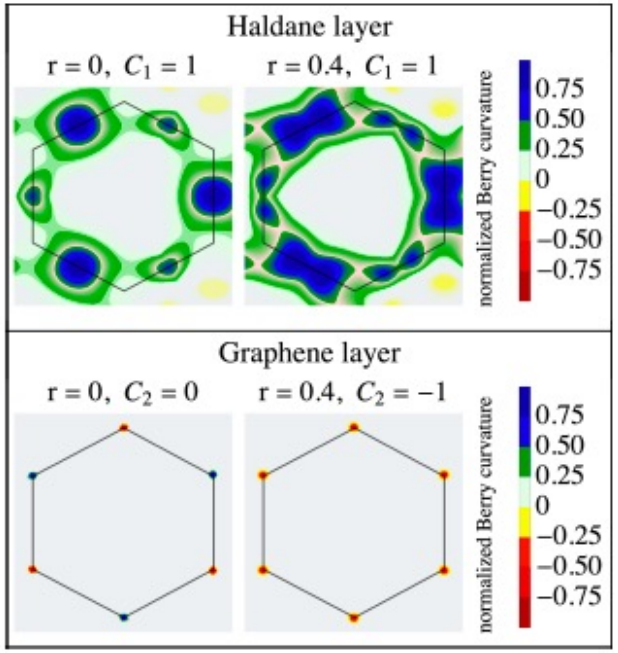
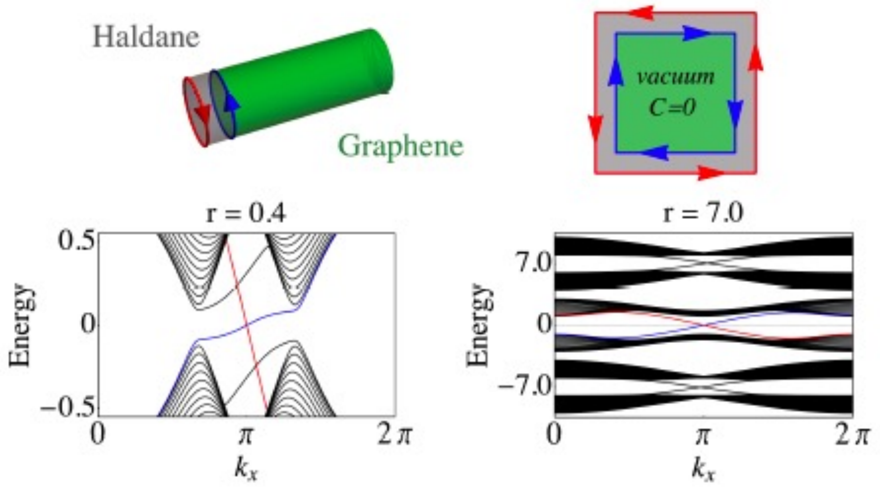
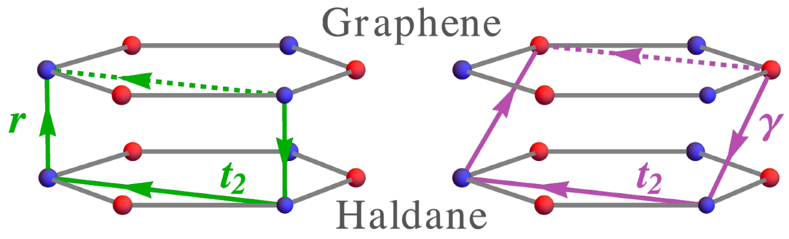


FIG. 1. Berry curvature in the Brillouin zone for the Haldane and graphene layers at $r = 0$ and small r, showing the Berry phase jump effect [35]. Here, $t_1 = 1$ and $t_2 = 1/3$.





Class IV

Ising Interaction $\tilde{r}\,\sigma_1^z\sigma_2^z$

interaction in k-space or hopping

Peng Cheng, Philipp Klein, K. Plekhanov, K. Sengstock, M. Aidelsburger, C. Weitenberg and Karyn Le Hur, Phys. Rev. B 100, 08110 (R) (2019). Collaboration with Munich and Hamburg.

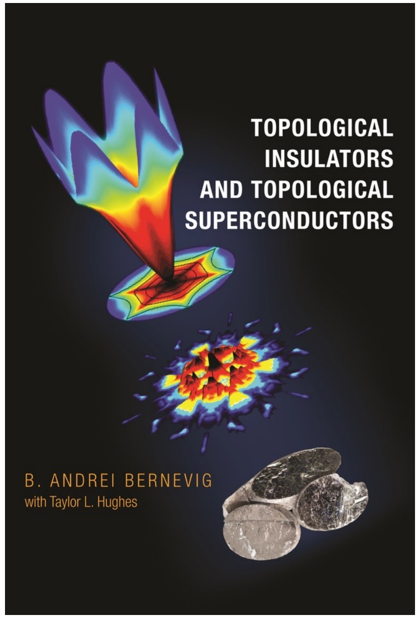
Periodic Table of Topological Invariants

QHE

Topological insulators

Topological p-wave SCs

	$\mathcal{T}^2$	$\mathcal{P}^2$	$\mathcal{C}^2$	d 1	2	3	4	5	6	7	8
A	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}$
AIII	$\varnothing$	$\varnothing$	$+$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}$	$\varnothing$
AII	$-$	$\varnothing$	$\varnothing$	$\varnothing$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\varnothing$	$\varnothing$	$\varnothing$	$\mathbb{Z}$
DIII	$-$	$+$	$+$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\varnothing$	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\varnothing$
D	$\varnothing$	$+$	$\varnothing$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\varnothing$	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}_2$
BDI	$+$	$+$	$+$	$\mathbb{Z}$	$\varnothing$	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AI	$+$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	$+$	$-$	$+$	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\varnothing$
C	$\varnothing$	$-$	$\varnothing$	$\varnothing$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\varnothing$	$\varnothing$
CII	$-$	$-$	$+$	$\mathbb{Z}$	$\varnothing$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\varnothing$	$\varnothing$	$\varnothing$



arXiv:1506.05805

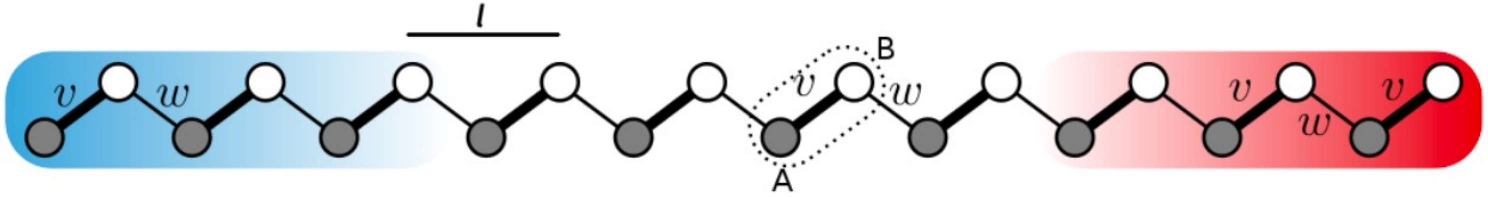
Topological states in one dimension?

Felicien Appas, Stage PRL Ecole Polytechnique 2017

Review:

J. K. Asboth, L. Oroszlany, and A. Palyi, A Short Course on Topological Insulators (Springer, 2016).

Su-Schrieffer-Heeger (SSH) model of polyacetylene



$$\hat{\mathcal{H}} = \sum_{m=1}^{N} v\hat{a}_m^\dagger \hat{b}_m + w\hat{a}_m^\dagger \hat{b}_{m-1} + h.c$$

$$\hat{a}_k = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} \hat{a}_m e^{imkl}$$

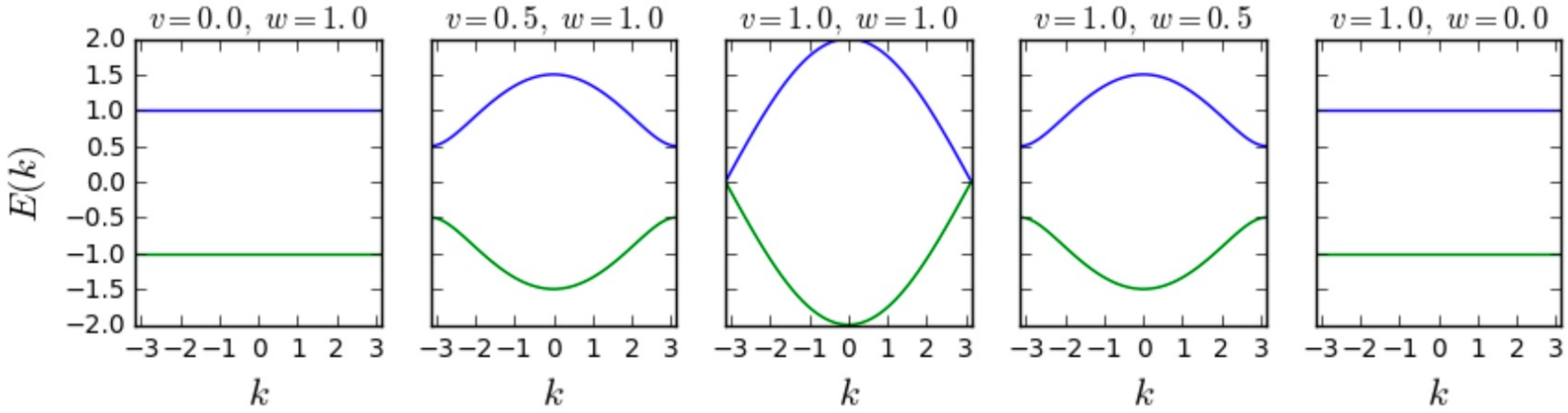
$$\hat{b}_k = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} \hat{b}_m e^{imkl}$$

$$\hat{\mathcal{H}} = \sum_k \left[(v + we^{-ikl})\hat{a}_k^\dagger \hat{b}_k + (v + we^{ikl})\hat{b}_k^\dagger \hat{a}_k \right]$$

$$= \sum_k (\hat{a}_k^\dagger, \hat{b}_k^\dagger) \hat{H}(k) \begin{pmatrix} \hat{a}_k \\ \hat{b}_k \end{pmatrix}$$

$$\hat{H}(k) = \begin{pmatrix} 0 & v + we^{ikl} \\ v + we^{-ikl} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & h(k) \\ h^*(k) & 0 \end{pmatrix}$$

$$\boxed{E(k) = \pm\sqrt{v^2 + w^2 + 2vw\cos kl} \equiv \pm\epsilon(k)}$$



$$\begin{aligned} d_x(k) &= v + w\cos(kl) \\ d_y(k) &= w\sin(kl) \\ d_z(k) &= 0 \end{aligned}$$

$$\boxed{H(k) = \vec{d}(k)\cdot\vec{\sigma}}$$

Zak Phase: $\varphi^{\pm}_{Zak} = \frac{1}{2}\int_{-\pi/l}^{\pi/l} \partial_k \varphi(k)\, dk$

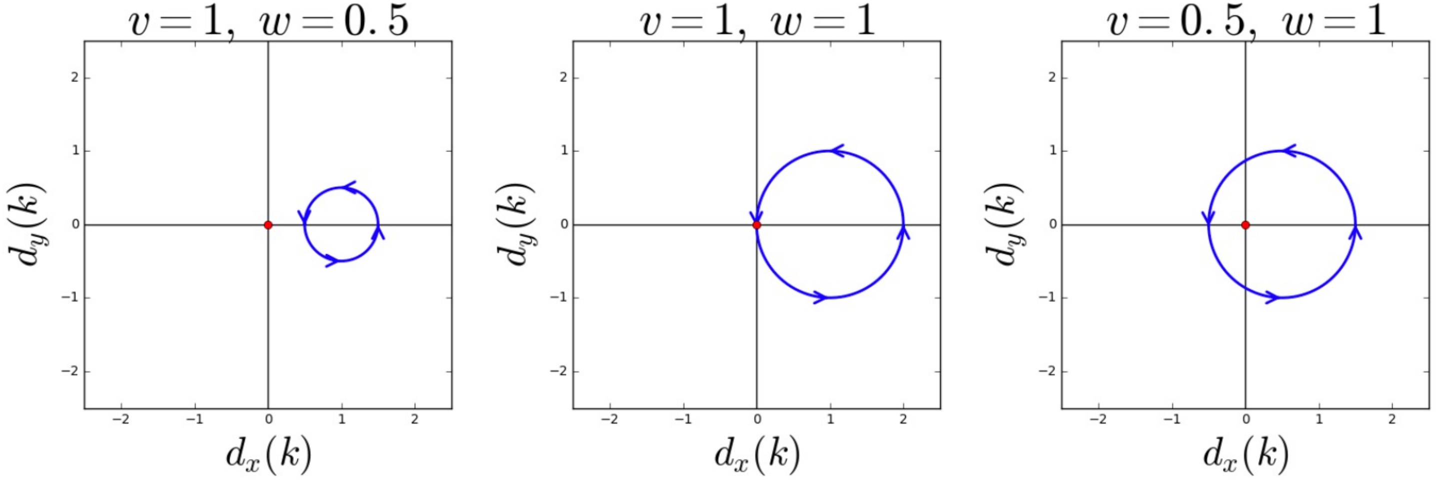
$\varphi = 0$

$\varphi = \pi$

$$M(k) = \begin{pmatrix} 0 & h(k)\, e^{-i\varphi(k)} \\ h(k)\, e^{i\varphi(k)} & 0 \end{pmatrix}$$

$$h(k)e^{i\varphi(k)} \equiv v + we^{-ikl}$$

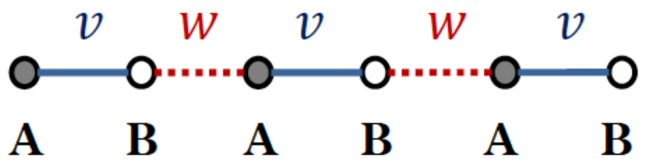
Winding number defined in the xy-plane

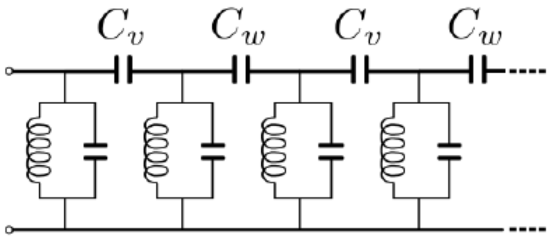


(a) (b) (c)

Su-Schrieffer-Heeger Model: Strong-Coupling Limit of Polyacetylene

T. Goren, K. Plekhanov, F. Appas, KLH **arXiv:1711.02034 and PRB RC 2018**





$$\epsilon_0 = \sqrt{\frac{1}{LC}}\sqrt{\frac{C+C_v+C_w}{C}} \qquad v,w = \epsilon_0\sqrt{\frac{C_{v,w}}{C+C_v+C_w}}$$

$$H = \sum_n \epsilon_0\left(a_n^{\dagger}a_n + b_n^{\dagger}b_n\right)$$

$$+\sum_k v\left(a_n^{\dagger} + a_n\right)\left(b_n^{\dagger} + b_n\right) + w\left(a_{n+1}^{\dagger} + a_{n+1}\right)\left(b_n^{\dagger} + b_n\right)$$

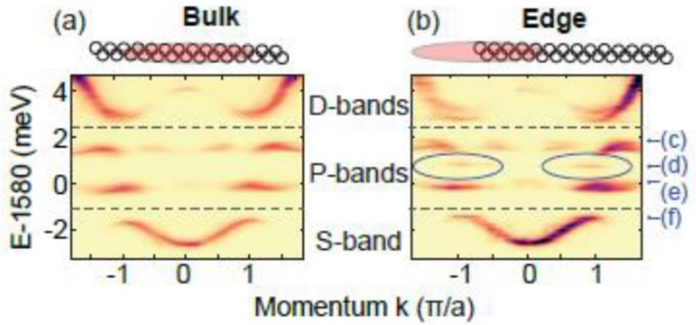
Counter-rotating terms indicate strong coupling, Dicke model

Various measurements of LDOS and edge wave functions : weak-coupling

P. St Jean et al. 2017 (Jacqueline Bloch & Alberto Amo)

C. Poli, M. Bellec et al. Nature 2015 (Lancaster and Nice)

E. J. Meier et al. (Brice Gadway's group), Nature comm. 2016



Similar implementations at Wurzburg, L. Molenkamp's group, Impedance measurements in circuits

And Zurich, S. Huber, T. Neupert; Recent experiment in Boulder, Zak phase

Zak Phase

$$\gamma_{k}^{\pm} = \psi_{\mathbf{k}}^{\pm} \cdot (a_k, a_{-k}^{\dagger}, b_k, b_{-k}^{\dagger})$$

$$\psi_{\mathbf{k}}^{\pm} = \frac{1}{\sqrt{2}} \left(\pm e^{i\varphi(k)} \mathbf{v}_{\mathbf{k}}^{\pm}, \mathbf{v}_{\mathbf{k}}^{\pm} \right)$$

$$\mathbf{v}_{\mathbf{k}}^{\pm} = (\cosh \eta_{k}^{\pm}, \sinh \eta_{k}^{\pm}),$$

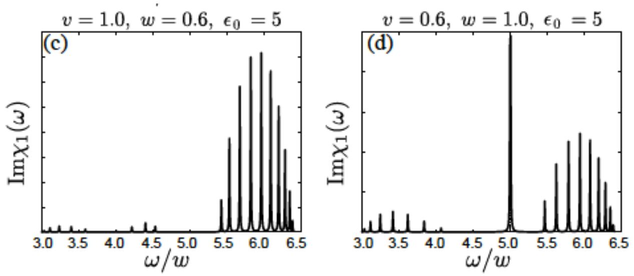
Berry phase

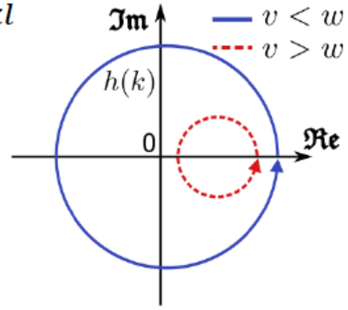
$$\phi_{Zak} = \frac{i}{\pi} \int_{-\pi}^{\pi} \psi_{k}^{\dagger} * \partial_k \psi_k dk$$

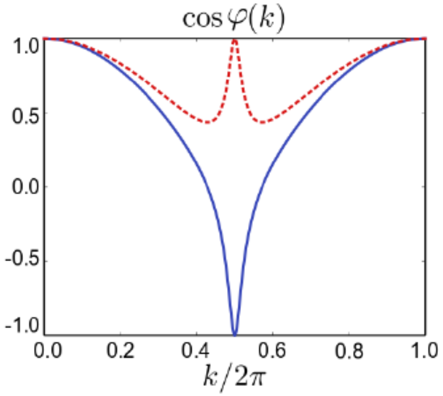
Symmetries (inversion, sub-lattice) and « symplectic » properties of Bogoliubov transformation (similar to measurements in cold atoms, M. Atala et al. (2013))

$$\phi_{Zak} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_k \varphi(k) dk = \begin{cases} 1 & v < w \quad \text{topological} \\ 0 & v > w \quad \text{trivial} \end{cases}$$

$$h(k)e^{i\varphi(k)} = v + w^{-ikl}$$







Bogoliubov Transformation: Vacuum is not the Vacuum…

Squeezing (capacitance model below the super-radiant transition)…

$$|GS\rangle = \prod_k \exp(-\tanh\eta_k^+ {\alpha_k}^\dagger {\alpha_{-k}}^\dagger - \tanh\eta_k^- {\beta_k}^\dagger {\beta_{-k}}^\dagger)|0\rangle$$

$$\alpha_k/\beta_k = \frac{1}{\sqrt{2}}(\pm e^{i\varphi(k)} a_k + b_k)$$

$$\langle a_n^\dagger a_n\rangle = \langle b_n^\dagger b_n\rangle = \frac{1}{2N}\sum_k \sinh^2\eta_k^+ + \sinh^2\eta_k^-$$

$$\xrightarrow{v,w \ll \epsilon_0} \frac{v^2+w^2}{2\epsilon_0^2}$$

T. Karzig, C.-E. Bardyn, N. Lindner, G. Refael PRX 2015

"Resolving photon number states in a superconducting circuit"
Schuster, Houck et al Nature 2007

$$H = \sum_k \epsilon_+(k)\gamma_k^{+\dagger}\gamma_k^+ + \epsilon_-(k)\gamma_k^{-\dagger}\gamma_k^-$$

$$\boldsymbol{\epsilon_\pm(k) = \sqrt{\epsilon_0^2 \pm 2\epsilon_0 h(k)}}$$

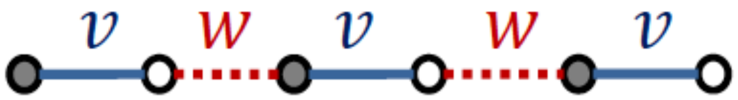
$$h(k)e^{i\varphi(k)} \equiv v + w^{-ikl}$$

$$\varphi(k) = \arg(v + w^{-ikl})$$

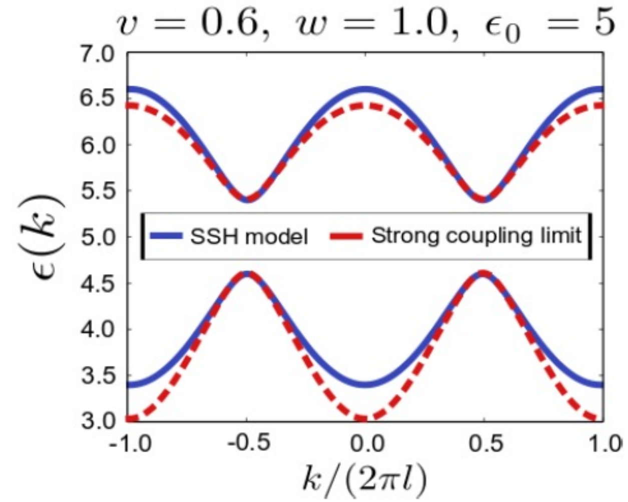
$$\tanh 2\eta_k^\pm = \pm\frac{h(k)}{\epsilon_0 \pm h(k)}$$

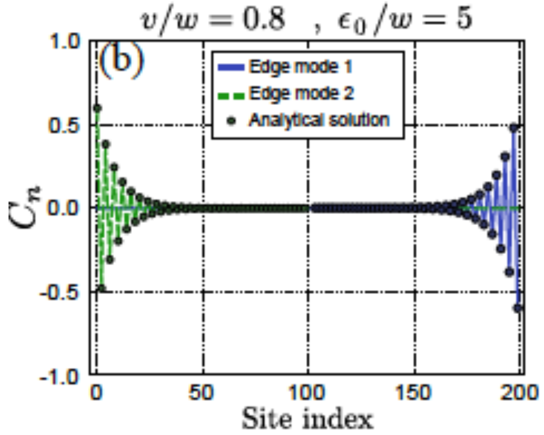
$$\gamma_k^\pm = \pm\frac{e^{i\varphi(k)}}{\sqrt{2}}\left(\cosh\eta_k^\pm a_k + \sinh\eta_k^\pm a_{-k}{}^\dagger\right) +$$

$$\frac{1}{\sqrt{2}}\left(\cosh\eta_k^\pm b_k + \sinh\eta_k^\pm b_{-k}{}^\dagger\right)$$



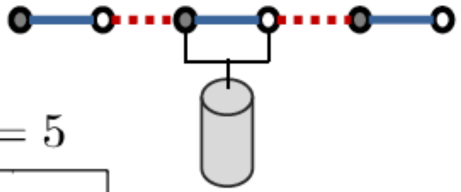
$$\mathcal{W}^- - \mathcal{W}^+ = \frac{4wv}{\epsilon_0},$$

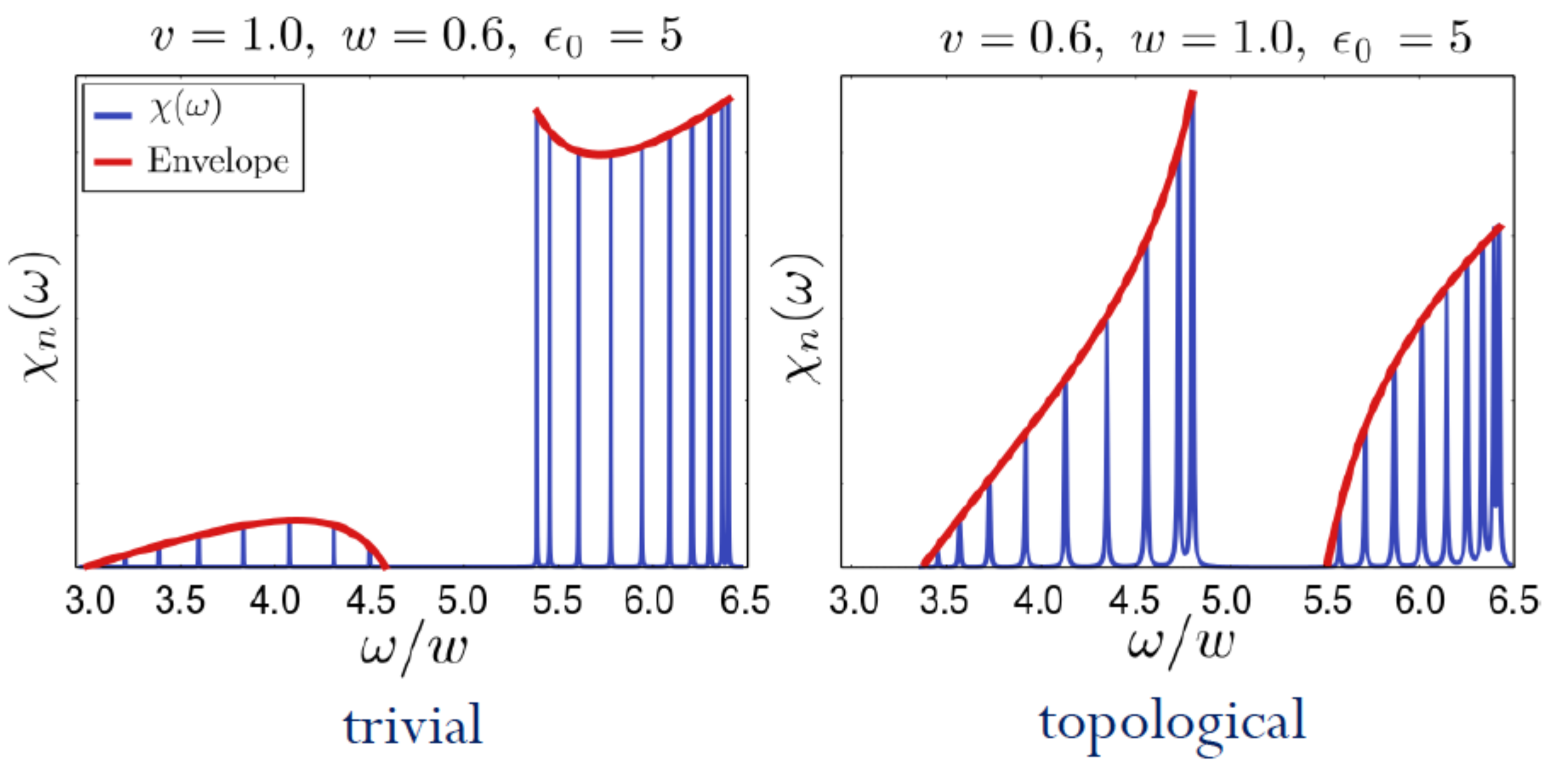




Topological Measurement with Light

$$\chi_n(\omega) = \frac{1}{N}\sum_k \left(1+\cos\varphi(k)\right)\frac{\epsilon_0}{\epsilon_+(k)}\frac{1}{\omega-\epsilon_+(k)} + \frac{1}{N}\sum_k \left(1-\cos\varphi(k)\right)\frac{\epsilon_0}{\epsilon_-(k)}\frac{1}{\omega-\epsilon_-(k)}$$

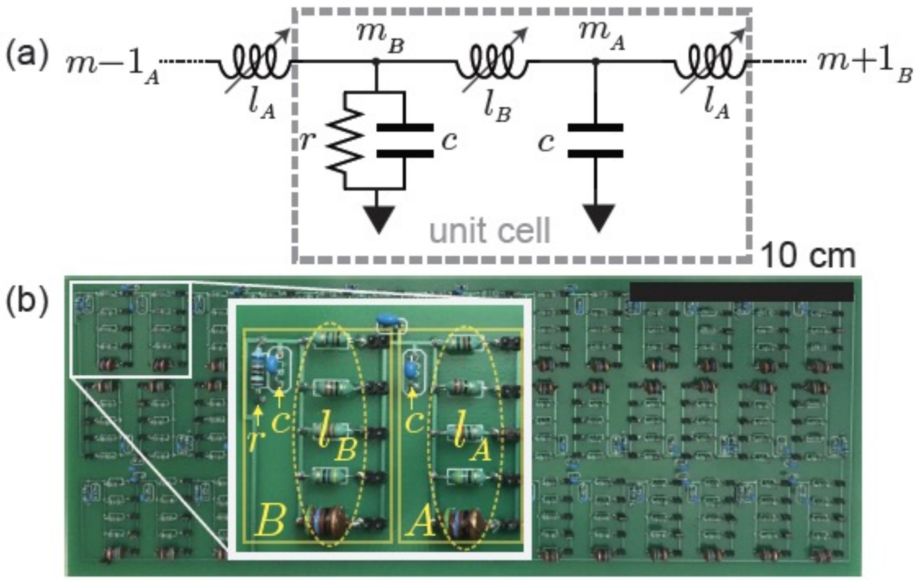


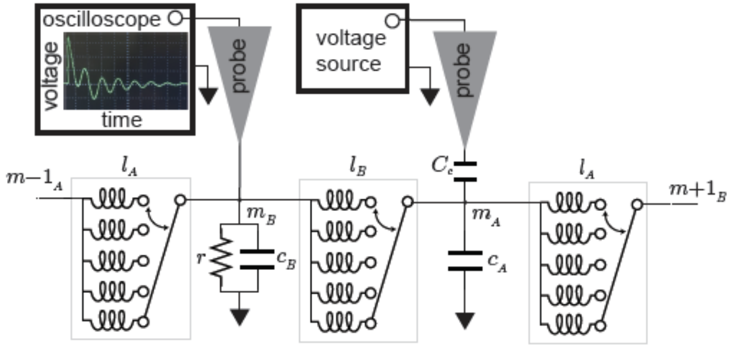


Measurements of impedance at Wurzburg, L. Molenkamp's group on LC devices (theory Ronny Thomale's group)

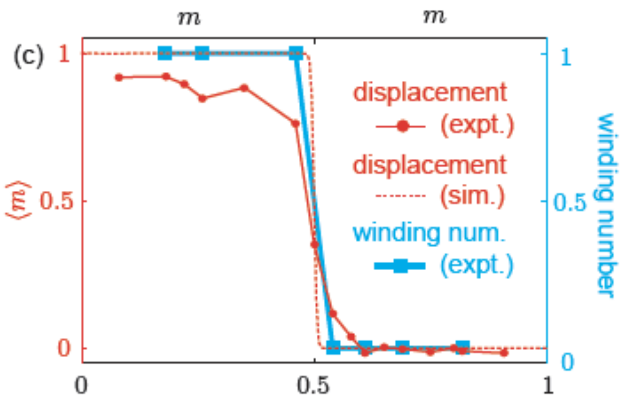
Colorado Boulder: K. Lehnert's group

Eric Rosenthal et al. arXiv:1802.02243





Measurement of energy or Potential



$$\alpha = l_A/(l_A + l_B)$$

$$\langle m \rangle = \oint \frac{dk}{2\pi}\frac{\partial \theta_k}{\partial k} = \begin{cases} 1 & \text{if } \alpha < 1/2 \\ 0 & \text{if } \alpha > 1/2 \end{cases},$$

2001

Kitaev p-wave Superconductor

$$H = -\mu \sum_x c_x^\dagger c_x - \frac{1}{2}\sum_x \left(t c_x^\dagger c_{x+1} + \Delta e^{i\phi} c_x c_{x+1} + h.c.\right),$$

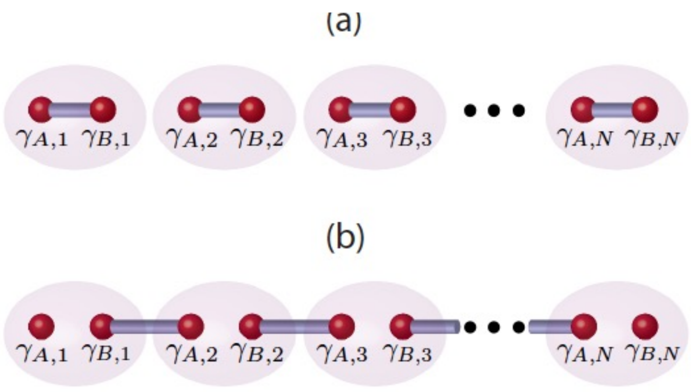
$$\boxed{\mathcal{H}_k = \mathrm{h}(k)\cdot\boldsymbol{\sigma}}$$

$$\boxed{C_k^\dagger = (c_k^\dagger, c_{-k})}$$

$$H = \frac{1}{2}\sum_{k\in BZ} C_k^\dagger \mathcal{H}_k C_k, \quad \mathcal{H}_k = \begin{pmatrix} \epsilon_k & \tilde{\Delta}_k^* \\ \tilde{\Delta}_k & -\epsilon_k \end{pmatrix},$$

with $\epsilon_k = -t\cos k - \mu$ the kinetic energy and $\tilde{\Delta}_k = -i\Delta e^{i\phi}\sin k$ the Fourier-transformed pairing potential. The

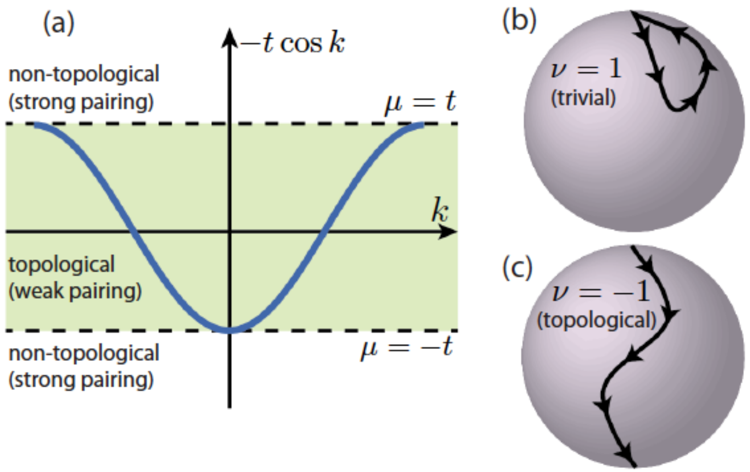
$$c_x = \frac{e^{-i\phi/2}}{2}(\gamma_{B,x} + i\gamma_{A,x}).$$



1937

Majorana fermions

class IV



class I:

$$C = \frac{1}{2}\left(\langle\sigma_z(0)\rangle - \langle\sigma_z(\pi)\rangle\right)$$

Review: J. Alicea, arXiv:1202.1293

Majorana Fermion at an edge stabilized from an impurity

class I

Emery-Kivelson solution of the "2-channel" Kondo model (1992): simple view

Nozieres & Blandin 1980

$$\hat{a} = \hat{a}^{+}$$
$$\hat{b} = \hat{b}^{+}$$

$$H = H_{kin} + J\left[\psi(0) + \psi^{+}(0)\right]\hat{a}$$

$$S^{x} = \frac{\hat{a}}{\sqrt{2}}$$
$$S^{y} = \frac{\hat{b}}{\sqrt{2}}$$
$$S^{z} = i\hat{a}\hat{b}$$

1 free Majorana fermion $x=0$

$$G_b = \frac{1}{2}\,\mathrm{sgn}\,\tau = \langle b(\tau)\,b(0)\rangle$$

$$S_{imp} = \frac{1}{2}\ln 2$$

charge : superfluid

spin : Δ

1D Superconductor

Quasiparticles $e^{\pm ipx}$

$$\mathcal{E}_p = \sqrt{(vp)^2 + \Delta^2}$$

zero-energy mode at $x=0$

$$\mathcal{E}_p = 0 \qquad p = \frac{\pm i\Delta}{v}$$

Impurity Effect : $\hat{a}$, gap

$\hat{b}$ free

Karyn Le Hur, Europhys. Lett., 49 (6), pp. 768-774 (2000)