Dirac Particles, Spin-$\frac{1}{2}$ Formalism and Quantum Hall Effect on Graphene[1]

The quantum Hall effect in Mosfets has been observed by K. von Klitzing, G. Dorda and M. Pepper in 1980.[2] Here, we study the quantum Hall effect on graphene. The magnetic field in z direction is described similarly through the vector potential in a one-dimensional gauge. We have $\mathbf{B} = B\mathbf{e}_z$ such that the vector potential reads $\mathbf{A} = (-By, 0, 0)$ with $\mathbf{B} = \nabla \times \mathbf{A}$. We can now absorb the vector potential in the 2×2 matrix of the graphene model modifying $-i\hbar\nabla \to -i\hbar\nabla - qA_x$ in the Dirac equation. Here, q describes the charge of the particle. This is equivalent to modify $p_x \to p_x - qA_x$ in the reciprocal space where p_x measures a small deviation from the K or K' Dirac points. The Hamiltonian takes the form

$$H = \begin{pmatrix} 0 & -i\hbar v_F\partial_x + v_F qBy - \zeta\hbar v_F\partial_y \\ -i\hbar v_F\partial_x + v_F qBy + \zeta\hbar v_F\partial_y & 0 \end{pmatrix}.$$

Here, $\zeta = \pm 1$ at the K and K' points in the Brillouin zone. The solutions read

$$\Phi(\mathbf{r}) = e^{ikx}\Phi(y) = e^{ikx}\begin{pmatrix} \Phi_A(y) \\ \Phi_B(y) \end{pmatrix}.$$

Therefore, our starting point is the Hamiltonian

$$H = \begin{pmatrix} 0 & \hbar v_F k + v_F qBy - \zeta\hbar v_F\partial_y \\ \hbar v_F k + v_F qBy + \zeta\hbar v_F\partial_y & 0 \end{pmatrix}. \tag{1}$$

We can write the matrix for an electron with charge $q = -e < 0$ introducing the cyclotron length $l_B = \sqrt{\frac{\hbar}{|q|B}}$ and frequency $\omega_c = \frac{v_F}{l_B}$:

$$H = \hbar\omega_c \begin{pmatrix} 0 & -l_B\zeta\partial_y + \left(kl_B - \frac{y}{l_B}\right) \\ l_B\zeta\partial_y + \left(kl_B - \frac{y}{l_B}\right) & 0 \end{pmatrix}.$$

The matrix can be written in terms of the dimensionless position operator $\hat{r} = -\frac{y}{l_B} + kl_B$ and momentum operator $-i\hbar\partial_r = l_B(i\hbar\partial_y)$ such that $[\hat{r}, -i\hbar\partial_r] = i\hbar$.

Following the Dirac approach for the harmonic oscillator in quantum mechanics, we introduce the ladder operators

$$\mathcal{O} = \frac{1}{\sqrt{2}}(\hat{r} + \partial_r) = \mathcal{O}_K = \mathcal{O}^\dagger_{K'} \tag{2}$$

$$\mathcal{O}^\dagger = \frac{1}{\sqrt{2}}(\hat{r} - \partial_r) = \mathcal{O}^\dagger_K = \mathcal{O}_{K'} \tag{3}$$

such that $[\mathcal{O}, \mathcal{O}^\dagger] = 1$. The matrix can be re-written as

$$H = \hbar\omega_c^* \begin{pmatrix} 0 & \mathcal{O}^\dagger \\ \mathcal{O} & 0 \end{pmatrix} = \hbar\omega_c^* \left(\mathcal{O}^\dagger\sigma^+ + \mathcal{O}\sigma^-\right). \tag{4}$$

We have the identfication $\omega_c^* = \sqrt{2}\omega_c$. Here, the raising and lowering spin operators are written in terms of the Pauli spin operators (matrices) $\sigma^\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

We have the equations

$$\hbar\omega_c^*\mathcal{O}^\dagger\Phi_B(r) = E\Phi_A(r)$$
$$\hbar\omega_c^*\mathcal{O}\Phi_A(r) = E\Phi_B(r).$$

It is important to be meticulous when solving these equations since we have operators. We can multiply the second equation by $\mathcal{O}^\dagger$ such that

$$\hbar\omega_c^*\mathcal{O}^\dagger\mathcal{O}\Phi_A(r) = E\mathcal{O}^\dagger\Phi_B(r).$$

[1] This additional material is from the recent review K. Le Hur, arXiv:2209.15381

[2] https://journals.aps.org/prl/pdf/10.1103/PhysRevLett.45.494

Then, we use the first equation and find

$$\hbar\omega_c^*\mathcal{O}^\dagger\mathcal{O}\Phi_A(r) = E\mathcal{O}^\dagger\Phi_B(r) = \frac{E^2}{\hbar\omega_c^*}\Phi_A(r). \tag{5}$$

Introducing $\hat{N} = \mathcal{O}^\dagger\mathcal{O}$ with associated (integer) eigenvalues N, we have the equation

$$\hat{N}\Phi_A(N) = N\Phi_A = \frac{E^2}{(\hbar\omega_c^*)^2}\Phi_A(N)$$

which shows the solutions $E_\pm(N) = \pm\hbar\omega_c^*\sqrt{N}$. We also have the other equation

$$\hbar\omega_c^*\mathcal{O}\mathcal{O}^\dagger\Phi_B = E_\pm(N)\mathcal{O}\Phi_A(N) = \frac{E_\pm^2}{\hbar\omega_c^*}\Phi_B.$$

To satisfy the same eigen-energies $E_\pm(N)$, then the eigenstate should be $\Phi_B = \Phi_B(N-1)$ such that $\mathcal{O}^\dagger\Phi_B(N-1) = \sqrt{N}\Phi_B(N)$ and therefore $\mathcal{O}(\mathcal{O}^\dagger\Phi_B(N-1)) = \sqrt{N}\sqrt{N}\Phi_B(N-1)$. In this way, $\Phi_A(N)$ and $\Phi_B(N-1)$ admit the same energy solutions $E_\pm(N) = \pm\hbar\omega_c^*\sqrt{N}$. The energy at the two Dirac points becomes quantized in units of $\hbar\omega_c^*$ corresponding to infinite ladders.[3] The main difference compared to the solution of the Schrödinger equation is that at $N = 0$, then $E = 0$ and we have a zero-energy state corresponding to the formation of a Landau level. The spectrum is also doubly degenerate since the K and K' points give identical solutions.

The ladder starts at $N = 0$ and then we progressively fill the states ±1 ... such that the ladder progressively becomes infinite. We have the same ladder structure at the K' Dirac point.

For $N = 0$, then we have a zero-energy mode shared between the K and K' Dirac points. The ground state at K is projected on Φ_A and satisfies $\mathcal{O}\Phi_A = 0$ and equivalently $\Phi_B = 0$. The associated wave-function is a Gaussian

$$\Phi_A(y) = c.e^{-\frac{(y-k)^2}{2l_B^2}}.$$

Due to the inversion of $\mathcal{O}(K)$ and $\mathcal{O}^\dagger(K')$, at the K' point, the ground state is projected on Φ_B, similarly as for the Haldane model.

What are the eigenstates at the K and K' Dirac points for $N \neq 0$?

For fixed $N \neq 0$, we have 2 possible energies $E_\pm = \pm\sqrt{N}(\hbar\omega_c^*)$ corresponding then to two orthogonal eigenstates. Since $\Phi_A(N)$ and $\Phi_B(N-1)$ are solutions of the energy equations, we can build the following orthogonal eigenstates:

$$\Phi_\pm(K) = \pm\Phi_B(N-1)\otimes B + \Phi_A(N)\otimes A. \tag{6}$$

The $\pm$ then refers to energies $\pm\hbar\omega_c^*\sqrt{N}$. Here, B corresponds to the unit vector $(1,0)^T$ and A to $(0,1)^T$. In general, $\Phi_i(N)$ with $i = A, B$ corresponds to the Gaussian wave-function times the Hermite polynomial of degree N such that $\Phi_\pm(K)$ can be correctly normalized. At the K' Dirac point, due to the inversion between $\mathcal{O}$ and $\mathcal{O}^\dagger$ this leads to

$$\Phi_\pm(K') = \pm\Phi_B(N)\otimes B + \Phi_A(N-1)\otimes A. \tag{7}$$

To study the transport, we can use the following analogy with the charged particle seen in quantum class

$$\mathcal{O}^\dagger\mathcal{O} = \frac{1}{2}(\hat{r}^2 - \partial_r^2 + [\hat{r}, \partial_r]) = \frac{1}{2}(\hat{r}^2 - \partial_r^2 - 1).$$

[3] J. W. McClure, Phys. Rev. **104**, 666 (1956).

Therefore, Eq. (5) leads to

$$\frac{\hbar\omega_c^*}{2}\left(\hat{r}^2-\partial_r^2+[\hat{r},\partial_r]\right)\Phi_A(y)=\frac{\hbar\omega_c^*}{2}\left(\hat{r}^2-\partial_r^2-1\right)\Phi_A(y)=\frac{E^2}{\hbar\omega_c^*}\Phi_A(y).$$

This leads to

$$\hat{H}_{eff}\Phi_A(r)=\left(\frac{E^2}{\hbar\omega_c^*}+\frac{\hbar\omega_c^*}{2}\right)\Phi_A(r)=\hbar\omega_c^*\left(N+\frac{1}{2}\right)\Phi_A(r).$$

We add an electric field along y direction corresponding to a potential term $-eV(y)=eEy$ in the Hamiltonian. Introducing the mass $m=\frac{\hbar}{\omega_c^* l_B^2}$, we can re-write $\hat{H}_{eff}$ as

$$\hat{H}_{eff}=\frac{p_y^2}{2m}+eEy+\frac{1}{2}m\omega_c^*\left(y-l_B^2k\right)^2.$$

We can absorb the effect of the electric field completing the square such that it modifies the oscillator coordinate

$$k\to k-\frac{eE}{m\omega_c^{*2}l_B^2}.$$

This is equivalent to a drift velocity in x direction

$$\langle v_x\rangle=\frac{\hbar k}{m}=-\frac{E}{B}.$$

This velocity can be justified from physical arguments. If we include both a Coulomb and Lorentz force for a charge q, $q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$ then this is equivalent to modify the electric field along y direction such that $E_y\to E_y-v_xB$.

Our goal is now to show that the quantum Hall conductivity is quantized in units of $\frac{e^2}{h}$. For this purpose, this requires to calculate the current density $j_x=ne|\langle v_x\rangle|$ transverse to the applied electric field. We have a general relation between the number of particles N_e and the density n through $N_e=nA$ with $A=L_xL_y$ being the area of the two-dimensional plane. We assume $(2N+1)$ filled energy Landau levels at a Dirac point. We need to evaluate the number of states in each level.

The number of states can be simply understood from the number of possibilities to place the cyclotron orbits coordinate $y_0=kl_B^2$.

The number of accessible states in each Landau level can be written as

$$\mathcal{N}=L_x\int_0^{|k|_{max}}\frac{d|k|}{2\pi}.$$

We have seen above that the wave-vector k has the dimension of y/l_B^2 with $y_{max}=L_y$ such that $|k|_{max}=L_y/l_B^2$. We discuss here the effect of the drift velocity where $k<0$. We can re-write $\mathcal{N}$

$$\mathcal{N}=\frac{ABe}{2\pi\hbar}=\frac{\Phi}{\Phi_0},$$

where $\Phi=AB$ corresponds to the magnetic flux in the area and $\Phi_0=\frac{h}{e}$ is the flux quantum. The total number of states is equal to the number of flux quanta in the sample. Within the definitions, then we have 1 state per Landau level if the flux is equal to the flux quantum.

What is then the total number of electrons N_e in the system?

If we count correctly, we have

$$N_e=2(2N+1)\mathcal{N}$$

particles in the system. For each Landau level, we can have $\mathcal{N}$ electrons and we have $2(2N+1)$ Landau levels if we take into account the fact that the K and K' points double the number of

Landau levels.

We can now deduce the quantum Hall conductivity σ_{xy} and compare with experimental results[4].

If we summarize the results, we have

$$j_\perp = j_x = \frac{2(2N+1)\mathcal{N}}{A}\frac{E}{B}e = \frac{2(2N+1)e^2}{h}E,$$

which leads to a quantization of the quantum Hall conductivity

$$\sigma_{xy} = 2(2N+1)\frac{e^2}{h}.$$

In the experiments, we observe both positive and negative plateaus associated to particles and holes.

[4]https://journals.aps.org/rmp/abstract/10.1103/RevModPhys.81.109