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Lecture 2: Physical implementation of qubits and harmonic oscillators

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The first lecture reviewed the main principles of quantum physics. It also introduced in an abstract way the idea of a quantum bit, which is nothing but a two-level system that can be placed in any superpositions of two states $|0\rangle$ and $|1\rangle$. Here we discuss some physical implementations of qubits in real systems.

The simplest example of an exact realization of a two-level system is an electron, which carries a spin 1/2. If it is at rest, one ignores its external degrees of freedom and only considers the spin. The projection of the spin along any axis takes two values $m = \pm 1/2$, corresponding to two states $|\uparrow\rangle = |0\rangle$ and $|\downarrow\rangle = |1\rangle$. A similar example is that of photons in any mode with wave vector $\vec{k}$. In this case, the qubit may be encoded in two orthogonal polarization state, e.g. the linear polarizations $|\mathrm{H}\rangle$ and $|\mathrm{V}\rangle$ or the circular polarizations $|\sigma^+\rangle$ and $|\sigma^-\rangle$. Any other physical system in which one can identify two orthogonal states may serve as a qubit. However, almost all examples other that a spin 1/2 or a $\boldsymbol{k}$-photon are approximate implementations, e.g. two-states of an atomic structure selected by a resonant laser, two states of a particle in a square well potential, a double-well potential, ...

For a physical system to be suitable to encode a qubit, one requires five criteria, introduced around 2000 by David DiVincenzo:

1. The physical system encoding the qubit can be well isolated, and can be replicated (scalability);
2. It can be initialized in one of the two states;
3. It can be measured;
4. It can be manipulated to prepare coherent superpositions;
5. It should have a long enough coherence time to keep the superposition state $\alpha\,|0\rangle + \beta\,|1\rangle$ coherent during the whole manipulations.

For most tasks in quantum technologies, one must also be able to make the qubits interact to generate entanglement. This is almost invariably done using a *quantum mediator*, which is an excitation of a field shared by the qubits and equivalent to a harmonic oscillator. These excitations have various names depending on the context: photons, phonons,... This lecture also describes physical realizations of quantum harmonic oscillators.

1 Atomic physics in a nutshell

Because their quantum properties are well documented, they can be isolated and precisely controlled, atomic system – namely atoms, ions, and molecules – are popular systems to encode qubits. We thus start by a brief review of their general properties and the possible choices of qubit states.

Hydrogene atom. The simplest atomic system is the hydrogen atom. It consists of an electron bound to a nucleus, made of a single proton. If we first only consider the Coulomb interaction, the Hamiltonian of the system in the center-of-mass frame of the atom is

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m_\mathrm{r}} - \frac{e^2}{|\hat{\boldsymbol{r}}|} \ , \tag{1}$$

where $\hat{\boldsymbol{p}}$ is the momentum operator, $\hat{\boldsymbol{r}}$ the electron-proton relative position operator, m_r the reduced mass, and $e^2 = q_\mathrm{e}^2/(4\pi\epsilon_0)$, with q_e the electron charge and ϵ_0 the vacuum permittivity. In principle, $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{p}}$, and m_r refer to the reduced particle, but since the proton is much heavier than the electron, they can be assimilated to those of the electron itself. The Hamiltonian may be diagonalized by noting that, owing to rotation invariance, $\hat{H}$ commutes with the angular momentum operator $\hat{\boldsymbol{L}}$ and one of its components, say $\hat{L}_z$. The eigenstates may be classified using 3 quantum numbers, say $|\psi_{nlm}\rangle$, where $n \in \mathbb{N}^*$ is the principal number, $\ell \in [0..n-1]$ is the angular number, and $m \in [-\ell, +\ell]$ is the azimutal number. The solution of the stationary Schrödinger equation yields the energies

$$E_n = -\frac{E_\mathrm{I}}{n^2} \ , \tag{2}$$

where $E_\mathrm{I} \simeq 13.6\,\mathrm{eV}$ is the ionization energy. It may be written as $E_\mathrm{I} = mc^2\alpha^2/2$ with $\alpha = e^2/\hbar c \simeq 1/137$ the fine-structure constant. It follows that the energy is only determined by the principal number and each energy state is n^2 times degenerated. Including the twofold electonic spin degeneracy, it yields $g_n = 2n^2$ electronic states per energy level.

Alkali atoms. This description is also valid to a good approximation for hydrogen-like systems, namely alkali atoms, i.e. those of the first column or the periodic table (Li, Na, K, Rb, Cs, ...), and ions obtained by removing an electron from elements of the second column, such as for instance Ca^+ commonly used in laboratories. These systems are characterized by the fact that they have a single electron in the valence shell, pretty much as the hydrogene atom. The $Z-1$ electrons in the inner shells are essentially frozen and partially screen the Z positive charges of the nucleus. They are thus similar to the hydrogene atom but the effective charge of the nucleus is slightly larger than $+|q_\mathrm{e}|$. It yields the energies

$$E_{n,\ell} \simeq -\frac{E_\mathrm{I}}{(n-\delta_{n,\ell})^2} \ , \tag{3}$$

with $\delta_{n,\ell}$ the so-called *quantum defect*, of the order of a few units. Apart from the slight correction to the energy levels, the main difference with the hydrogene atom is that the ℓ-degeneracy is lifted since the defect depends on the angular number ℓ. This is because this degeneracy in the hydrogene atom is due to a special symmetry of the $1/r$ Coulomb potential, which is broken by the screening of the inner electronic shells. All in all, this formula already tells us that the transitions between low lying n states have energies of the order of a few eV. It corresponds to frequencies of the order of $\omega \sim 1.5 \times 10^{15}\,\mathrm{s}^{-1}$ and wavelengths of the order of $\lambda \sim 1\,\mu$m. This is nothing but the optical spectrum, and such transitions can be addressed by laser beams.

Fine and hyperfine structures. To go beyond the electrostatic approximation, we must first include the spin-orbit coupling. It originates from the interaction between the magnetic moment of the electron $\boldsymbol{\mu}$, proportional its spin $\mathbf{S}$, with the motional magnetic field $\mathbf{B}_\mathrm{m}$ induced by the motion of the electron in the Coulomb field $\mathbf{E} \propto \mathbf{r}/r^3$ created by the nucleus. Since $\mathbf{B}_\mathrm{m} = \mathbf{v} \times \mathbf{E}/c^2 \propto \mathbf{L}$ with $\mathbf{L} = m\,\mathbf{r} \times \mathbf{v}$ the orbital angular momentum of the electron, the Hamiltonian describing this interaction reads as

$$\hat{H}_\mathrm{so} = -\hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{B}}_m \propto \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}\ . \tag{4}$$

It yields an energy scale much smaller than the eV and this term can be included using perturbation theory. Noting that $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ commute, we may write $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = [(\hat{\mathbf{L}} + \hat{\mathbf{S}})^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{S}}^2]/2$. Since the values of $\mathbf{L}^2 = \hbar^2\ell(\ell+1)$ and $\mathbf{S}^2 = \hbar^2 S(S+1)$ are fixed within each level of the principal structure, we see that the spin-orbit coupling induces level splitting corresponding to the different values of $\mathbf{J}^2 = \hbar^2 J(J+1)$ with $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$. The structure hence obtained is known as the *fine structure*. We recall that J spans the discrete set of values $|L-S|$, $|L-S|+1$, ..., $|L+S|$. Typically, the splitting corresponds to transitions with wavelengths of a few nm.

The second effect we may consider is the coupling between the magnetic moment $\boldsymbol{\mu} \propto \mathbf{J}$ of the electron in orbit and the magnetic moment of the nucleus $\boldsymbol{\mu}_\mathrm{n} \propto \mathbf{I}$, where $\mathbf{I}$ is the angular momentum of the nucleus (if it has one, of course). The corresponding Hamiltonian is

$$\hat{H}_\mathrm{hf} \propto \hat{\mathbf{I}} \cdot \hat{\mathbf{J}} \tag{5}$$

and leads to the so-called *hyperfine structure*. It is again much smaller than the previous one and it can be treated in perturbation of the fine structure. A similar approach as above suggests to introduce the new angular momentum operator $\hat{\mathbf{F}} = \hat{\mathbf{J}} + \hat{\mathbf{I}}$. It shows that the hyperfine term induces a splitting of the fine structure energy states corresponding to the different values of $\mathbf{F}^2 = \hbar^2 F(F+1)$, where F spans the discrete set $|J-I|, |J-I|+1, ..., |J+I|$. The corresponding transition frequencies lie in the 10 MHz-1 GHz range.

These various corrections lead to a hierarchy of transition frequencies, ranging from optical to microwaves, as illustrated in Fig. 1 for the sodium atom. Although the exact

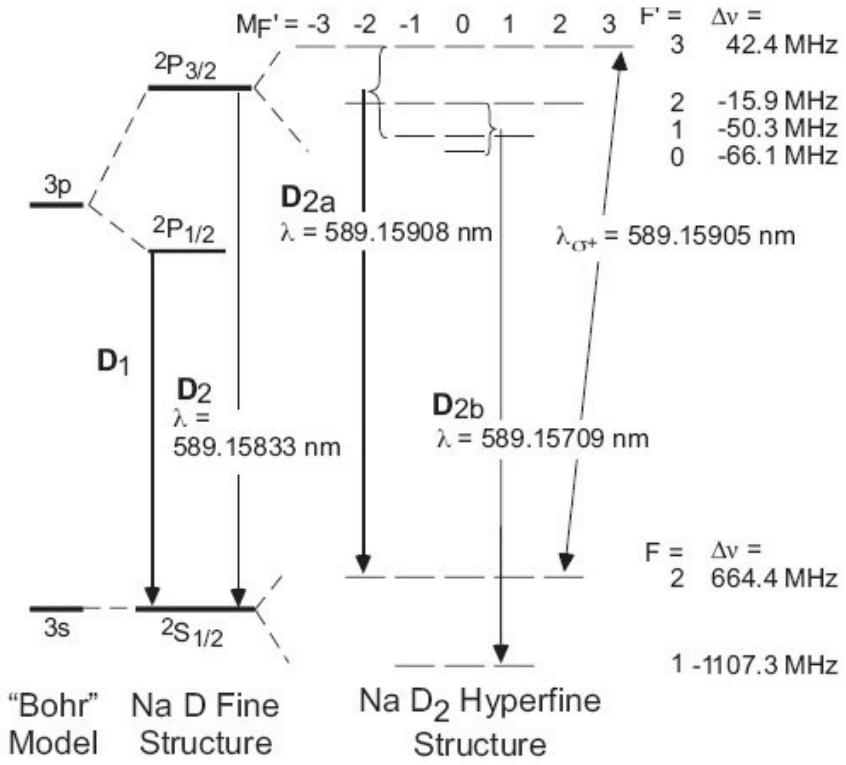

Figure 1: Electronic structure of the sodium atom (Na, atomic number $Z = 11$) in the low lying states. Principal structure ("Bohr model"): The ground state is $n = 3$ for the valence electron since the two lowest shells are saturated with, respectively, $g_1 = 2$ and $g_2 = 8$ electrons. In contrast to the hydrogene atom, the principal levels with $\ell = 0$ (shell "s") and $\ell = 1$ (shell "p") are nondegenerate owing to different values of the quantum defects ($\delta_s \neq \delta_p$). The principal level with $\ell = 2$ (shell "d") is not shown. The corresponding transition is called the "D" line. Fine structure: The spin-orbit coupling lifts the $3p$ degeneracy in two levels $3^2p_{1/2}$ ($J = 1/2 = |\ell - S|$ with $\ell = 1$ and $S = 1/2$) and $3^2p_{3/2}$ ($J = 3/2 = |\ell + S|$). The degeneracy of the $3s$ level is not lifted since $\ell = 0$. Hyperfine structure: Each J level degeneracy is lifted by the interaction with the nucleus with angular moment $I = 3/2$. The ground state $3^2s_{1/2}$ ($J = 1/2$) splits in two hyperfine levels with $F = 1 = |3/2 - 1/2|$ and $F = 2 = |3/2 + 1/2|$. The level $3^2p_{1/2}$ ($J = 1/2$) also splits into two hyperfine levels (not shown). The level $3^2p_{3/2}$ ($J = 3/2$) splits into four hyperfine levels with $F = 0 = |3/2 - 3/2|$, $F = 1$, $F = 2$, and $F = 3 = |3/2 + 3/2|$.

features depend on the atom or ion, they are generic of all atomic structures. If now one considers molecules, new degrees of freedom appear, such as their rotations and vibrations, once again with characteristics energy scales.

Atomic qubits. For atomic systems, two choices for qubit encoding are usually made, as illustrated in Fig. 2. The first choice uses two Zeeman states $|F, M_F\rangle$ (with $-F \leq$

$M_F \leq F$) of the hyperfine structure of atoms or ions from two different levels. For ^{87}Rb, one may use the states $|0\rangle = |F = 1, M_F = 0\rangle$ and $|1\rangle = |F = 2, M_F = 0\rangle$ as shown in Fig. 2(a). To lift the degeneracy between the Zeeman states, one applies a magnetic field of a few Gauss ($1\,\mathrm{G} = 10^{-4}\,\mathrm{T}$), thus isolating various two-level systems with frequencies separated by an energy of the order of $\Delta E \sim \mu_\mathrm{B} B$. An asset of this encoding is the essentially infinite lifetime of the qubit states. The second choice uses an excited state connected to the ground state by a nearly forbidden optical transition from two different fine levels, see Fig. 2(b). In this case the excited qubit state $|1\rangle$ has a lifetime which can exceed seconds.

To prepare the system in the state $|0\rangle$, one uses optical pumping. It consists in transferring the angular momentum of the light onto the atom. For instance, a $\sigma^\pm$ circularly polarized light changes the Zeeman state from M to $M \pm 1$. To read out the qubit state, one uses an auxiliary optical transition connecting one of the two qubit states to an excited state $|\mathrm{e}\rangle$ with large decay rate Γ. This is realized using a laser field resonant with, say, the transition $|0\rangle \leftrightarrow |\mathrm{e}\rangle$ but sufficiently far from resonance from the transition $|1\rangle \leftrightarrow |\mathrm{e}\rangle$, see Fig. 2(b) [Note that it is the opposite in the configuration of Fig. 2(a)]. Then, if the qubit is in state $|0\rangle$ coupled to $|\mathrm{e}\rangle$, the atom fluoresces when the laser light is switched on, while it remains dark if the qubit in the other state $|1\rangle$. The detection of fluorescence light therefore projects the atom into the bright qubit state $|0\rangle$. To get an idea of the number of photons collected, remember that for an optical transition $\Gamma \sim 2\pi \times 10\,\mathrm{MHz}$. Illuminating the atom for $100\,\mu$s, and collecting 1% of the emitted light, one gets ~ 60 photons, which are easily detected with a CCD camera.

2 Manipulating a single-qubit state: Interaction between atoms and classical fields

Having identified two atomic states to encode the qubit, we need to manipulate its state in the general form $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. This may be realized by tuning the frequency of an electromagnetic field at the transition frequency between the two qubit states $|0\rangle$ and $|1\rangle$. We therefore describe here how an atom or a molecular (which we will call "atom" for the sake of conciseness) couples to a classical electromagnetic field with an angular frequency ω and a wavelength $\lambda = 2\pi c/\omega$.

Electric dipole optical interaction. When the atom is placed in an electromagnetic field, the electric field $\mathbf{E}(\mathbf{r}, t)$ distorts the electronic cloud of the atom. The electrons and the nucleus are pulled apart, and as a consequence a dipole $\mathbf{d}$ appears. In turn, the dipole interacts back with the field. The Hamiltonian describing this interaction is

$$\hat{H}_\mathrm{int} = -\hat{\mathbf{d}} \cdot \mathbf{E}(\mathbf{R}, t)\ , \tag{6}$$

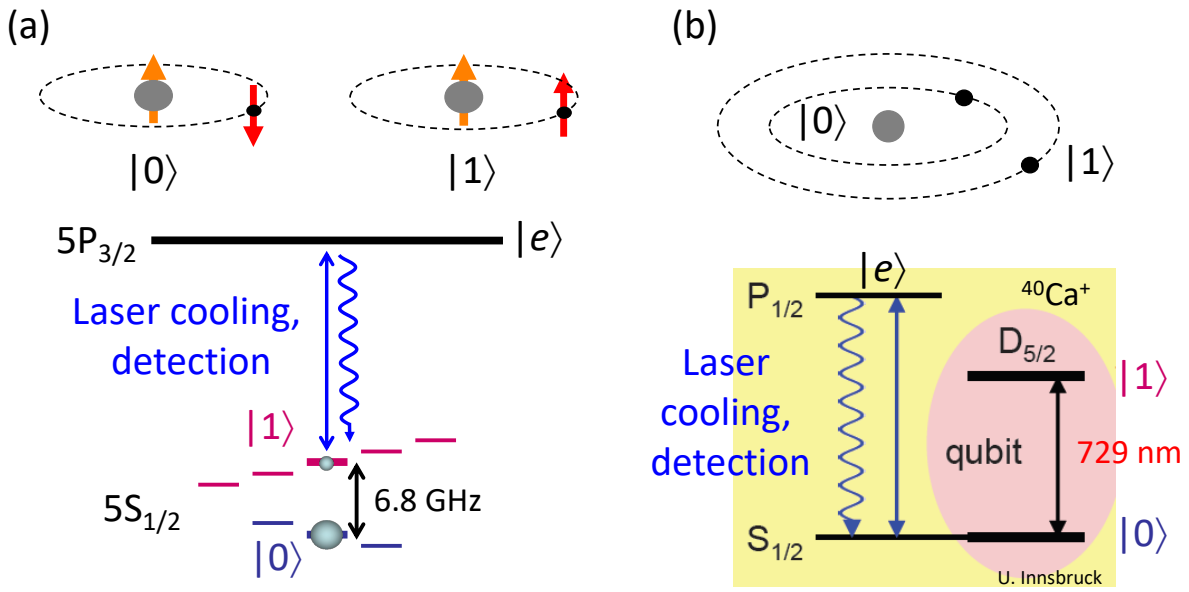

Figure 2: Possible choices of atomic qubit. (a) Hyperfine encoding, with the example of ^{87}Rb: $|0\rangle = |F = 1, M_F = 0\rangle$ and $|1\rangle = |F = 2, M_F = 0\rangle$, separated by a transition with a frequency at 6.8 GHz. (b) Optical encoding, with the example of Ca^+.

with $\mathbf{R}$ the position of the center of mass of the atom, and not that of the electron, as it should *a priori* be the case. This is valid when the radiation wavelength λ is much larger than the size of the atom so that all electrons see the same amplitude of the field. Here one neglects the field propagation phase $k\Delta r$, with $k = 2\pi/\lambda$, across the atomic size Δr. In practice, $\Delta r \sim 0.1$ nm and $\lambda \sim 1\,\mu$m, so that this long-wavelength approximation is very well justified.

If we consider for simplicity a single-electron atom, the electric dipole operator is $\hat{\mathbf{d}} = q\hat{\mathbf{r}}$, with $\mathbf{r}$ the electron-nucleus relative position operator. The dipole operator couples states with opposite parities, i.e. the matrix element $\mathbf{d}_{01} = \langle 0|\,\hat{\mathbf{d}}\,|1\rangle$ between the states $|0\rangle = |n, \ell\rangle$ and $|1\rangle = |n', \ell'\rangle$ is non-zero only if $\ell' = \ell \pm 1$. The distortion of the electronic cloud described above is a consequence of the electromagnetic field admixing these two states. In terms of electronic wave functions, we have

$$\mathbf{d}_{01} = \int d^3\mathbf{r}\, \psi^*_{n\ell}(\mathbf{r})\, q\mathbf{r}\, \psi_{n'\ell'}(\mathbf{r}) \qquad \text{and} \qquad d_{01} \sim qa_0\ . \tag{7}$$

Since a state is not coupled to itself by the dipole operator, the dipole operator may alternatively be written as

$$\hat{\mathbf{d}} = \mathbf{d}_{01}\hat{\sigma}_x = \mathbf{d}_{01}(\hat{\sigma}^+ + \hat{\sigma}^-)\ , \tag{8}$$

with $\hat{\sigma}^+ = |1\rangle\langle 0|$ and $\hat{\sigma}^- = |0\rangle\langle 1|$, since $\mathbf{d}_{01}$ is real.

If we take a monochromatic plane wave with angular frequency ω and the atom at $\mathbf{R} = 0$, we have $\mathbf{E}(\mathbf{R}, t) = \mathbf{E}_0 \cos\omega t$ and $\langle 0| H_{\rm int} |1\rangle = -\langle 0| \mathbf{d} |1\rangle \cdot \mathbf{E}_0 \cos\omega t$, i.e.

$$\langle 0| H_{\rm int} |1\rangle = \hbar\Omega \cos\omega t \ , \tag{9}$$

with $\Omega = -\mathbf{d}_{01} \cdot \mathbf{E}_0/\hbar$ the *Rabi frequency*.

Microwave transition. Consider now a qubit encoded in the two spin states of an atom in ground-state level $nS_{1/2}$, $|0\rangle = |\downarrow\rangle$ and $|1\rangle = |\uparrow\rangle$. The qubit state may be manipulated using an oscillating magnetic field of the form $\mathbf{B} = B_0\mathbf{e}_z + B_1 \cos\omega t\, \mathbf{e}_x$. The magnetic moment $\boldsymbol{\mu} = -g\mu_{\rm B}\mathbf{S}/\hbar$ associated to the electronic spin couples to the magnetic field and the Hamiltonian reads as

$$\hat{H}_{\rm MW} = -\boldsymbol{\mu} \cdot \mathbf{B} = (g\mu_{\rm B}/2)[B_0\, \hat{\sigma}_z + B_1 \cos\omega t\, \hat{\sigma}_x] \ . \tag{10}$$

The matrix element coupling the states $|0\rangle = |\downarrow\rangle$ and $|1\rangle = |\uparrow\rangle$ is thus

$$\langle 0| \hat{H}_{\rm MW} |1\rangle = \hbar\Omega \cos\omega t \ , \tag{11}$$

with $\Omega = g\mu_{\rm B}B_1/2\hbar$ the Rabi frequency. We thus recover the same Hamiltonian as for the electric dipole interaction, see Eqs. (9) and (11), although, obviously, with different Rabi frequencies since they correspond to different physical processes.

Coherent manipulation and Rabi oscillations. As shown above, be the transition dipole electric or magnetic, the Hamiltonian describing the interaction of an atom with an electromagnetic field has the general form

$$\hat{H}(t) = -\frac{\hbar\omega_0}{2}\, \hat{\sigma}_z + \hbar\Omega \cos\omega t\, \hat{\sigma}_x = \hbar \begin{pmatrix} -\omega_0/2 & \Omega \cos\omega t \\ \Omega \cos\omega t & \omega_0/2 \end{pmatrix}_{|0\rangle,|1\rangle} \ , \tag{12}$$

with $\mp\hbar\omega_0/2$ the energies of the states $|0\rangle$ and $|1\rangle$, respectively. Starting from an initial atomic state $|\psi(0)\rangle$, the Schrödinger equation $i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)\,|\psi(t)\rangle$ allows us to calculate the state $|\psi(t)\rangle$ at any time. However, as $H(t)$ depends explicitly on time, the exact calculation is often difficult. It can be considerably simplified by using the following standard procedure:

1. Apply the transformation $|\tilde{\psi}(t)\rangle = \hat{\mathcal{R}}(t)\,|\psi(t)\rangle$ with

$$\mathcal{R}(t) = \exp\left(-i\frac{\omega t}{2}\hat{\sigma}_z\right) = \begin{pmatrix} e^{-i\frac{\omega t}{2}} & 0 \\ 0 & e^{i\frac{\omega t}{2}} \end{pmatrix}_{|0\rangle,|1\rangle} \ . \tag{13}$$

   This transformation amounts to applying a rotation around the Oz axis of angle $\omega t/2$, and hence "move to the frame rotating at frequency ω".

2. The Hamiltonian governing the evolution of the new state $|\tilde{\psi}(t)\rangle$ is found by writing the Schrödinger equation and applying the transformation. It yields the new Schrödinger equation

$$i\hbar\frac{d}{dt}|\tilde{\psi}(t)\rangle = \tilde{H}(t)\,|\tilde{\psi}(t)\rangle \quad \text{with} \quad \tilde{H} = \mathcal{R}H\mathcal{R}^{-1} + i\hbar\frac{d\mathcal{R}}{dt}\mathcal{R}^{-1}\ . \tag{14}$$

Applying the transformation and using $i\hbar\frac{d\mathcal{R}}{dt} = (\hbar\omega/2)\hat{\sigma}_z\mathcal{R}$, we find

$$\tilde{H} = \frac{\hbar}{2}\begin{pmatrix} \delta & \Omega\,(1+e^{-2i\omega t}) \\ \Omega\,(1+e^{+2i\omega t}) & -\delta \end{pmatrix}_{|0\rangle,|1\rangle}\ , \tag{15}$$

with $\delta = \omega - \omega_0$ the laser detuning wit respect to the atomic transition. This new Hamiltonian is still time-dependent. However, the terms $e^{\pm 2i\omega t}$ oscillate rapidly with respect to all the other frequency scales δ or Ω in the problem, and can therefore be neglected.

Apart from a numerical simulation to check this, it is not so easy to justify precisely this fact. However, the following hand-waving argument helps. Write an arbitrary qubit state as $|\tilde{\psi}\rangle = c_0(t)\,|0\rangle + c_1(t)\,|1\rangle$. The Schrödinger equation (14) then yields the coupled differential equations

$$\dot{c}_0 = -i\frac{\delta}{2}c_0 - i\frac{\Omega}{2}\left(1+e^{-2i\omega t}\right)c_1 \tag{16}$$

$$\dot{c}_1 = i\frac{\delta}{2}c_1 - i\frac{\Omega}{2}\left(1+e^{+2i\omega t}\right)c_0\ . \tag{17}$$

If $\Omega \ll \delta$, the second equation yields $c_1(t) \simeq e^{i\delta t/2}c_1(0)$ and the first equation becomes

$$\dot{c}_0 \simeq -i\frac{\delta}{2}c_0 - i\frac{\Omega}{2}(1+e^{-2i\omega t})e^{i\frac{\delta t}{2}}c_1(0)\ , \tag{18}$$

which can be solved by using the variation of constant method. Write $c_0(t) = A(t)\,e^{-i\delta t/2}$. Then we get $\dot{A} = -i\frac{\Omega}{2}(1+e^{-2i\omega t})e^{i\delta t}c_1(0)$ and

$$A(t) \simeq A(0) - i\frac{\Omega}{2}\left[\frac{e^{i\delta t}-1}{i(\omega-\omega_0)} - \frac{e^{-i(\omega+\omega_0)t}-1}{i(\omega+\omega_0)}\right]c_1(0)\ . \tag{19}$$

In the quasi-resonant case, $|\omega-\omega_0| \ll \omega+\omega_0$, the second term, which comes from the rapidly oscillating factor $e^{-2i\omega t}$, is negligible.

At the end of the procedure, and under the quasi-resonant approximation, we are then left with the effective time-independent Hamiltonian

$$\tilde{H} = \frac{\hbar}{2}\begin{pmatrix} \delta & \Omega \\ \Omega & -\delta \end{pmatrix}_{|0\rangle,|1\rangle} = \frac{\hbar\delta}{2}\,\hat{\sigma}_z + \frac{\hbar\Omega}{2}\,\hat{\sigma}_x\ . \tag{20}$$

The latter has been solved in the first lecture (see also homework 1). We know that if $|\psi(0)\rangle = |0\rangle$, then the probability to find the system in $|1\rangle$ at a later time t is given by the Rabi formula

$$p_1(t) = \frac{\Omega^2}{\Omega^2 + \delta^2} \sin^2\left(\sqrt{\Omega^2 + \delta^2}\frac{t}{2}\right) . \tag{21}$$

The transfer to state $|1\rangle$ is maximal at resonance, i.e. $\delta = 0$, in which case the solution of the Schrödinger equation is $|\psi(t)\rangle = \cos\frac{\Omega t}{2}|0\rangle - i\sin\frac{\Omega t}{2}|1\rangle$. It is negligible when $|\delta| \gg \Omega$. This last condition gives the criterion to isolate a two-level system from a multi-level atomic structure: The driving frequency ω should be close to the transition frequency ω_0 between $|0\rangle$ and $|1\rangle$ ($\delta \approx 0$), whereas the frequency detuning with respect to all other neighboring states should be much larger than the corresponding Rabi frequency Ω to avoid their excitation.

Finally, two remarks are in order. First, we have implicitly assumed in our treatment above that the states $|0\rangle$ and $|1\rangle$ have an infinite lifetime, which is rarely the case in practice. In real atoms, apart from the ground state, all states have a finite lifetime $\tau = 1/\Gamma$ with Γ the decay rate. For hyperfine transitions, τ can be as long as billions of years, but for optical transitions, τ lies in the sub-microsecond range and $\Gamma/(2\pi)$ is above $0.1 - 10\,\mathrm{MHz}$. Hence, to be able to ignore this lifetime, one needs $\Omega \gg \Gamma$.

Second, as explained in the first section, the measurement usually occurs in the $\{|0\rangle, |1\rangle\}$ z-basis, e.g. using fluorescence measurement. But what if we want to measure in a different basis, say the x-basis? If we were measuring spin components in a Stern and Gerlach apparatus, we would simply rotate the magnet setting the direction of the magnetic field gradient along the axis for which we want to find the spin components. For atomic systems, we cannot rotate the measurement basis. In turn, we may rotate the atomic state right before the measurement. For instance, if you want to measure along the x-axis, you apply a $\pi/2$ rotation around y (matrix $\mathcal{R}_y(\pi/2)$ with the notations of the first lecture) just before reading out the state of the qubit, now in the z-basis.

3 Trapping ions and atoms

In this section, we briefly review some experimental techniques to trap and isolate individual ions or atoms.

Trapped ions. Ions (or electrons), carrying a charge, are *a priori* the easiest to trap using electric fields. However, the Maxwell-Gauss equation in free space, $\nabla \cdot \mathbf{E} = 0$, implies that if the field points towards a given point in along x and y, it has to point away from it in the z direction. The electrostatic potential thus realizes a saddle point and can therefore not trap the charge in all three dimensions. Two solutions to overcome this issue have been devised: (i) Applying a magnetic field to prevent the ions from escaping

the saddle point region (Penning traps); (ii) Rotating the saddle point around its axis by applying an AC voltage on auxiliary electrodes (Paul trap). In the second case, if the rotation frequency is high enough and well chosen, the ion sees an average potential with a minimum. In both cases the effective potential close to the minimum is well approximated by a harmonic trap, with oscillation frequencies in the MHz range.

When loading several ions in such traps, the Coulomb repulsion between the ions tend to separate them and counteracts the trapping potential. It leads to a so-called *artificial ion crystal*, where the ions are separated by a few micrometers. An asset of such a configuration is that the ions can therefore be detected and manipulated individually by focussed laser beams. The typical trap depth is 1 eV, corresponding to 12000 K. However to allow for the formation of the crystal, the ions have to be laser-cooled to very low temperature. In state-of-the-art experiments, up to 50-100 ions are cooled down to the lowest vibrational state of the ion crystal.

Trapped atoms. Atoms being neutral, the action of an electric field on them is much weaker than for charged particles, and occurs at leading order through the dipolar interaction. Using a laser is a convenient way to produce large electric fields. The oscillating field $\mathbf{E}$ of the laser induces a proportional atomic dipole $\mathbf{d} = \epsilon_0 \alpha \mathbf{E}$ and the interaction of the dipole with field gives an interaction potential energy averaged over many oscillation periods of the field $U(\boldsymbol{R}) = \langle -\mathbf{d} \cdot \mathbf{E}/2 \rangle \propto -\alpha \langle E(\boldsymbol{R})^2 \rangle$. Hence, depending on the sign of the polarizability α, the energy minimum is located at either the minimum or maximum of intensity of the light. For instance, when the frequency of the light ω is tuned below the atomic resonance (so-called *red detuning*), the dipole and the driving field are in phase, hence $\alpha > 0$. In this case, a tightly focused laser beam creates a trapping potential around its focalization point where atoms can be trapped. This configuration is called *optical tweezers*. One can also interfere several laser beams, producing regularly spaced intensity maxima in 1D, 2D or 3D, where atoms can be localized. This setting forms so-called *optical lattices*. Over the last 10 years, several methods have been devised to trap individual atoms at the intensity maxima and to observe and manipulate them optically. The typical trap depth ranges between 100 μK and 10 mK, making it necessary to pre-cool the atoms with lasers before loading them in these optical traps.

4 Entangling atoms and photons

In Sec. 2, we discussed the interaction between atoms and *classical* electromagnetic fields. We showed that it allows us to prepare any superposition state of a qubit. However, we did not worry about the state of the electromagnetic field itself, which was assumed to have an amplitude independent of the qubit state. Here, we describe the interaction between an atom and a quantum field. We will see that the coupling leads to an entanglement between atom and light, which can ultimately be turned into an entanglement between

two qubits.

The use of a cavity is a fruitful way of enhancing the quantum character of the electromagnetic field. Its aim is essentially to select quantized modes, sufficiently separated from each other for the atom to resolve its discrete character. We consider a two-level atom with states $|0\rangle$ and $|1\rangle$ placed at rest inside a linear optical cavity delimited by two mirrors separated by a distance L. The system consists of two parts coupled to each other: the atom and the cavity field, which we will assume to have a frequency ω close to resonance with the qubit frequency ω_0.

The atom. Restricting the atom to its two qubit states, the Hamiltonian of the two-level atom may be written as

$$\hat{H} = -(\hbar\omega_0/2)\,\hat{\sigma}_z \ , \tag{22}$$

where zero energy has been set to the middle of the transition line.

The cavity field. The electromagnetic field inside the cavity is a standing wave along the axis Oz perpendicular to the mirrors. Using absorbing boundary conditions, the electric field cancels at the mirrors and its expression is

$$E_x = E_0 \sin kz \sin \omega t, \quad \text{with} \quad k \in \frac{\pi}{L}\mathbb{N}^* \ , \tag{23}$$

assuming it is polarized along the axis x. The Maxwell-Faraday equation, $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, yields a magnetic field along the y axis with amplitude

$$B_y = -\frac{E_0}{c} \cos kz \cos \omega t \ . \tag{24}$$

As the electromagnetic field oscillates sinusoidally with time, it is equivalent to an harmonic oscillator. To make the analogy more evident, let us write $E_x = E_0 \sin(kz)\, q(t)$ and $B_y = -\frac{E_0}{c} \cos(kz)\, p(t)$ with $q(t) = \sin(\omega t)$ and $p(t) = \cos(\omega t)$. These are the equations of motion of a harmonic oscillator with unit mass, and in particular we have $\dot{q}(t) = \omega p(t)$ and $\dot{p}(t) = -\omega q(t)$. These are the equations of motion of an harmonic oscillator with frequency ω and mass $m = 1/\omega$. Thus E_x and B_y appear as the position and the momentum of the oscillator (in some units). Moreover, the energy contained in the cavity field with volume $V = LA$, with A the transverse surface of the mode, averaged over an oscillation period, is

$$H_{\text{EM}} = \int_V dV \left(\epsilon_0 \frac{E_x^2}{2} + \frac{B_y^2}{2\mu_0} \right) = \frac{\epsilon_0 E_0^2 V}{2} \ \frac{p^2 + q^2}{2} \ . \tag{25}$$

Hence, a mode of the electromagnetic field is equivalent to an harmonic oscillator described by $H = \frac{\hbar\omega}{2}(p^2 + q^2)$, with $\epsilon_0 E_0^2 V/2 = \hbar\omega$, i.e. $E_0 = \sqrt{\frac{2\hbar\omega}{\epsilon_0 V}}$. Note that there is a factor $1/2$ compared to the formula derived in quantum optics. This is due to the cavity, which changes the boundary conditions.

Quantization of the field and photons. To quantize the cavity field we make q and p operators, $q \to \hat{q}$ and $p \to \hat{p}$, and impose the canonical commutation relations between momentum and position, i.e. $[\hat{q}, \hat{p}] = i$ (the absence of $\hbar$ comes from the units for p and q). The quantization is performed following the method proposed by Dirac: One introduces the annihilation and creation operators

$$\hat{a} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}} \quad \text{and} \quad \hat{a}^\dagger = \frac{\hat{q} - i\hat{p}}{\sqrt{2}} \ . \tag{26}$$

The commutation relation between $\hat{q}$ and $\hat{p}$ implies $[\hat{a}, \hat{a}^\dagger] = 1$. Then, the Hamiltonian reads as

$$\hat{H}_{\rm EM} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) \ . \tag{27}$$

One shows that the eigenvalues of the operator $\hat{a}^\dagger\hat{a}$ are all the positive integer numbers n starting from 0, the corresponding states being $|n\rangle$. The operator therefore counts the number of excitations with energy $\hbar\omega$ in the field. These excitations are called *photons*. The operators $\hat{a}$ and $\hat{a}^\dagger$ respectively remove or add one photon from the cavity field. Their action on the energy eigenstates $|n\rangle$ are

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \quad \text{and} \quad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \ . \tag{28}$$

The state $|n = 0\rangle$ corresponds to the vacuum (i.e. empty of any photons), and $\hat{a}\,|0\rangle = 0$.

The electric and magnetic fields are now operators, and their expressions in terms of annihilation and creation operators are:

$$\hat{E}_x = \frac{E_0}{\sqrt{2}} \sin kz\,(\hat{a} + \hat{a}^+) \ \text{ and } \ \hat{B}_y = -i\frac{E_0}{\sqrt{2}c} \cos kz\,(\hat{a} - \hat{a}^+) \ \text{ with } \ E_0 = \sqrt{\frac{2\hbar\omega}{\epsilon_0 V}} \ . \tag{29}$$

Atom-field coupling. For an electric dipole transition as described in Sec. 2, the Hamiltonian is again $\hat{H}_{\rm int} = -\hat{d}_x\hat{E}_x$ with the notable difference that the field is now an operator. This Hamiltonian acts in the tensor product space $\mathcal{E}_{\rm at} \otimes \mathcal{E}_{\rm EM}$ spanned by the basis $\{|0\rangle, |1\rangle\} \otimes \{|n\rangle, n \in \mathbb{N}\}$. Introducing $\hbar\Omega = -\sqrt{2}\,\langle 0|\,\hat{d}_x\,|1\rangle\,E_0$, and recalling that $\hat{d}_x = \langle 0|\,\hat{d}_x\,|1\rangle\,(\hat{\sigma}^+ + \hat{\sigma}^-)$ with $\hat{\sigma}^+ = |1\rangle\langle 0|\,, \hat{\sigma}^- = |0\rangle\langle 1|$, we may write

$$\hat{H}_{\rm int} = \frac{\hbar\Omega}{2}(\hat{\sigma}^+ + \hat{\sigma}^-)(\hat{a} + \hat{a}^\dagger) \ , \tag{30}$$

having assumed that the atom seats at a position z such that $\sin kz = 1$.

Jaynes-Cummings Hamiltonian. The total Hamiltonian of the atom+field system, is thus

$$\hat{H} = -\frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\omega(\hat{a}^+\hat{a} + 1/2) + \frac{\hbar\Omega}{2}(\hat{\sigma}^+ + \hat{\sigma}^-)(\hat{a} + \hat{a}^+) \ . \tag{31}$$

The coupling terms are $\hat{a}\sigma^+$, $\hat{a}^\dagger\sigma^-$, $\hat{a}^\dagger\sigma^+$, and $\hat{a}\sigma^-$. The first one describes the absorption of a photon from the cavity field by the atom initially in its ground state $|0\rangle$ and then promoted to its excited state $|1\rangle$. The second one corresponds to the reverse process consisting of the emission of a photon into the cavity field by an atom initially in $|1\rangle$ and then decaying to its ground state $|0\rangle$. The last two terms correspond to processes that are energetically very unlikely, for instance the excitation of an atom from ground to excited state with the emission of a photon. As a consequence these terms may be neglected (and one can indeed show that their effect is negligible with respect to the two other, more standard, processes). Under this approximation, one gets the Hamiltonian introduced initially by Edwin Jaynes and Fred Cummings in the 1960's:

$$\hat{H}_{\rm JC} = -\frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\omega(\hat{a}^+\hat{a} + 1/2) + \frac{\hbar\Omega}{2}(\hat{\sigma}^+\hat{a} + +\hat{\sigma}^-\hat{a}^\dagger) \ . \tag{32}$$

Let us diagonalize this Hamiltonian, keeping in mind that the cavity is tuned near resonance ($\omega \approx \omega_0$). In the absence of coupling ($\Omega = 0$), the Hamiltonian, $\hat{H}_0 = \hat{H}_{\rm JC}(\Omega = 0)$, is diagonal in the decoupled basis, with the energies

$$|1,n\rangle \quad \leftrightarrow \quad E_{1,n} = \langle 1,n|\,\hat{H}_0\,|1,n\rangle = -\frac{\hbar\delta}{2} + \hbar\omega(n+1) \tag{33}$$

$$|0,n+1\rangle \quad \leftrightarrow \quad E_{0,n} = \langle 0,n+1|\,\hat{H}_0\,|0,n+1\rangle = +\frac{\hbar\delta}{2} + \hbar\omega(n+1) \tag{34}$$

$$|0,0\rangle \quad \leftrightarrow \quad E_{0,0} = \langle 0,0|\,\hat{H}_0\,|0,0\rangle = +\frac{\hbar\delta}{2} \ , \tag{35}$$

with $\delta = \omega - \omega_0$. The two states $|1,n\rangle$ and $|0,n+1\rangle$ have thus nearly equal energies, and the spectrum splits in pairs of states, see Fig. 3. On the one hand, the ground state $|0,0\rangle$ is isolated (not coupled to any other state). On the other hand, the interaction term only couples the states $|1,n\rangle$ and $|0,n+1\rangle$ with an amplitude $(\hbar\Omega/2)\sqrt{n+1}$. The restriction of the Hamiltonian to the manifold corresponding to a given value of n is thus

$$H^{(n)} = \hbar\omega(n+1)\,\hat{\rm Id} + \frac{\hbar}{2}\begin{pmatrix} \delta & \Omega\sqrt{n+1} \\ \Omega\sqrt{n+1} & -\delta \end{pmatrix}_{|0,n+1\rangle,|1,n\rangle} \ . \tag{36}$$

The diagonalization of these 2×2 matrices yield the eigenenergies of the atom+field system considered as a whole,

$$E_\pm^{(n)} = \hbar\omega(n+1) \pm \frac{\hbar}{2}\sqrt{\delta^2 + \Omega^2(n+1)} \ . \tag{37}$$

In particular, at resonance ($\delta = 0$), the corresponding eigenstates are $|n,\pm\rangle = (|0,n+1\rangle \pm |1,n\rangle)\sqrt{2}$.

This offers a way to entangle the atom and cavity states. For instance, if one prepared an atom in the excited $|1\rangle$ in a initially empty cavity, the state of the atom+field system is $|\psi_{10}(0)\rangle = |1,n=0\rangle$. Due to the coupling, it evolves into

$$|\psi_{10}(t)\rangle = \cos\frac{\Omega t}{2}\,|1,n=0\rangle - i\sin\frac{\Omega t}{2}\,|0,n=1\rangle \ \ , \tag{38}$$

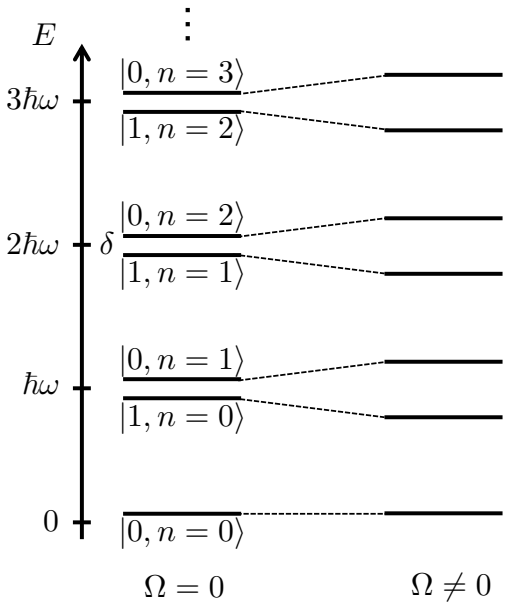

Figure 3: Spectrum of the Jaynes-Cummings Hamiltonian for $\delta > 0$.

up to an irrelevant dynamical phase. This is often referred to as the vacuum Rabi oscillation or single-photon Rabi oscillations: The photon emitted by the atom bounces back on the cavity mirrors and is reabsorbed by the atom periodically. The decay of the atom is thus reversible! Moreover, the atom+field state is almost always entangled (except for $\Omega t \equiv 0\ [\pi]$)! Maximum entanglement is found for $\Omega t \equiv \pi/2\ [\pi]$. Similarly, if the atom is instead prepared in its ground state $|0\rangle$ and the cavity with one photon $|n = 1\rangle$, the state $|\psi_{01}(0)\rangle = |0, n = 1\rangle$ evolves into

$$|\psi_{01}(t)\rangle = \cos\frac{\Omega t}{2}\,|0, n = 1\rangle - i\sin\frac{\Omega t}{2}\,|1, n = 0\rangle \ , \tag{39}$$

again up to an irrelevant dynamical phase.

This entanglement between atom and field can also be used to entangle two atoms, without a direct interaction between the two atoms. Assume that you send a first atom a, initially in its excited state $|1\rangle$, through an empty cavity such that its interaction with the cavity field is $\Omega t = \pi/2$. After it has left the cavity, we have $|\psi_{a,\text{field}}\rangle = (|1_a, n = 0\rangle - i\,|0_a, n = 1\rangle)/\sqrt{2}$. If you then send as second atom b, now initially in its ground state $|0\rangle$, through the cavity containing now one atom, tuned such that $\Omega t = \pi$, the states of the cavity+atom a+atom b becomes

$$|\psi_{a,b,\text{field}}\rangle = \frac{|1_a, 0_b\rangle - \mathrm{e}^{i\theta}\,|0_a, 1_b\rangle}{\sqrt{2}} \otimes |n = 0\rangle \ , \tag{40}$$

where the phase θ is due to the evolution of atom a after it has left the cavity. The two atoms are thus maximally entangled, and separated from the cavity state.

5 Photons in free space

Photons are also a popular choice to encode quantum bits, as they couple very weakly to the environment thus limiting decoherence. In addition, they propagate either in free space or, even better, in optical fibers over large distances. They can thus mediate quantum information between separated locations. There exist different ways to encode qubits on individual photons (assuming you have a single-photon source). We describe two of them below.

Polarization qubits. A photon propagating in a direction $\mathbf{k}$ has two orthogonal polarizations perpendicular to $\mathbf{k}$. For example horizontal and vertical linear polarizations $|H\rangle \equiv |1\rangle$ and $|V\rangle \equiv 0$, or right and left circular polarizations. Each choice can be used as a qubit basis. In this first approach, creating superposition states is done by using waveplates. For example, a $\lambda/2$-plate with axis at 22.5° of the state $|0\rangle = |V\rangle$ prepares $(|0\rangle + |1\rangle)/\sqrt{2}$. A $\lambda/4$-plate with axis at 45° with respect to $|0\rangle = |V\rangle$ leads to a circular photon $(|0\rangle + i\,|1\rangle)/\sqrt{2}$. The equivalent of a Stern and Gerlach device is a polarizer making an axis θ with respect to say $|0\rangle$ followed by a single photon counter. The probability to detect a click is given by the Malus law, $|\langle\theta|0\rangle|^2 = \cos^2\theta$.

Mode/path qubits. Another possible encoding consists in placing photons in the two input modes a and b of a beam-splitter: A photon in mode a realizes the qubit state $|0\rangle = |1_a, 0_b\rangle$ while a photon in mode b realizes $|1\rangle \equiv |0_a, 1_b\rangle$. The beamsplitter creates in the output modes c and d the superposition $t\,|0\rangle + r\,|1\rangle$ or $-r\,|0\rangle + t\,|1\rangle$, with t and r the transmission and reflexion coefficients of the beam splitter, with $|t|^2 + |r|^2 = 1$.

Single photon sources. The simplest example is made of one atom prepared in its excited state, placed at the focal point of a collection optics defining a solid angle $\Delta\Omega$. The atom decays in a typical time $1/\Gamma$ and the photon is collected by the lens with a probability $\Delta\Omega/(4\pi)$. It is hard to build optics with collection efficiency larger than $\sim 10\%$, thus limiting the emission rate in this method. Modern approaches therefore rely on solid-state emitters placed in optical cavities in the strong coupling regime: The photons are then emitted in the cavity mode with nearly 100% efficiency at a rate which can reach $10^6 - 10^7\,\mathrm{s}^{-1}$.

One drawback of the purely photonic approach though is that photons interact with each other extremely weakly in free space, making the generation of entanglement between two of them challenging. Approaches combining beamsplitters and measurement have been developed, as well as media presenting very strong non-linearity. These are still a very active field of research with many developments.

6 Quantum circuits: From harmonic oscillators to qubits

Quantum circuits coupled by microwave photons propagating in waveguides are at the heart of the approach followed by companies like Google or IBM to develop quantum computers. Before explaining their choice of encode qubits, let us discuss how to quantize an electrical circuit.

Consider the LC circuit made of a capacitor C and an inductance L. If Q is the charge of the capacitor and ϕ the flux through the inductance, the current is $I = -dQ/dt$, the voltage drop is $V = -Q/C$, and we have $\phi = LI$. Moreover, the equation of motion for the charge results from the induction law $V = -LdI/dt$. It yields the equation of motion

$$\dot{\phi} = Q/C \quad \text{and} \quad \dot{Q} = -\phi/L \ . \tag{41}$$

These equations are again those of a harmonic oscillator. The frequency is found by writing $\ddot{Q} = -\omega_0^2 Q$ with $\omega_0 = 1/\sqrt{LC}$. The electromagnetic energy in the circuit is shared between the inductance and the capacitor, and reads as

$$H_{\rm EM} = \frac{Q^2}{2C} + \frac{1}{2}LI^2 = \frac{Q^2}{2C} + \frac{\phi^2}{2L} = \frac{\hbar\omega_0}{2}\left(\frac{Q^2}{\hbar\omega_0 C} + \frac{\phi^2}{\hbar\omega_0 L}\right) \ . \tag{42}$$

This is indeed a harmonic oscillator with generalized position $x = \phi/\sqrt{\hbar\omega_0 L}$ and momentum $p = Q/\sqrt{\hbar\omega_0 C}$. The quantization is realized by making x and p operators with the canonical commutation relation $[\hat{x}, \hat{p}] = i$. This is equivalent to making ϕ and Q operators with a similar commutation relation, $[\hat{\phi}, \hat{Q}] = i\hbar$, owing to the relation $\omega_0 = 1/\sqrt{LC}$. We then introduce the annihilation and creation operators

$$\hat{a} = \frac{\hat{\phi}}{\sqrt{2\hbar\omega_0 L}} + i\frac{\hat{Q}}{\sqrt{2\hbar\omega_0 C}} \quad \text{and} \quad \hat{a}^\dagger = \frac{\hat{\phi}}{\sqrt{2\hbar\omega_0 L}} - i\frac{\hat{Q}}{\sqrt{2\hbar\omega_0 C}} \tag{43}$$

We then find the canonical commutation relation rule $[\hat{a}, \hat{a}^\dagger] = 1$, and the Hamiltonian is

$$H_{\rm EM} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + 1/2) \ . \tag{44}$$

The operator $\hat{a}^\dagger\hat{a}$ counts the number of photons of energy $\hbar\omega_0$ in the circuit, while the charge and flux are now operators,

$$\hat{\phi} = \sqrt{\frac{\hbar\omega_0 L}{2}}(\hat{a} + \hat{a}^\dagger) \quad \text{and} \quad \hat{Q} \propto -i\sqrt{\frac{\hbar\omega_0 C}{2}}(\hat{a} - \hat{a}^\dagger) \ . \tag{45}$$

As regards orders of magnitude of what microfabrication achieves in practice, we have $C \sim 10\,\text{pF}$ and $L \sim 100\,\text{pH}$. It yields $\omega_0/(2\pi) \sim 5\,\text{GHz}$ and $\lambda \sim 5\,\text{cm}$. For the circuit to behave quantum mechanically, one has to operate at a temperature T such that

$k_{\rm B}T \ll \hbar\omega_0$ to avoid thermal excitations. This means in practice $T \leq 200\,{\rm mK}$, and the circuits are placed in a dilution fridge operating below this temperature.

Since the energy spacing between the states of a harmonic oscillators are all equal, any driving would couple all the states to its neighbours, which does not allow us to select two isolated states and realize a useful qubit. The idea to create a suitable qubit is to introduce a nonlinearity so as to modify the spacing between states $n = 0, 1, 2, ...$ It turns out that a Josephson junction made of two superconductors separated by a thin isolating layer behaves as a nonlinear inductance. Moreover, using superconductor prevents dissipation. In the superconductor, two electrons attract each other via their interaction with the phonons of the metallic crystal and form Cooper pairs of charge $Q = -2e$. These Cooper pairs tunnel through the thin indulating barrier (for a good description, see Feynman lectures in Physics vol. 3, Chapter 21-9). Two Josephson junctions placed in parallel form a ring (the device is called a SQUID, for Superconducting Quantum Interference Device): A magnetic field perpendicular to the ring creates a magnetic flux $\phi = BS$. The Hamiltonian describing this device junction can be shown to be $\hat{H}_{\rm JJ} = -E_{\rm J}\cos(2\pi\hat{\phi}/\phi_0)$, with $\phi_0 = h/(2e)$ the fluc quantum. Placing a squid in series with a capacitor carrying a charge Q leads to the Hamiltonian

$$\hat{H} = \frac{\hat{Q}^2}{2C} - E_{\rm J}\cos\left(2\pi\frac{\hat{\phi}}{\phi_0}\right) \ . \tag{46}$$

The charge again appears as the conjugate momentum of the flux. This equation is analogous to the one of a mechanical pendulum with position ϕ. The potential $\cos(2\pi\phi/\phi_0)$ has energy levels which are no longer equidistant. In practice, for $\omega_{01}/2\pi \approx 5\,{\rm GHz}$, $(\omega_{01} - \omega_{12})/2\pi \approx 200\,{\rm MHz}$. The two lowest energy states therefore realizes a good two-level system.

Several such qubits can be coupled to each other by connecting them to a waveguide, which is nothing but a harmonic oscillator, analogous to the cavity of Sec. 4. The Hamiltonian describing a qubit and a waveguide is thus also the Jaynes-Cummings one of Eq. (32), and the phenomenology is one to one identical.

A Problem set for Lecture 2: Quantum gates

In the perspective of building a quantum computer, physicists have developed techniques to manipulate the state of quantum bits. These are called quantum gates and a quantum computer will be a machine running sequences of such gates. We study here a few examples of important gates that we will also use to illustrate some concepts in the coming lectures.

A.1 Single-qubit gates

These operations act on a single qubit. They rely on the driving of the qubit (transition frequency ω_0) by a classical field at a frequency ω.

1. An X-gate consists of a $\pi/2$-rotation of the qubit state around the Ox axis of the Bloch sphere. Write the matrix $\hat{U}_X$ acting on the qubit state.

2. Take one example of physical implementation discussed in the lecture and explain (in less than 5 lines) how such a gate can be realized.

3. If the Rabi frequency corresponding to the coupling between the qubit and field is $\Omega/(2\pi) = 1$ MHz, what is the duration of the gate?

4. The Hadamard gate is defined by the matrix:

$$\hat{U}_{\mathrm{H}}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{47}$$

   Show that it corresponds to a rotation of the Bloch vector around $\mathbf{n} = (\mathbf{x} + \mathbf{z})/\sqrt{2}$, How do you have to choose Ω with respect to $\Delta = \omega - \omega_0$. What is the duration of the gate?

5. Estimate the laser intensity necessary to drive an optical dipole transition with a Rabi frequency of $\Omega/(2\pi) = 10$ MHz.

6. Estimate the magnetic field of a microwave necessary to drive the hyperfine transition of Rb atom at 6.8 GHz with a Rabi frequency $\Omega/(2\pi) = 1$ kHz.

A.2 Entangling gates

These gates operate on two qubits, the first being called target, the second control. Assume you know how to realize the two-qubit π-phase gate represented by the matrix in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis:

$$U_{\pi}^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{48}$$

1. Prepare the target and control qubits in $(|0\rangle + |1\rangle)\sqrt{2}$. Calculate the two-qubit state at the output.

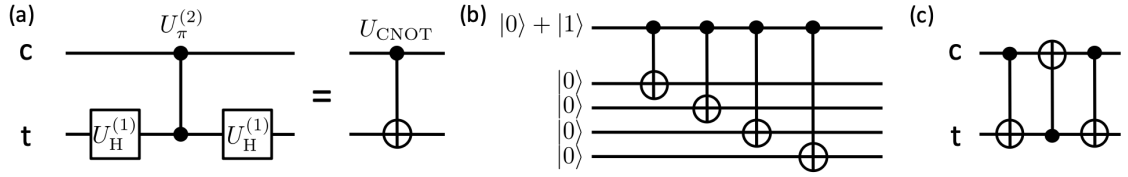

Figure 4: *(a) Quantum circuit to produce a CNOT gate from a π-Phase gate. Here $U_{\rm H}^{(1)}$ is the Hadamard gate; c, t are the control and target qubits. (b) Quantum circuit to prepare a Greenberger-Horne-Zeilinger state. (c) Swap gate consisting of three CNOT gates.*

2. Is it an entangled state and why ?

3. Consider the elementary quantum circuit shown in Fig. 4(a). Show that it is equivalent to a CNOT gate, whose matrix is:

$$U_{\rm CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{49}$$

4. Consider the circuit shown in Fig. 4(b). What is the state at the output?

5. **S**wap gate. Consider finally the circuit represented in Fig. 4(c) and show that it swaps the states of the control and target qubit.

A.3 Examples of implementation of entangling gates

1. **G**ate with neutral atoms. Consider two atoms trapped in optical tweezers and distant by $R = 5\,\mu$m. When they are in their ground state $|g\rangle = |0\rangle$, they do not interact at such a distance. When excited to a state $|r\rangle = |1\rangle$ with very large principal quantum number n (Rydberg state), their interaction is considerably enhanced and is of the form $V = C_6/R^6$.

   (a) Explain why the corresponding Hamiltonian is $\hat{H} = V\hat{n}_1 \otimes \hat{n}_2$, with $\hat{n}_i = (1 + \sigma_i^z)/2$.

   (b) Show that this Hamiltonian can realize a two-qubit π-phase gate an give its duration T.

2. **G**ate with superconducting circuits. Consider two superconducting circuits as described in the lecture, each with states $|0\rangle$ and $|1\rangle$ (frequency ω_0), coupled to a microwave cavity with resonant frequency ω. We have seen in the Lecture on Approximation Methods (example H) that the two qubits exchanging virtually a photon

via the cavity are described by the Hamiltonian:

$$\hat{H} = J(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) \quad \text{with} \quad J = \frac{\hbar\Omega^2}{\Delta} \ . \tag{50}$$

Write the matrix of the gate (called iSWAP) for $JT/\hbar = \pi$. Using single gates acting on the qubits, this iSWAP can generate a CNOT gate.