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Protected quasilocality in quantum systems with long-range interactions

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We study the out-of-equilibrium dynamics of quantum systems with long-range interactions. Two different models describing, respectively, interacting lattice bosons and spins are considered. Our study relies on a combined approach based on accurate many-body numerical calculations as well as on a quasiparticle microscopic theory. For sufficiently fast decaying long-range potentials, we find that the quantum speed limit set by the long-range Lieb-Robinson bounds is never attained and a purely ballistic behavior is found. For slowly decaying potentials, a radically different scenario is observed. In the bosonic case, a remarkable local spreading of correlations is still observed, despite the existence of infinitely fast traveling excitations in the system. This is in marked contrast to the spin case, where locality is broken. We finally provide a microscopic justification of the different regimes observed and of the origin of the protected locality in the bosonic model.

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It is common wisdom that the propagation of a signal through a classical medium presents a distinct notion of causality, characterized by the progressive time growth of the spatial region explored by the signal. In spite of the intrinsically nonlocal nature of quantum theory, this familiar notion of locality is preserved in a wide class of quantum systems with short-range interactions. A milestone example is provided by the Lieb-Robinson (LR) bounds, which set a ballistic limit to the propagation of information, with exponentially small leaks outside the locality cone [1,2]. The existence of LR bounds has many fundamental implications for thermalization, entanglement scaling laws, and information transfer in quantum systems [3]. A renewed interest in these topics is currently sparked by the impressive progress in the time-dependent control of ultracold-atom systems. Direct observation of cone spreading of correlations was reported in Refs. [4,5].

The extension of the notion of locality to quantum systems with long-range interactions constitutes a fundamental challenge. The paradigmatic model of long-range interactions considers an algebraic decay of some coupling term of the form $V(R) \sim 1/R^{\alpha}$ [6–10]. It applies either to the exchange coupling term in spin systems, as realized in cold ion crystals [11,12], or to the two-body interactions in particle systems, as realized in ultracold gases of polar molecules [13], magnetic atoms [14], and Rydberg atoms [15]. A remarkable feature of long-range systems is that instantaneous propagation of information, in violation of locality, can take place when the exponent α is smaller than some threshold. This possibility is supported by the known extensions of the LR bounds to long-range interactions [16–18]. The latter yield "quasilocal" superballistic bounds for $\alpha > d$, where d is the dimension of the system, whereas for $\alpha < d$ no known generalized bounds exist, hence suggesting the breaking of quasilocality. Evidence of the breaking of quasilocality in one-dimensional (1D) Ising spin systems has been reported theoretically [6,7] and experimentally in cold ion crystals [12,19]. However, many questions remain open. For instance, although the observations are compatible with the known long-range bounds, the propagation was found to be much slower than expected [7]. Hence, the bounds are usually not saturated and it is not clear that they provide a universal criterion for the breaking of quasilocality. Moreover the threshold value for the breaking of locality in these systems is debated, and contrasting results have been put forward [6,7]. To make progress on answering these questions, it is of crucial importance to provide a unified understanding of a wider class of systems and, at the same time, to understand the microscopic origin of the breaking of quasilocality.

In this Rapid Communication, we study the out-of-equilibrium dynamics, induced by an interaction quench, of homogeneous 1D quantum systems with long-range algebraic interactions. We consider two different models, namely, the long-range transverse Ising (LRTI) and long-range Bose-Hubbard (LRBH) models. A quantitative analysis of quasilocality in these systems is realized upon studying the earliest times at which information arrives at some fixed distance. On the one hand, we perform *ab initio* quantum many-body calculations based on the time-dependent variational Monte Carlo (t-VMC) approach [20]. On the other hand, we provide a unified analytical framework based on quasiparticle (QP) analysis. Both approaches consistently show that the two systems behave dramatically different. For $\alpha > 2$ the LRTI model shows ballistic spreading of correlations with exponentially small leaks in time, hence leading to a strong form of quasilocality. For $1 < \alpha < 2$ quasilocality is still found. However, algebraic leaks in time appear, which can be traced back to a divergent group velocity at low momenta. For $\alpha < 1$ quasilocality is instead completely broken. This effect is traced back to the divergences of both the QP energy and velocity, which induce infinitely fast oscillations and a response time scale that goes to zero with the system size. Conversely, for the LRBH model, we find ballistic spreading of correlations for any value of α, analogous to what was found in the short-range Bose-Hubbard model and in marked contrast with expectations based on the lack of LR bounds for these systems. This effect is traced back to the fact that the QP energy remains finite, which cancels nonlocal contributions for any value of α. All the observed regimes are explained by the unified QP analysis and shed new light on the microscopic origin of locality in long-range quantum systems.

Long-range transverse Ising model. We start with the long-range transverse Ising (LRTI) model, whose Hamiltonian

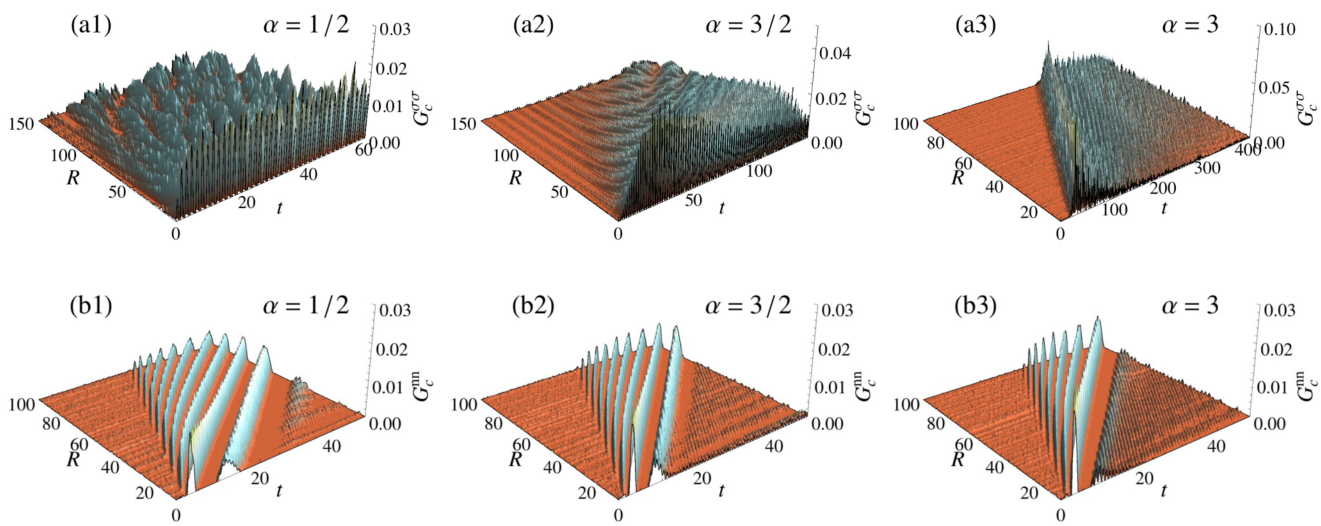

FIG. 1. (Color online) Correlation spreading in long-range spin and boson models for various values of α. (a) Connected spin-spin correlation function in the LRTI model for the quench $V_i = h/2 \rightarrow V_f = h/10$. (b) Connected density-density correlation function in the LRBH model for the quench $U_i = V_i = J \rightarrow U_f = V_f = J/4$. Results were obtained using the t-VMC approach for systems of $L = 400$ sites (for visibility, only a part is shown). The length unit is the lattice spacing and the time units are $\hbar/h$ for (a) and $\hbar/J$ for (b).

reads

$$\mathscr{H} = -h \sum_i \sigma_i^x + \frac{V}{2} \sum_{i \neq j} \frac{\sigma_i^z \sigma_j^z}{|i-j|^\alpha}, \tag{1}$$

where σ_i^x, σ_i^z are the Pauli matrices, h is the transverse field, V is the strength of the long-range spin exchange term, and in the following we set $\hbar = 1$ for convenience. Hamiltonian (1) is the prototype for long-range interacting quantum systems [6,7,21]. Moreover, it is experimentally implemented in cold ion crystals [22]. Evidence of the breaking of quasilocality in information spreading has been reported for the 1D LRTI model for sufficiently small exponents α [6,7,12,19], consistently with the absence of a long-range LR bound for $\alpha < 1$. It was pointed out, however, that a model-dependent form of quasilocality may occur for specific initial states [7] with a complete understanding of the possible scenarios being debated.

Asymptotically reliable results to reveal quasilocality require sufficiently long propagation times and sufficiently large systems. This is particularly crucial to determine precisely the nature of the dynamical regimes. To achieve this goal, we compute the unitary evolution of the correlation functions by means of the t-VMC approach [20,23] (see Supplemental Material [28]). The latter permits us to simulate the dynamics of correlated quantum systems with an accuracy comparable to tensor-network methods and proved numerically stable for unprecedented long times and large sizes.

In the t-VMC calculations, we use a Jastrow wave function with long-range spin-spin correlations at arbitrary distance [24]. To avoid misleading finite size effects, which are usually strong in these issues [25], periodic boundary conditions (PBCs) are used. For a lattice of size L with PBCs, the interaction potential is taken as the sum of the contributions resulting from all the periodic images of the finite system. The Fourier components of the effective interaction potential are then $P(k) = 2\sum_{n=1}^{\infty} \frac{\cos(kn)}{n^\alpha} = 2\mathrm{Cl}_\alpha(k)$, where we have used the Poisson summation formula over the periodic images, k is an integer multiple of $2\pi/L$, and $\mathrm{Cl}_\alpha(k)$ is the Clausen cosine function. To have a well-behaved potential in the thermodynamic limit, we set $P(k=0) = 0$. It is the equivalent of the standard regularization procedure ensuring charge neutrality in the presence of electrostatic interactions [26].

We consider global quenches of the strength of the long-range interaction, $V_i \rightarrow V_f$. The results for the time-connected average $G_c^{\sigma\sigma}(R,t) = G^{\sigma\sigma}(R,t) - G^{\sigma\sigma}(R,0)$ of the spin-spin correlation function $G^{\sigma\sigma}(R) = \langle \sigma_i^z \sigma_{i+R}^z \rangle$ are shown in Fig. 1(a). We find three qualitatively different regimes. For $\alpha < 1$, Fig. 1(a1), the propagation of correlations takes place on extremely short time scales and no conelike structure emerges. This is the signature of an efficient microscopic mechanism leading to the breaking of locality in the system. For $\alpha > 2$, Fig. 1(a3), a correlation cone with a well-determined velocity v clearly emerges. It is marked by a strong suppression of leaks in the region defined by $R/t > v$ and space-time oscillations in the region $R/t < v$. In the intermediate regime where $1 < \alpha < 2$, Fig. 1(a2), a correlation conelike structure is still visible but prominent nonlocal leaks are also appearing.

To quantify more precisely the time dependence of the leaks in the quasilocal regimes we study the time-integrated absolute value of the correlation function $\bar{G}_c^{\sigma\sigma}(R,t) = \frac{1}{t}\int_0^t dt' |G_c^{\sigma\sigma}(R,t')|$. While it retains all the features of the signal propagation, it is less sensitive to time oscillations. In Fig. 2(a1) we show the behavior of $\bar{G}_c^{\sigma\sigma}(R,t)$ for $\alpha = 3$. It clearly shows the sharp boundary of a ballistic cone. This is further assessed introducing a small cutoff ε and computing the first propagation time $t^\star(R)$ such that $\bar{G}_c^{\sigma\sigma}(R,t^\star) > \varepsilon$. The result is almost independent of ε and we find the scaling $vt^\star = R^\beta$ with finite v and $\beta \simeq 1$ to very good precision and up to large system sizes and long propagation times. The presence of a ballistic spreading, with exponentially suppressed leaks in time, is a stronger realization of locality than what expected from the looser long-range LR bound, which instead allows

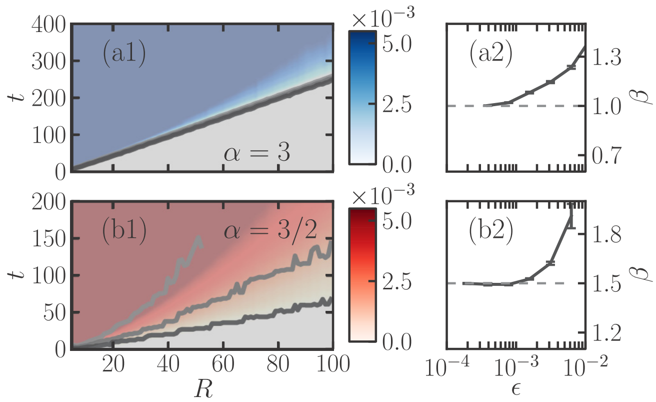

FIG. 2. (Color online) Panels (a1,b1): Time-integrated spin-spin correlation functions $\bar{G}_c^{\sigma\sigma}(R,t)$ for the same data as in Fig. 1(a). Superimposed lines show the activation time $t^\star(\varepsilon)$ (see text), for ε ranging from 2×10^{-2} (lighter lines) down to 5×10^{-3} (darker lines). Panels (a2,b2): Behavior of the fitted exponent β, computed within linear spin-wave theory, as a function of the cutoff parameter ε and for the system size $L = 2^{14}$. The length unit is the lattice spacing and the time unit is $\hbar/h$.

for polynomially suppressed leaks in time.[1] For $1 < \alpha < 2$ the same analysis of the leaks, shown in Fig. 2(b1), reveals instead that polynomial leaks in time appear with an exponent $\beta \simeq \alpha$, and a velocity v that vanishes with ε. This is compatible with the long-range LR bound [16]. Remarkably, the regimes we find here for a global quench are the same qualitative regimes that have been identified for a local quench in the LRTI model in Ref. [6].

Long-range Bose-Hubbard model. We now turn to the long-range Bose-Hubbard (LRBH) model, which describes interacting spinless bosons in a periodic potential with nearest-neighbor tunneling and long-range two-body interactions. The Hamiltonian reads

$$\mathscr{H} = -J \sum_{\langle i,j \rangle} (b_i^\dagger b_j + \text{H.c.}) + \frac{U}{2} \sum_i n_i(n_i - 1) + \frac{V}{2} \sum_{i \neq j} \frac{n_i n_j}{|i-j|^\alpha}, \tag{2}$$

where $b_i(b_i^\dagger)$ destroys (creates) a boson on site i, $n_i = b_i^\dagger b_i$ is the particle number operator, J is the tunneling amplitude, U is the on-site interaction energy, V is the strength of the interaction potential, and we set $\hbar = 1$ for convenience. The short-range ($V = 0$) case is now routinely realized in ultracold-atom experiments and the long-range ($V \neq 0$) case applies to polar molecules [13], magnetic atoms [14], and Rydberg atoms [15].

We perform t-VMC calculations using a Jastrow wave function incorporating density-density correlations at arbitrary large distances [27] (see Supplemental Material [28]). For simplicity we choose $U = V$, fix the density at half filling ($n = \frac{1}{2}$), and consider the connected density-density correlation function $G_c^{nn}(R) = \langle n_i n_{i+R} \rangle - n^2$. We study global quenches in the interaction strength $V_i \to V_f$. The results for various values of the exponent α are shown in Fig. 1(b). Surprisingly, we find here that the LRBH model exhibits the same qualitative behavior for all values of α, in marked contrast with the LRTI model. Within numerical precision, we always find a purely ballistic cone spreading of correlations at some velocity v. The long-range LR bound is therefore never saturated and, for every value of the exponent α, the spreading is qualitatively identical to the short-range case. Hence, in the LRBH model, quasilocality appears to be strongly protected even for very long-range interactions. This is further confirmed by a precise analysis of the leaks, along the same lines as for the LRTI model. It always yields a scaling of the form $vt^\star = R^\beta$ with $\beta = 1$ and the signal is exponentially suppressed for times out of the locality cone, i.e., when $t < R/v$.

Quasiparticle analysis. The radically different behaviors of the LRTI and LRBH models are particularly striking because they share the same class of long-range interactions and are therefore subjected to the same universal long-range LR bounds [16]. To understand the different behaviors of the two models, at a microscopic level, we use a general QP approach. The latter has a broad range of applications, e.g., universal conformal theories [29], spin systems [6], superfluids [30], and Mott insulators [4]. A generic time-dependent, two-body, connected correlation function in a translation invariant model with well-defined QP excitations can be written as

$$G_c(R,t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \mathscr{F}(k) \left\{ \cos(kR) - \frac{1}{2}\left[\cos\left(kR - 2E_k^f t\right) + \cos\left(kR + 2E_k^f t\right)\right]\right\}, \tag{3}$$

where E_k^f is the k-momentum QP energy of the post-quench Hamiltonian and $\mathscr{F}(k)$ is the weight associated with each QP. This general form states that the collective excitations of the system are coherent superpositions of pairs carrying excitations of momentum k and traveling in opposite directions. Whereas E_k^f depends only on the post-quench Hamiltonian, in general $\mathscr{F}(k)$ instead depends both on the pre- and post-quench Hamiltonians.[2]

In the LRTI model and in the regime of large transverse field ($h \gg V$), considered in the t-VMC calculations, we can apply linear spin-wave theory [11]. The QP energy and weight read, respectively, $E_k^f = 2\sqrt{h[h + VP(k)]}$ and $\mathscr{F}^{\sigma\sigma}(k) = \frac{2P(k)(V_i - V_f)}{E_k^i[1+P(k)V_f/h]}$, where $P(k)$ are the Fourier components of the interaction potential [28]. Let us then analyze the outcome of Eq. (3).

The ballistic behavior observed for $\alpha > 2$ can be understood from stationary phase analysis. Along the line $R = vt$, it yields

[1]We focus here exclusively on the time dependence of the leaks, and we find regimes with exponentially suppressed leaks in time. Notice that the spatial dependence of the correlation function, even for short-range Hamiltonians, can instead exhibit algebraic behavior, associated with quasi-long-range order in 1D.

[2]The validity of the QP picture in this context is corroborated by the good quantitative agreement between the QP dynamics given by Eq. (3) and the t-VMC results [28].

the dominant contribution

$$G_{\mathrm{c}}^{\sigma\sigma}(R,t) \simeq \frac{\mathscr{F}(k^*)}{4\sqrt{\pi\left|\partial_k^2 E_{k^*}^f\right|t}} \cos\left(k^* R - 2E_{k^*}^f t + \phi\right), \quad (4)$$

where $k^\star$ is the solution of $v = 2v_g(k^\star)$, v_g is the group velocity, and ϕ is a time-independent phase. For $\alpha > 2$, the group velocity is bounded and we find a ballistic cone spreading of correlations with a velocity that is given by twice the maximum group velocity. We have checked that it agrees well with the numerics where a cone propagation is also found. A quantitative analysis of the leaks within the spin-wave approach confirms that they are exponentially suppressed in time. Moreover, the exponent β is found to smoothly approach unity upon reducing the cutoff parameter ε [see Fig. 2(a2)], in agreement with the t-VMC results.

For $1 < \alpha < 2$ the group velocity is instead unbounded, since it exhibits the infrared divergence $v_g(k) \propto k^{-|2-\alpha|}$. Hence the correlation front is no longer given by a well-determined velocity but by the coherent superposition of infinitely fast modes at low momenta. More precisely, the behavior of the leaks can be found from the asymptotic $R \to \infty$ expansion of Eq. (3), retaining only the contributions of the divergent velocities. For $\alpha = 3/2$, it can be computed exactly, and yields $G_{\mathrm{c}}^{\sigma\sigma}(R \to \infty,t) \simeq F(t) \times t/R^\alpha$, where $F(t)$ is a bounded function of time. This scaling agrees with and explains the t-VMC results. It is further confirmed by the analysis of the leaks within the spin-wave approach. The exponent β is found to smoothly approach the interaction exponent α upon reducing the cutoff parameter ε [see Fig. 2(b2)]. The same was found for other values of α between 1 and 2.

For $\alpha < 1$, both the QP energy and the group velocity diverge for $k \to 0$, respectively, as $E_k^f = e_0 k^{-|\frac{1-\alpha}{2}|}$, $v_g(k) \propto k^{-|\frac{3-\alpha}{2}|}$, and $e_0 = 2\sqrt{h_f V_f}$. In particular, it is the energy divergence which sets the breaking of quasilocality in this case. The latter gives rise to a rapidly oscillating factor of the form $\cos(e_0 t/k^{\frac{1-\alpha}{2}})$ in Eq. (3), which leads to a nonanalytic $t = 0$ delta-kick in the thermodynamic limit. More precisely, an exact asymptotic expansion of the correlation function (3) can be derived in the limit of small propagation times t and large distances R. Keeping the relevant small quasimomenta, it yields

$$G_{\mathrm{c}}^{\sigma\sigma}(R,t \to 0) \simeq \lim_{L\to\infty} A \frac{\sin\left(L^{\frac{1-\alpha}{2}} e_0 t\right)}{e_0 t} \frac{\cos(R/L)}{R^{2-\alpha}} + B\frac{(e_0 t)^2}{R^{\frac{1+\alpha}{2}}}, \quad (5)$$

where A and B are finite numerical constants, which depend on the microscopic parameters of the model. A remarkable consequence of this expression is that the first term yields an instantaneous contribution to the signal, on a time scale $e_0\tau = 1/L^{\frac{(1-\alpha)}{2}}$, independent of the distance R and with an exponent set by the divergence of the QP energy. This implies that the system reacts on a time scale inversely proportional to the system size, yielding an efficient mechanism for the breaking of locality.

An analogous microscopic analysis can be carried out for the LRBH model in the weakly interacting superfluid regime, considered in the t-VMC calculations. In this regime the quantities E_k^f and $\mathscr{F}(k)$ are found by Bogoliubov analysis [30], which yields $E_k^f = \sqrt{\varepsilon_k\{\varepsilon_k + 2n[U_f + V_f P(k)]\}}$ and $\mathscr{F}^{nn}(k) = n^2 \frac{[(U_f - U_i) + (V_f - V_i)P(k)]\varepsilon_k}{\{\varepsilon_k + 2n[U_f + V_f P(k)]\}E_k^i}$, where $\varepsilon_k = 4J\sin^2(k/2)$ is the free-particle lattice dispersion and n is the particle density [28].

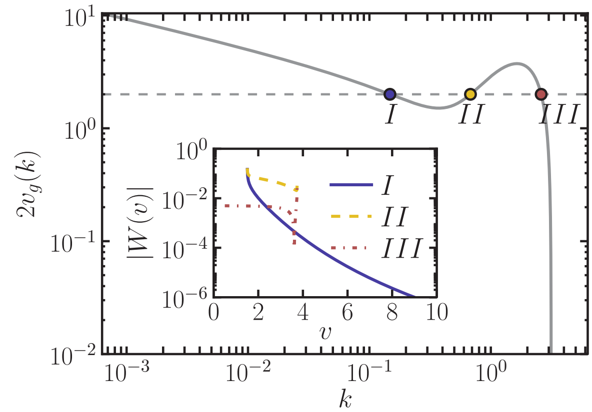

FIG. 3. (Color online) QP analysis of the group velocity for the LRBH model with exponent $\alpha = 1/2$, density $n = 1/2$, and $U = V = J/4$. In the inset, the stationary phase weights corresponding to the three solution branches of the equation $2v_g(k) = v$ are shown. Wave vectors are in units of the inverse lattice spacing and velocities are in units of the hopping amplitude J.

For $\alpha > 1$, the origin of the observed ballistic behavior is traced back to the fact that QP velocities are bounded. The propagation of correlations is dominated by the stationary-phase points of Eq. (4) and the correlation cone velocity is given by twice the maximum group velocity.

For $\alpha < 1$, the group velocity diverges as $v_g(k \to 0) \propto k^{-|\frac{1-\alpha}{2}|}$, whereas the QP energy is always finite. Hence, at variance with the LRTI model, the correlation function does not exhibit any instantaneous kick at $t = 0$ such as that of Eq. (5) and quasilocality is preserved. Moreover, although this case is formally analogous to the intermediate regime, with polynomial leaks in time, found for the LRTI model, a purely ballistic spreading is instead found within numerical precision in the LRBH model. To understand this, let us come back to the stationary-phase approach of Eq. (4). Due to the specific form of the group-velocity dispersion in the LRBH model, shown in Fig. 3, the equation $v = 2v_g(k^\star)$ has up to three separate solutions for a given velocity v. The correlation function is thus dominated by three contributions (I, II, and III) of the form of Eq. (4). The behavior of the corresponding weights, $W(v) = \mathscr{F}^{nn}(k^\star)/\sqrt{|\partial_k^2 E^f(k^\star)/J|}$, along the three branches is shown in the inset of Fig. 3. The largest weights, corresponding to the velocities dominating the propagation, belong to the regular branches (II and III). The latter extend up to a certain maximum velocity $v_{\max}$, which effectively sets the correlation cone velocity v_{c}. The infinitely fast modes, corresponding instead to branch I, have a weight which is polynomially suppressed at large velocities. The exponent for the algebraic decay can be derived from the known small k behavior of $P(k)$, which leads to $W(v \to \infty) \propto v^{-|\frac{9-3\alpha}{2(1-\alpha)}|}$. For $v \simeq v_{\max}$, the weights of these modes are already few orders of magnitude smaller than the quasilocal, ballistic modes. This separation of scales is responsible for the effective suppression of the infinitely fast nonlocal modes in the LRBH model. Notice that in the LRTI model the irregular branch of the infinitely fast modes is never protected by a finite-velocity branch. This is a consequence of the monotonic behavior of the QP group velocity in the spin model, yielding a single branch of solutions for the stationary phase equation.

Discussion. In summary, the radically different behaviors of the LRTI and LRBH models, found both in t-VMC and QP analysis, show that specific microscopic properties of the system, not accounted for in universal LR-like bounds, play a major role in quasilocality. This result sheds new light on the dynamics of long-range quantum systems. Yet many questions remain open and are worth being investigated in the future. For instance, it is expected, on the basis of universal bounds, that the critical exponents for the breaking of locality depend on the system dimension [16]. It would thus be of utmost interest to study the counterparts of the effects discussed here in dimensions higher than one, which could be done by a straightforward application of the present approach [23]. Moreover, it would be interesting to study the same LRTI and LRBH models in a regime of stronger interactions, where an extension of the t-VMC and QP analysis taking into account relevant excitations can be developed. In this regime, the LRBH model maps onto an effective spin model and might therefore exhibit radically different properties than those found in this work.

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–*Supplemental Material*–
Protected quasi-locality in quantum systems with long-range interactions

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In this Supplemental Material we give additional details on the methods and the analysis we have performed in the main Paper. In particular we provide a description of the time-dependent variational Monte Carlo method (Sec. I) and a microscopic derivation of the quasi-particle picture for both the long-range Ising and the long-range Bose-Hubbard models (Sec. II).

PACS numbers:

I. TIME-DEPENDENT VARIATIONAL MONTE CARLO

We consider the unitary dynamics of lattice systems with long-range interactions after a quantum quench in the interaction strength is realized. In order to obtain an accurate description of the dynamics, we treat the time evolution by means of the time-dependent variational Monte Carlo (t-VMC) method [1]. This method is a general framework to obtain the optimal time evolution of a given variational state, which is conveniently parametrized as

$$\Psi(\mathbf{X},t) \;=\; \exp\left[\sum_k \alpha_k(t)\mathcal{O}_k(\mathbf{X})\right]\times\Phi(\mathbf{X}), \qquad (1)$$

where $\mathbf{X}$ is a given many-body basis on which the wave-function is projected, α_i is a set of complex-valued variational parameters, $\mathcal{O}_k(\mathbf{X}) = \frac{1}{\Psi(\mathbf{X},t)}\frac{\partial\Psi(\mathbf{X},t)}{\partial\alpha_k(\mathbf{X})}$, and $\Phi(\mathbf{X})$ is some time-independent state.

By means of the Dirac-Frenkel time-dependent variational principle, it can be shown that the optimal variational parameters have to satisfy, at each time, the following equations of motion

$$i\sum_{k'}\langle\mathcal{O}_k^\star\mathcal{O}_{k'}\rangle_t^{\mathrm{c}}\,\dot{\alpha}_{k'}(t) \;=\; \langle\mathcal{O}_k^\star\mathcal{H}\rangle_t^{\mathrm{c}}\,, \qquad (2)$$

where $\mathcal{H}$ is the system hamiltonian, $\langle AB\rangle_t^{\mathrm{c}} \equiv \langle AB\rangle_t - \langle A\rangle_t\langle B\rangle_t$ are two-point connected averages, and $\langle\dots\rangle_t \equiv \frac{\langle\Psi(t)|\dots|\Psi(t)\rangle}{\langle\Psi(t)|\Psi(t)\rangle}$ denote expectation values over the variational state at time t.

Since the variational state is typically taken to be non-locally entangled, the former expectation values cannot be computed exactly with the approaches used, for example, within Matrix Product States techniques. The expectation values are therefore obtained from Monte Carlo sampling of the (sign-problem-free) square modulus of the variational wave-function, and the equations of motion (2) are then solved.

The t-VMC approach has been so-far used to simulate the dynamics of both one and two-dimensional lattice systems [1, 2]. In general, with a sensible choice of the variational states, it allows to simulate both the short and the long-time dynamics of correlated quantum systems with an accuracy comparable to methods based on tensor-network variational states.

A. Jastrow states for spins and bosons

For the lattice systems studied here, we consider time-evolved wave-functions of the general form given by the Jastrow-Feenberg (JF) correlations expansion,

$$\Psi_{\mathrm{JF}}(\mathbf{X},t) = \exp\left[\sum_i J_i^{(1)}(t)\mathcal{D}_i + \frac{1}{2}\sum_{i,j}J_{ij}^{(2)}(t)\mathcal{D}_i\mathcal{D}_j + \frac{1}{3!}\sum_{ijk}J_{ijk}^{(3)}(t)\mathcal{D}_i\mathcal{D}_j\mathcal{D}_k + \dots\right]\times\Phi(\mathbf{X}), \qquad (3)$$

where the time-dependent variational parameters are the complex amplitudes of the m-body Jastrow tensors $J_{i_1 i_2\dots i_m}^{(m)}(t)$, $\mathcal{D} = \{\mathcal{D}_1(\mathbf{X}),\mathcal{D}_2(\mathbf{X})\dots D_L(\mathbf{X})\}$ is the set of L operators in which the expansion is performed, and $\Phi(\mathbf{X})$ is a time-independent state solution of the non-interacting problem. This expansion accurately describes equilibrium properties of a variety of prototypical correlated quantum systems. It provides an accurate description of the Mott transition both in the bosonic and fermionic Hubbard models [3], and of the equilibrium properties of superfluid Helium 4 [4], to name a few classical references. Moreover, it can be shown that an ansatz containing only up to the second order correlation tensor is an exact description of important prototypical models, both with short-range (Luttinger liquids) [5] and long-range (Calogero-Sutherland) interactions [6].

For spin Hamiltonians, we consider an expansion in the local spin operators, i.e. $\mathcal{D}_{\mathrm{LRTI}} = (\sigma_1^z,\dots\sigma_L^z)$. In this case the non-interacting state is taken to be a mere constant. For bosonic systems, the JF expansion is performed in the density operators, namely $\mathcal{D}_{\mathrm{LRBH}} = (n_1,\dots n_L)$, therefore systematically including high-order density-density correlations in the wave-function. In this latter case the non-interacting state is taken to be the superfluid state in the absence of interactions. In both cases, due to the homogeneity and the translation invariance of the system, we have $J^{(1)}_i = 0$ and $J^{(2)}_{i,j} = J^{(2)}_{|i-j|}$.

For the quantum quenches we consider here, where the system is prepared in relatively weakly interacting initial states, we have checked that the inclusion of 3-body and higher terms in the JF expansion doesn't change quantitatively our conclusions on the locality behavior. In the main Paper we therefore present results where the JF expansion includes up to 2-body tensors.

II. QUASI-PARTICLE APPROACH

In our Paper we consider two types of system: a bosonic and a spin one. Even if their Hamiltonians are of different nature, we will show in the following that both of them can be reduced, in specific regimes, in the following general form

$$\mathcal{H} = \frac{1}{2}\sum_k \left[\mathcal{A}_k\left(b^\dagger_k b_k + b_{-k}b^\dagger_{-k}\right) + \mathcal{B}_k\left(b^\dagger_k b^\dagger_{-k} + b_{-k}b_k\right)\right], \quad (4)$$

where the quantities $\mathcal{A}_k$ and $\mathcal{B}_k$ are real-valued even functions of k, and the $b_k(b^\dagger_k)$ are bosonic annihilation (creation) operators. Using a matrix representation where $V^\dagger_k = \begin{pmatrix} b^\dagger_k & b_{-k}\end{pmatrix}$ and $\mathbb{M}_k = \begin{pmatrix}\mathcal{A}_k & \mathcal{B}_k \\ \mathcal{B}_k & \mathcal{A}_k\end{pmatrix}$, the Hamiltonian reads

$$\mathcal{H} = \frac{1}{2}\sum_k V^\dagger_k \mathbb{M}_k V_k.$$

This Hamiltonian can be diagonalized with the standard transformation $V_k = \mathbb{A}_k W_k$ where $W^\dagger_k = \begin{pmatrix}\beta^\dagger_k & \beta_{-k}\end{pmatrix}$ is the vector composed by Bogoliubov quasi-particles, the matrix $\mathbb{A}_k = \begin{pmatrix} u_k & v_k \\ v_k & u_k\end{pmatrix}$ diagonalizes $\mathbb{M}_k$ and it has $\det \mathbb{A}_k = 1$. These last two conditions are sufficient to determine the coefficients:

$$\begin{aligned} u_k &= \sqrt{\frac{1}{2}\left(\mathcal{A}_k/\sqrt{\mathcal{A}^2_k - \mathcal{B}^2_k} + 1\right)}, \\ v_k &= -\mathrm{sign}(\mathcal{B}_k)\sqrt{\frac{1}{2}\left(\mathcal{A}_k/\sqrt{\mathcal{A}^2_k - \mathcal{B}^2_k} - 1\right)}. \end{aligned}$$

To study the out-of-equilibrium dynamics we assume that the system is prepared in the ground state of an initial Hamiltonian defined by $\mathbb{M}^i_k$ and that a sudden change, a quantum quench, is performed in this matrix to a final matrix $\mathbb{M}^f_k$. This change induces a non-trivial time evolution of the particle operators $b_k(t)$. The natural basis to study this evolution is that of $V_k = \mathbb{A}^f_k W^f_k$, since in this basis the final Hamiltonian is diagonal, $\mathcal{H}_f = \sum_k E^f_k \beta^{f\dagger}_k \beta^f_k$. The dispersion relation of the quasi-particles in the final basis is

$$E^f_k = \sqrt{\mathcal{A}^2_k - \mathcal{B}^2_k} \quad (5)$$

and the time evolution of the quasi-particle operator is $\beta^f_k(t) = \mathrm{e}^{-iE^f_k t}\beta^f_k(0)$. At time $t = 0$ the operators b_k can be expanded on both the initial and final bases $V_k = \mathbb{A}^i_k W^i_k = \mathbb{A}^f_k W^f_k$. It yields a linear relation between pre- and post-quench operators, $W^f_k = \left(\mathbb{A}^f_k\right)^{-1}\mathbb{A}^i_k W^i_k$. This relation is useful because we know that W^i_k acts trivially on the initial state, the ground state of the initial Hamiltonian, namely $\beta^i_k|0\rangle = 0$.

We can use the previous relations to compute the time evolution of some observables. We will focus on observables that are quadratic in the particles operators, for example the $\langle n_i(t)n_j(t)\rangle$ correlation function in the LRBH model and the $\langle\sigma^z_i(t)\sigma^z_j(t)\rangle$ in the LRTI model. One of the most general, real quadratic operators that conserve the translational invariance of the system and the particles number takes the form

$$\begin{aligned} g(R;t) = \frac{1}{N}\sum_k e^{-ikR}\left(\langle b^\dagger_k b_k\rangle + \langle b_{-k}b^\dagger_{-k}\rangle + \right. \\ \left. + \langle b_{-k}b_k\rangle + \langle b^\dagger_k b^\dagger_{-k}\rangle\right) \quad (6) \end{aligned}$$

where the expectation value is taken over the initial ground state $|0\rangle$. The time evolution is due to the operators α^f_k and it involves only the final dispersion relation E^f_k, namely $2E^f_k$ because $b^\dagger_k b_k$ is the product of two $\beta_k(t)$ and $\beta^\dagger_k(t)$. The amplitude of oscillations is given by the transformation between pre- and post- quench operators $\beta^i_k(0)$ and $\beta^f_k(0)$ that depends on all the coefficients $\mathcal{A}^i_k$, $\mathcal{B}^i_k$, $\mathcal{A}^f_k$ and $\mathcal{B}^f_k$. We then find:

$$\begin{aligned} g(R,t) - g(R,0) = \int_{-\pi}^{+\pi}\frac{dk}{2\pi}\cos(kR)\times \\ \times\left[\frac{\mathcal{A}^i_k\mathcal{B}^f_k - \mathcal{A}^f_k\mathcal{B}^i_k}{E^i_k\left(\mathcal{A}^f_k + \mathcal{B}^f_k\right)}\right]\times\left[1 - \cos\left(2E^f_k t\right)\right], \end{aligned}$$

which is Eq. (3) of the main Paper with:

$$\mathcal{F}(k) = \frac{\mathcal{A}^i_k\mathcal{B}^f_k - \mathcal{A}^f_k\mathcal{B}^i_k}{E^i_k\left(\mathcal{A}^f_k + \mathcal{B}^f_k\right)}.$$

Note that we can see a time dependent part, that goes to zero as $1/\sqrt{t}$ for large t, as predicted by the stationary phase argument, and the time independent part which is the thermalization value.

A. Long-Range Transverse Ising model

We first study the long-range transverse Ising chain (LRTI), the Hamiltonian of which reads

$$\mathcal{H}_{LRTI} = -h\sum_i \sigma_i^x + \frac{V}{2}\sum_{i\neq j}\frac{\sigma_i^z\sigma_j^z}{|i-j|^\alpha}.$$

Close to a polarized phase, this Hamiltonian can be written in the form of Eq. 4 using linear spin wave theory (LSWT). We first replace the spin operators by classical spins $S_i^\alpha = \frac{1}{2}\sigma_i^\alpha$. Then, we find the minimum of the classical energy rotating the reference frame around the y axis with an angle γ, $S_i' = \mathcal{R}_i(\gamma)S_i$. As a function of the rotated spin operators the Hamiltonian reads

$$\begin{aligned}\mathcal{H}_{LRTI} = 2V\sum_{i\neq j}\frac{1}{|i-j|^\alpha}\left[\cos^2(\gamma)\, S_i^{z\prime}S_j^{z\prime} + \sin^2(\gamma)\, S_i^{x\prime}S_j^{x\prime} + \right.\\ \left. -\sin(\gamma)\cos(\gamma)\left(S_i^{z\prime}S_j^{x\prime} + S_i^{x\prime}S_j^{z\prime}\right)\right] + \\ + 2h\sum_i\left(\sin(\gamma)\, S_i^{z\prime} + \cos(\gamma)\, S_i^{x\prime}\right).\end{aligned}$$

In order to introduce bosonic operators, we use the Holstein-Primakoff transformations for spin one-half, $S_i^{z\prime} \approx \frac{1}{2}\left(b_i^\dagger + b_i\right)$, $S_i^{x\prime} = b_i^\dagger b_i - \frac{1}{2}$ and treat the Hamiltonian perturbatively in the b_i operators. The zero order term is the classical energy per particle, $E_{cl} = \left[\frac{V}{2}\sin^2(\gamma)\,\bar{P} + h\cos(\gamma)\right]$, where $\bar{P} = \sum_{R>0} 1/R^\alpha$ is the average interaction energy. We fix the rotation angle γ imposing the minimum of the classical energy. In the case of a quasi-classical state, $V \ll h$, the minimum corresponds to $\gamma = 0$ and we will use this value to simplify our expressions. The linear term in the b_i operators vanishes on the minimum of the classical energy. Then the quantum corrections appear at quadratic order, that in momentum space reads

$$\begin{aligned}\mathcal{H}_{LRTI} - NE_{cl} = \frac{1}{2}\sum_k\left[\left(b_k^\dagger b_k + b_{-k}b_{-k}^\dagger\right)(VP(k) + 2h) + \right.\\ \left. + VP(k)\left(b_k^\dagger b_{-k}^\dagger + b_{-k}b_k\right)\right],\end{aligned}$$

where $P(k)$ is the Fourier transform of the long-range potential and the quasi-particle energy is

$$E_k = 2\sqrt{h(h + VP(k))}.$$

Note that the latter is well behaved for $V \ll h$, where our expressions hold.

In the main Paper we consider the correlations along the z-axis, which reads $G(R,t) = \langle\sigma_i^z\sigma_j^z\rangle$, in real space. Using the Holstein-Primakoff transformation it becomes $G(R,t) = \langle\left(b_i^\dagger + b_i\right)\left(b_j^\dagger + b_j\right)\rangle$, in terms of bosonic operators, which is exactly Eq. 6 in real space. Taking the Fourier transform we get $G(R,t) - G(R,0) = g(R,t)$. The main parameters in the LRTI model are

$$\begin{aligned}\mathcal{A}_k &= VP(k) + 2h\\ B_k &= VP(k)\\ \mathcal{F}^{\sigma\sigma}(k) &= 2\,\frac{(h_fV_i - h_iV_f)\,P(k)}{(h_f + V_fP(k))\,E_k^i}.\end{aligned}$$

Setting $h_i = h_f = 1$ we find the same expression found in the Paper.

B. Long-Range Bose-Hubbard model

For the long-range Bose-Hubbard Hamiltonian we can write the Hamiltonian directly in Fourier space as

$$\mathcal{H}_{LRBH} = \sum_k \epsilon_k b_k^\dagger b_k + \frac{1}{2L}\sum_{k,p,q} V(q) b_{k-q}^\dagger b_{p+q}^\dagger b_p b_k,$$

in the standard second quantization form, where $\epsilon_k = 4J\sin^2\left(\frac{k}{2}\right)$ is the usual dispersion for the free lattice system. The Fourier transform of the potential is $V(q) = U + VP(q)$ where U is the on-site short-range interaction strength, V is the long-range interaction strength and $P(q)$ its the Fourier transform with the regularization condition $P(q=0) = 0$. Since the $k=0$ component are macroscopically populated we can separate them from the other momenta and take this expansion up to the second order in b_0 and $b_0^\dagger$. It yields

$$\begin{aligned}\mathcal{H}_{LRBH} \approx \frac{V(0)}{2L} b_0^\dagger b_0^\dagger b_0 b_0 + \sum_{k\neq0}\epsilon_k b_k^\dagger b_k + \\ + \frac{n_0}{2L}\sum_{k\neq0}\left\{2\left[V(k) + V(0)\right] b_k^\dagger b_k + b_k^\dagger b_{-k}^\dagger + b_{-k}b_k\right\},\end{aligned}$$

using the relation $\left(b_0^\dagger b_0\right)^2 \approx N^2 - 2N\sum_{k\neq0} b_k^\dagger b_k$ and we get, up to a constant, the Hamiltonian

$$\begin{aligned}\mathcal{H}_{LRBH} = \frac{1}{2}\sum_{k\neq0}\left[\left(b_k^\dagger b_k + b_{-k}b_{-k}^\dagger\right)(\epsilon_k + n_0V(k)) + \right.\\ \left. + n_0V(k)\left(b_k^\dagger b_{-k}^\dagger + b_{-k}b_k\right)\right],\end{aligned}$$

where n_0 is the condensate fraction. In this case we can therefore identify $\mathcal{A}_k = \epsilon_k + n_0V(k)$ and $\mathcal{B}_k = n_0V(k)$. Inserting these expressions into Eq. 5, we then find the dispersion relation of the quasi-particles $E_k = \sqrt{\epsilon_k\left(\epsilon_k + 2n_0V(k)\right)}$.

In the LRBH case, we consider the density-density correlations, $G(R;t) = \langle n_i(t)n_j(t)\rangle = \frac{1}{N}\sum_k e^{-\imath kR}\langle n_k(t)n_{-k}(t)\rangle$, where we have introduced the Fourier transform of the density $n_k(t) = \sum_q b_{k+q}^\dagger b_q$. Using the Bogoliubov expansion in powers of the $k=0$ mode, we write the correlation in the form of Eq. 6, namely $G(R,t) - G(R,0) = n_0 g(R,t)$. Hence the main

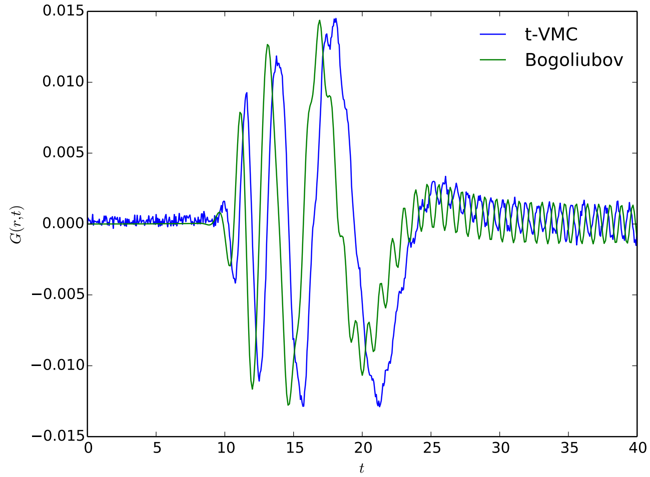

FIG. 1: Comparison between the density-density correlations obtained with the t-VMC approach (with the inclusion of the 2-body tensor, blue curve) and the Bogoliubov analysis (green curve) for $\alpha = 3, V_i = U_i = 1$, and $V_f = U_f = \frac{1}{4}$. Here the distance is $R = 40$.

parameters of the LRBH model are

$$\begin{aligned} \mathcal{A}_k &= \epsilon_k + n_0 V(k) \\ \mathcal{B}_k &= n_0 V(k) \\ E_k &= \sqrt{\epsilon_k \left(\epsilon_k + 2n_0 V(k)\right)} \\ \mathcal{F}^{nn}(k) &= n_0^2 \frac{\epsilon_k \left(V_i(k) - V_f(k)\right)}{E_k^i \left(\epsilon_k + 2n_0 V_f(k)\right)}. \end{aligned}$$

In the case $U_i = U_f$ we get the expression given in the Paper.

C. Validity of the quasi-particle picture

In the main Paper we use the quasi-particle picture, in order to interpret the various dynamical regimes we have observed in the many-body correlations. This picture is based on the assumption that, after a quantum quench, freely-propagating quasi-particles are released, whose interactions can be neglected. The quasi-particle picture can legitimately be questioned on the basis that it neglects weak residual interactions between the quasi-particles. In order to test the validity of the quasi-particle picture, we have systematically compared the space-time behavior of the correlation functions given by the quasi-particle and many-body t-VMC approaches.

In Fig. 1 we show a direct comparison of the time-dependent density-density correlations for the LRBH model. The two approaches (t-VMC on one hand and Bogoliubov analysis on the other hand) yield quantitatively similar results. The slight discrepancy is here due to a renormalized quasi-particle velocity in the correlated t-VMC approach, which results in slightly faster propagating signals. More precisely we have determined the cone velocities given by the two approaches in Table I. We find that the quasi-particle and many-body t-VMC approaches give very close values with a discrepancy of the order of 4%. This legitimates the quasi-particle picture for all values of α.

LRTI	$\alpha = 3$	LRBH	$\alpha = 1/2$	$\alpha = 3/2$	$\alpha = 3$
$v_c^{\text{t-VMC}}$	$0.37\,h$	$v_c^{\text{t-VMC}}$	$3.6\,J$	$3.5\,J$	$3.1\,J$
v_c^{qp}	$0.393\,h$	v_c^{qp}	$3.740\,J$	$3.389\,J$	$3.177\,J$

TABLE I: Comparison of the cone velocities obtained from both the t-VMC method and the quasi-particle picture. The t-VMC velocities are obtained as a fit of the activation time $vt^\star = R$ (see main Paper), with a statistical uncertainty of $\sim \pm 0.01\,h$ (for the LRTI model) and $\sim \pm 0.1\,J$ (for the LRBH model). The reported v_c^{qp} equals twice the maximum group velocity of the quasi-particles, i.e. $v_c^{\text{qp}} = 2 \times \max_k \partial_k E_k$.

In general, in the regime that precedes the arrival of the correlation front the scattering processes between the fastest traveling quasi-particles are to all purposes negligible, and the ballistic spreading is a direct consequence of this. Therefore, because of the almost non-interacting nature of the quasi-particles in the ballistic regime, the Bogoliubov dynamics captures qualitatively well the correlation front. Nevertheless, the picture of independent quasi-particles inevitably breaks down on time scales $t \gg t^\star$ when scattering processes coupling all the excited modes play a key role in the decay of the correlation function. We have indeed found regimes, for interactions significantly stronger than those studied in the Paper, where at large times the damping of the signal is substantially different between the t-VMC and the Bogoliubov approach.

For what concerns the LRTI model, the agreement between t-VMC and the simple LSW theory is less accurate than Bogoliubov theory for the LRBH model. This has been already observed in Ref [7]. We believe that the origin of the quantitative discrepancy is due to the order retained in the Holstein-Primakoff transformation, which effectively amounts to treat hard-core bosons as if they were soft-core bosons. This approximation is certainly crude in this case and breaks the full quantitative agreement. However, we conclude stressing that the qualitative features are still reproduced in a fair way.

Most important, we find that the correlation front studied in the Paper fairly reproduced by the LSW theory. As shown in Table I, the cone velocity given by the LSW theory is in good agreement with that of the many-body t-VMC approach in the ballistic regime ($\alpha = 3$).

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