Ultracold Bose Gases in 1D Disorder: From Lifshits Glass to Bose-Einstein Condensate

P. Lugan,[1] D. Clément,[1] P. Bouyer,[1] A. Aspect,[1] M. Lewenstein,[2] and L. Sanchez-Palencia[1]

[1]*Laboratoire Charles Fabry de l'Institut d'Optique, CNRS and Univ. Paris-Sud, Campus Polytechnique,
RD 128, F-91127 Palaiseau cedex, France*

[2]*ICREA and ICFO-Institut de Ciències Fotòniques, Parc Mediterrani de la Tecnologia, E-08860 Castelldefels (Barcelona), Spain*

We study an ultracold Bose gas in the presence of 1D disorder for repulsive interatomic interactions varying from zero to the Thomas-Fermi regime. We show that for weak interactions the Bose gas populates a finite number of localized single-particle Lifshits states, while for strong interactions a delocalized disordered Bose-Einstein condensate is formed. We discuss the schematic quantum-state diagram and derive the equations of state for various regimes.

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Disorder is present in nearly all condensed-matter systems due to unavoidable defects of the sustaining media. It is known not only to impair quantum flows but also to lead to spectacular effects such as Anderson localization [1–3]. In contrast to condensed-matter systems, ultracold atomic gases can be realized in the presence of controlled disorder [4], opening possibilities for investigations of localization effects [5–9] (for a review, see Ref. [10]). The first experimental studies of localization in disordered interacting Bose gases have been reported in Refs. [11–16].

One of the most fundamental issues in this respect concerns the interplay between localization and interactions in many-body quantum systems at zero temperature. Without interactions, a quantum gas in a random potential populates localized states [1], either a single state (in the case of bosons), or many (fermions). Weak repulsive interactions lead to delocalization, but strong interactions in confined geometries lead to Wigner-Mott-like localization [17]. Surprisingly, even for weakly interacting Bose gases, where the mean-field Hartree–Fock–Gross–Pitaevskii–Bogolyubov–de Gennes (HFGPBdG) description is expected to be valid, there exists no clear picture of the localization-delocalization scenario. Numerical calculations using the Gross-Pitaevskii equation (GPE) suggest that the Bose gas wave function at low densities is a superposition of localized states [15]. It is thus natural to seek the true ground state in the form of generalized HFGPBdG states, for which the Bose gas populates various low-energy single-atom states. In the presence of disorder, they correspond to so-called Lifshits states (LS) [18].

In this Letter we consider a *d*-dimensional (*d*D) Bose gas at zero temperature with repulsive interactions, and placed in a 1D random potential with arbitrary amplitude and correlation length. We show that generalized HFGPBdG states indeed provide a very good description of the many-body ground state for interactions varying from zero to the Thomas-Fermi (TF) regime [19]. We stress that the solution we find is different from that of noninteracting fermions which at zero temperature form a Fermi glass and occupy a large number of localized single-particle levels [10]. In contrast, many bosons may occupy

the same level and thus populate only a finite number of LSs forming what we call a Lifshits glass. In the following, we discuss the quantum states of the system as a function of the strength of interactions and the amplitude and correlation length of the random potential, and we draw the schematic quantum-state diagram (see Fig. 1). In the limit of weak interactions, the Bose gas is in the Lifshits glass state, whereas for stronger interactions the gas forms a (possibly smoothed) delocalized disordered Bose-Einstein condensate (BEC) [20]. Our theoretical treatment provides us with a novel, physically clear, picture of disordered, weakly interacting, ultracold Bose gases. This is the main result of this work. In addition, we derive analytical formulas for the boundaries (corresponding to crossovers) in the quantum-state diagram and for the equations of state in the various regimes. We illustrate our results using a speckle random potential [21].

Consider a *d*D ultracold Bose gas with weak repulsive interactions, i.e., such as $n^{1-2/d} \ll \hbar^2/mg$, where m is the atomic mass, n the density, and g the *d*D coupling constant.
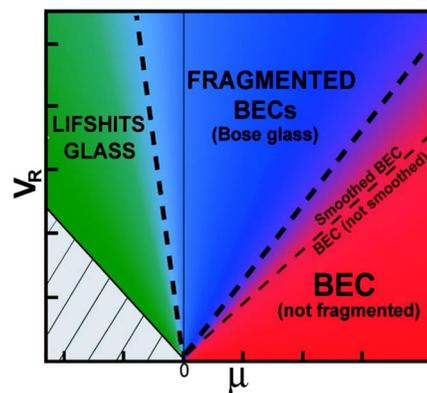

FIG. 1 (color online). Schematic quantum-state diagram of an interacting ultracold Bose gas in 1D disorder. The dashed lines represent the boundaries (corresponding to crossovers) which are controlled by the parameter $\alpha_R = \hbar^2/2m\sigma_R^2 V_R$ (fixed in the figure, see text), where V_R and σ_R are the amplitude and correlation length of the random potential. The hatched part corresponds to a forbidden zone ($\mu < V_{min}$).

The gas is assumed to be axially confined to a box of length $2L$ in the presence of a 1D random potential $V(z)$, and trapped radially in a 2D harmonic trap with frequency $\omega_\perp$. We assume that the random potential is bounded below $[V_{min} = \min(V)]$ and we use the scaling form $V(z) = V_R v(z/\sigma_R)$, where $v(u)$ is a random function with both typical amplitude and correlation length equal to unity [22].

For illustration, we will consider a 1D speckle potential [21] similar to that used in Refs. [11–14]. In brief, $v(u)$ is random with the probability distribution $\mathcal{P}(v) = \Theta(v + 1)\exp[-(v+1)]$, where Θ is the Heaviside function. Thus v is bounded below by $v_{min} = -1$ and we have $\langle v \rangle = 0$ and $\langle v^2 \rangle = 1$. In addition, for a square aperture, the correlation function reads $\langle v(u)v(u') \rangle = \mathrm{sinc}^2[\sqrt{3/2}(u - u')]$.

Below, we discuss the quantum states of the Bose gas, which are determined by the interplay of interactions and disorder.

BEC regime.—For strong repulsive interactions the Bose gas is delocalized and forms a BEC [12,20] (possibly quasi-BEC in 1D or elongated geometries [23]). The density profile is then governed by the GPE,

$$\mu = -\hbar^2 \nabla^2(\sqrt{n})/2m\sqrt{n} + m\omega_\perp^2 \rho^2/2 + V(z) + gn(\mathbf{r}), \quad (1)$$

where ρ is the radial coordinate and μ the chemical potential. This regime has been studied in the purely 1D case in Ref. [20]. Here, we focus on the case of a shallow radial trap ($\hbar\omega_\perp \ll \mu$) such that the radial profile is a TF inverted parabola. Proceeding as in Ref. [20], we find that the BEC density has a generalized TF profile [24]

$$\sqrt{n(\rho, z)} \simeq \sqrt{\mu(\rho)/g}[1 - \tilde{V}(\rho, z)/2\mu(\rho)], \quad (2)$$

where $\mu(\rho) = \mu[1 - (\rho/R_\perp)^2]$ is the local chemical potential, $R_\perp = \sqrt{2\mu/m\omega_\perp^2}$ is the radial TF half-size and $\tilde{V}(\rho, z) = \int dz' G(\rho, z')V(z - z')$ is a smoothed potential [20] with $G(\rho, z) = \frac{1}{\sqrt{2}\xi(\rho)}\exp(-\frac{\sqrt{2}|z|}{\xi(\rho)})$, and $\xi(\rho) = \hbar/\sqrt{2m\mu(\rho)}$ being the local healing length. For $\xi(\rho) \ll \sigma_R$, i.e., for

$$\mu(\rho) \gg \hbar^2/2m\sigma_R^2, \quad (3)$$

we have $\tilde{V}(\rho, z) \simeq V(z)$, and the BEC density follows the modulations of the random potential in the TF regime. For $\xi(\rho) \gtrsim \sigma_R$, the kinetic energy cannot be neglected and competes with the disorder and the interactions. The random potential is therefore smoothed [20]: $\Delta \tilde{V}(\rho) < \Delta V$, where ΔV [$\Delta \tilde{V}(\rho)$] is the standard deviation of the (smoothed) random potential. The solution (2) corresponds to a delocalized disordered BEC.

The perturbative approach is valid when $\mu(\rho) \gg \Delta \tilde{V}(\rho)$. From the expression for $\tilde{V}$, we write $\Delta \tilde{V}(\rho) = V_R\sqrt{\Sigma^0(\sigma_R/\xi(\rho))}$. For the speckle potential, we can approximate the correlation function to $V_R^2 \exp(-z^2/2\sigma_R^2)$ and we find [20]

$$\Sigma^0(\tilde{\sigma}_R) = \tilde{\sigma}_R^2 + (1 - 2\tilde{\sigma}_R^2)\tilde{\sigma}_R e^{\tilde{\sigma}_R^2}\int_{\tilde{\sigma}_R}^\infty d\theta e^{-\theta^2}, \quad (4)$$

with $\tilde{\sigma}_R = \sigma_R/\xi(\rho)$. In the center, i.e. $\rho = 0$ or in 1D, the validity condition of the BEC regime thus reduces to

$$\mu \gg V_R\sqrt{\Sigma^0(\sigma_R/\xi)} \quad \text{with} \quad \xi = \xi(0). \quad (5)$$

If condition (5) is not fulfilled, the Bose gas will form a fragmented BEC. The latter is a compressible insulator and thus can be identified with a Bose glass [17].

Noninteracting regime.—In the opposite situation, for vanishing interactions, the problem is separable and the radial wave function is the ground state of the radial harmonic oscillator. We are thus left with the eigenproblem of the single-particle 1D Hamiltonian $\hat{h} = -\hbar^2\partial_z^2/2m + V(z)$. In the presence of disorder, the eigenstates χ_ν are all localized [2] and are characterized by [18] (i) a finite localization length, (ii) a dense pure point density of states $\mathcal{D}_{2L}$, and (iii) a small participation length $P_\nu = 1/\int dz|\chi_\nu(z)|^4$ [25]. If $V(z)$ is bounded below, so is the spectrum and the low energy states belong to the so-called Lifshits tail, which is characterized by a stretched exponential cumulative density of states (cDOS), $\mathcal{N}_{2L}(\epsilon) = \int^\epsilon d\epsilon' \mathcal{D}_{2L}(\epsilon') \sim \exp(-c\sqrt{\frac{V_R}{\epsilon - V_{min}}})$, in 1D [26].

Numerical results for the speckle potential are shown in Fig. 2. As expected the cDOS shows a stretched exponential form, the lowest LSs are spatially localized, and $P(\epsilon)$ increases with energy indicating a weaker localization. However, $P(\epsilon)$ is almost constant at low energy. Note also that the lowest LSs hardly overlap if their extension is much smaller than the system size.

Lifshits regime.—We turn now to the regime of finite but weak interactions, where the chemical potential μ lies in the Lifshits tail of the spectrum. Owing to the fact that the lowest single-particle LSs hardly overlap, it is convenient to work in the basis of the LSs, $\{\chi_\nu, \nu \in \mathbb{N}\}$. These can be regarded as trapping microsites populated with N_ν bosons in the quantum state $\phi_\nu(\rho)\chi_\nu(z)$, where the longitudinal motion is frozen to χ_ν and ϕ_ν accounts for the radial extension in the microsite ν. Therefore, the many-body wave function is the Fock state

$$|\Psi\rangle = \prod_{\nu \geq 0}(N_\nu!)^{-1/2}(b_\nu^\dagger)^{N_\nu}|\mathrm{vac}\rangle, \quad (6)$$

where $b_\nu^\dagger$ is the creation operator in the state $\phi_\nu(\rho)\chi_\nu(z)$ [27]. Each ϕ_ν can be a transverse 2D BEC for $N_\nu \gg 1$. However, the quantum state (6) does not correspond generally to a single 3D BEC since it does not reduce to $(N!)^{-1/2}(b_0^\dagger)^N|\mathrm{vac}\rangle$. Rather, the Bose gas splits into several fragments whose longitudinal shapes are those of the LSs, χ_ν, and are hardly affected by the interactions.
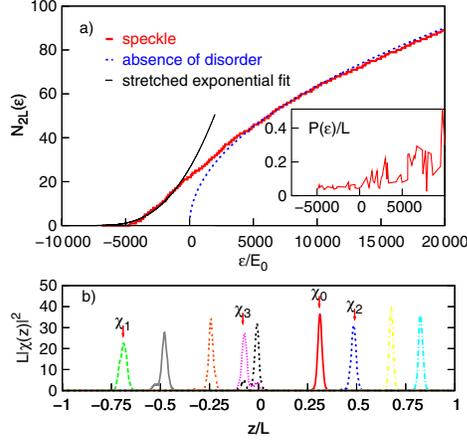
FIG. 2 (color online).   (a) Cumulative density of states of single particles in a speckle potential with $\sigma_R = 2 \times 10^{-3}L$ and $V_R = 10^4 E_0$, where $E_0 = \hbar^2/2mL^2$ ($V_{\min} = -V_R$). Inset: Participation length [25]. (b) Low-energy Lifshits eigenstates. For the considered realization of disorder, $\epsilon_0 \simeq -5 \times 10^3 E_0$.

The mean-field energy associated with state (6) reads

$$E[\Psi] = \sum_\nu N_\nu \int d\rho\, \phi_\nu^* \left(\frac{-\hbar^2 \nabla_\perp^2}{2m} + \frac{m\omega_\perp^2 \rho^2}{2} + \epsilon_\nu \right) \phi_\nu$$

$$+ \sum_\nu \frac{N_\nu^2}{2} \int d\rho\, U_\nu |\phi_\nu|^4, \tag{7}$$

where $U_\nu = g \int dz |\chi_\nu(z)|^4 = gP_\nu^{-1}$ is the local interaction energy in the LS χ_ν. Minimizing $E[\Psi]$ for a fixed number of atoms ($E[\Psi] - \mu \sum_\nu N_\nu \to \min$), we find the equation

$$(\mu - \epsilon_\nu)\phi_\nu = [-\hbar^2 \nabla_\perp^2/2m + m\omega_\perp^2 \rho^2/2$$

$$+ N_\nu U_\nu |\phi_\nu|^2]\phi_\nu. \tag{8}$$

Solving the 2D GPE (8) for each microsite ν, one finds the atom numbers N_ν and the wave functions ϕ_ν. As μ increases, $|\phi_\nu|^2$ will turn continuously from a Gaussian (for $\hbar\omega_\perp \gg \mu$) into an inverted parabola (for $\hbar\omega_\perp \ll \mu$).

To discuss the validity condition of the Lifshits regime let us call $\nu^{\max}$ the index of the highest LS such that all lower LSs hardly overlap. The Lifshits description requires the chemical potential μ to be small enough so that the number of populated LSs is smaller than $\nu^{\max}$, i.e., if

$$\mathcal{N}_{2L}(\mu) \leq \nu^{\max}. \tag{9}$$

If condition (9) is not fulfilled, several populated LSs will overlap and the Bose gas will start to form a fragmented BEC. Each fragment will be a superposition of LSs, and its shape will be modified by the interactions.

Although both $\mathcal{N}_{2L}$ and $\nu^{\max}$ may have complex dependencies versus V_R, σ_R and the model of disorder, general properties can be obtained using scaling arguments. We write the single-particle equation as

$$(\epsilon_\nu/V_R)\varphi_\nu(u) = -\alpha_R \partial_u^2 \varphi_\nu(u) + \upsilon(u)\varphi_\nu(u), \tag{10}$$

where $u = z/\sigma_R$, $\varphi_\nu(u) = \sqrt{\sigma_R}\chi_\nu(z)$ and $\alpha_R =$

$\hbar^2/2m\sigma_R^2 V_R$. Thus, all characteristics of the spectrum depend only on the parameter α_R after renormalization of energies and lengths. Scaling arguments show that in the Lifshits tail

$$\mathcal{N}_{2L}(\epsilon) = (L/\sigma_R)\zeta(\alpha_R, \epsilon/V_R)$$

$$\text{and} \quad \nu^{\max} = (L/\sigma_R)\eta(\alpha_R), \tag{11}$$

where ζ and η are υ-dependent functions. Finally, inserting these expressions into Eq. (9) and solving formally, we obtain the validity condition of the Lifshits regime:

$$\mu \lesssim V_R F(\alpha_R), \tag{12}$$

where F is the solution of $\zeta[\alpha_R, F(\alpha_R)] = \eta(\alpha_R)$, which can be computed numerically, for example.

We are now able to draw the schematic quantum-state diagram of the zero-temperature Bose gas as a function of μ and V_R (see Fig. 1). From the discussion above, it is clearly fruitful to fix the parameter α_R while varying V_R. The boundaries between the various regimes (Lifshits, fragmented BEC, BEC, and smoothed BEC) result from the competition between the interactions and the disorder and are given by Eqs. (3), (5), and (12). We stress that they are crossovers rather than phase transitions. Interestingly, all these boundaries are straight lines with slopes depending on the parameter α_R. This is clear from Eq. (12) for the boundary between the Lifshits and the fragmented regimes. In addition, since $V_R = (\hbar^2/2m\sigma_R^2)/\alpha_R$, the non-smoothing condition (3) reduces to $\mu \gg \alpha_R V_R$. Finally, since $\mu = \hbar^2/2m\xi^2$ and thus $\sigma_R/\xi = \frac{1}{\sqrt{\alpha_R}}\sqrt{\mu/V_R}$, the nonfragmented BEC condition (5) also corresponds to a straight line α_R in Fig. 1.

To finish with, we derive the equations of state of the Bose gas in the identified quantum states. It is important to relate the chemical potential μ which governs the crossovers between the various regimes to the mean atomic density $\bar{n} = N/2L$ and the coupling constant g. Both can be controlled in experiments with ultracold atoms.

Tight radial confinement.—For $\mu' - \epsilon_\nu \ll \hbar\omega_\perp$ where $\mu' = \mu - \hbar\omega_\perp$, the radial wave functions are frozen to zero-point oscillations, $\phi_\nu(\rho) = \exp(-\rho^2/2l_\perp^2)/\sqrt{\pi}l_\perp$ with $l_\perp = \sqrt{\hbar/m\omega_\perp}$ the width of the radial oscillator.

In the BEC regime, $\mu' \gg \Delta\tilde{V}$, we find from Eq. (2),

$$\mu' = \bar{n}g. \tag{13}$$

In the Lifshits regime, we find

$$N_\nu = [\mu' - \epsilon_\nu]/U_\nu \quad \text{for} \quad \mu' > \epsilon_\nu$$

$$\text{and} \quad N_\nu = 0 \quad \text{otherwise}, \tag{14}$$

by inserting the above expression for $\phi_\nu(\rho)$ into Eq. (8). Turning to a continuous formulation and using the normalization condition, $N = \int d\epsilon \mathcal{D}_{2L}(\epsilon)N(\epsilon)$, we deduce the equation of state of the Bose gas in the Lifshits regime:

$$Ng = \int_{-\infty}^{\mu'} d\epsilon \mathcal{D}_{2L}(\epsilon)(\mu' - \epsilon)P(\epsilon), \tag{15}$$
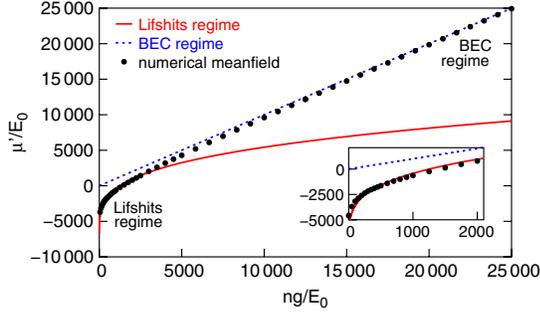
FIG. 3 (color online).   Chemical potential of a Bose gas in a speckle potential with the same parameters as in Fig. 2, in the case of a tight radial confinement ($\mu' - \epsilon_\nu \ll \hbar\omega_\perp$). The points are given by numerical calculations; the solid and dotted lines represent the analytical formulas Eqs. (13) and (16) derived for the Lifshits and BEC regime, respectively.

which relates the chemical potential μ' to the coupling constant g. The relation is in general nonuniversal. In the case of a speckle potential, $\mathcal{N}_{2L}(\epsilon) = A(\alpha_R)(L/\sigma_R) \times \exp[-c(\alpha_R)/\sqrt{\epsilon/V_R + 1}]$, and assuming that the participation length $P_\nu = \sigma_R p_\nu(\alpha_R)$ is independent of the energy in the Lifshits tail, we find

$$\bar{n}g \simeq A(\alpha_R)c^2(\alpha_R)p_0(\alpha_R)V_R\Gamma(-2, c(\alpha_R)/\sqrt{\mu'/V_R + 1}),$$
(16)

where Γ is the incomplete gamma function and $A(\alpha_R)$, $c(\alpha_R)$, and $p_0(\alpha_R)$ can be determined numerically.

Using a numerical minimization of the energy functional (7) in the Gross-Pitaevskii formulation, we compute the chemical potential of the Bose gas in a wide range of interactions. The result shown in Fig. 3 indicates a clear crossover from the Lifshits regime to the BEC regime as the interaction strength increases. The numerically obtained chemical potential μ agrees with our analytical formulas in both Lifshits and BEC regimes.

Shallow radial confinement.—The equations of state can also be obtained in the case of shallow radial confinement ($\mu - \epsilon_\nu \gg \hbar\omega_\perp$). In the BEC regime, for $\xi \ll \sigma_R$, we find $\mu \simeq \sqrt{\bar{n}gm\omega_\perp^2/\pi} - V_R^2$. In the Lifshits regime the 2D wave functions $\phi_\nu(\rho)$ are in the TF regime, $|\phi_\nu(\rho)|^2 = \frac{\mu - \epsilon_\nu}{N_\nu U_\nu}(1 - \rho^2/R_\nu^2)$, where $R_\nu = \sqrt{2(\mu - \epsilon_\nu)/m\omega_\perp^2}$ is the 2D-TF radius and $N_\nu = \pi(\mu - \epsilon_\nu)^2/m\omega_\perp^2 U_\nu$ for $\mu > \epsilon_\nu$ (0 otherwise). Proceeding as in the 1D case, we find

$$Ng = \frac{\pi}{m\omega_\perp^2} \int_{-\infty}^{\mu} d\epsilon \mathcal{D}_{2L}(\epsilon)(\mu - \epsilon)^2 P(\epsilon).$$
(17)

Applying this formula to the relevant model of disorder allows us to compute the populations N_ν of the various LSs χ_ν and the corresponding radial extensions ϕ_ν.

In summary, we have presented a complete picture of the quantum states of an interacting Bose gas in the presence of 1D disorder, including the novel description of the weakly interacting Lifshits glass state. We have provided analytical formulas for the boundaries (crossovers) in the quantum-state diagram and shown that they are determined by the coupling constant. Since this coupling constant can be controlled in cold gases, future experiments should be able to explore the whole diagram.

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