

# Exact solution and Luttinger liquid behavior of the quantum 1D hard rod model

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The quantum hard rod model, a one-dimensional extension of the Lieb-Liniger model, is exactly solved using an adapted Bethe ansatz. Our solution, benchmarked against path-integral quantum Monte Carlo calculations, reveals significant corrections to the excitation spectrum and thermodynamic properties, previously overlooked by the standard excluded-volume approach. We also show that the hard rod model exhibits Luttinger liquid behavior across a wide range of parameters, at zero and finite temperature, as unveiled by correlation functions. This work provides a comprehensive framework for understanding strongly correlated regimes in dilute 1D systems, with applications to quantum wires, spin chains, and ultracold atoms.

Hard-sphere models describe hard-core particles with a finite diameter. They form a central class of classical statistical models for determining universal thermodynamic properties of repulsive particle gases, with applications to liquid-gas transitions and fluid dynamics [1–6], as well as generalized hydrodynamics [7–11]. Classical models, as well as quantum counterparts, have been extensively studied with direct applications to rigid molecules and colloidal systems [12], strongly repulsive particles in gaseous helium on carbon nanotubes [13, 14], spin-ion compounds [15], polarized hydrogenoids [16], and ultracold atoms [17, 18]. Notably, hard spheres also describe the excluded volume associated with Rydberg blockade in atomic systems [19]. In one dimension (1D), the hard rod (HR) model is a natural extension of the Lieb-Liniger model, which describes 1D bosons with contact interactions and is exactly solvable [20, 21].

In quantum systems, the reduced dimensionality exaggerates quantum correlations, leading to unique phenomena in 1D [22, 23], such as absence of Bose-Einstein condensation, fermionization of interacting Bose gases, and the lack of quasiparticle excitations. Quantum systems in 1D are also highly susceptible to perturbations with characteristic lengths commensurate with the average interparticle distance, leading to the celebrated pinning transition in arbitrary weak periodic potentials [24–26] and Bose glass transitions in disordered or quasiperiodic systems [27–33]. Unlike in 3D, quantum correlations in 1D bosonic systems increase with decreasing particle density, making them an ideal testbed for studying strongly correlated regimes in dilute systems. This applies to a variety of physical systems, including electronic quantum wires, spin chains, organic conductors, and tightly confined ultracold atoms [22, 23]. Recent experimental observations have confirmed a number of phenomena, such as Tonks-Girardeau (TG) physics [34, 35], absence of thermalization in integrable systems [36], spin-charge separation [37, 38], Mott transitions in shallow periodic potentials [39, 40], disorder-induced enhancement of coherence [41], and the observation of the Tan contact [42, 43].

The standard approach to the HR model involves mapping onto the LL model with infinite repulsive interactions (TG gas) by excluding the volume occupied by the

HRs and shifting the ordered coordinates of the particles accordingly [44]. In this work, we provide an exact solution based on an adapted Bethe ansatz (BA) and benchmark it against path-integral quantum Monte Carlo (QMC) calculations, finding excellent agreement. We show that while the excluded-volume approach accurately describes the ground state, it fails to correctly capture the excited states. This discrepancy leads to significant corrections in the excitation spectrum and thermodynamic properties, which we discuss. Moreover, we show that the HR model exhibits Luttinger liquid (LL) behavior over a wide range of parameters, as evidenced by correlation functions. Universal properties of LL allow for a second validation of our BA solution as well as thermometry of 1D systems described by HRs. Finally, possible applications to Rydberg atom systems are outlined.

The Hamiltonian for 1D bosons with pairwise interactions is given by

$$\hat{H} = -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{\ell > j} V(x_\ell - x_j), \quad (1)$$

where  $m$  is the particle mass,  $N$  is the total number of particles, and  $x_j$  denotes the center position of particle  $j$ . Equation (1) describes a family of models determined by the scattering potential  $V(x)$ . For HRs, it reads as  $V(x) = \infty$  for  $|x| \leq a$  and  $V(x) = 0$  for  $|x| > a$ , which describes impenetrable particles with finite diameter  $a$ .

We first show that the HR model can be solved exactly using an adapted coordinate BA [45], beyond the excluded-volume approach. To proceed, we write the BA wavefunction as

$$\Psi(\mathbf{x}) = \sum_{\mathcal{P} \in S_N} A_{\mathcal{P}} \exp\left(i \sum_{j=1}^N k_{\mathcal{P}_j} x_j\right), \quad (2)$$

for  $|x_{j+1} - x_j| \geq a$ , and  $\Psi(\mathbf{x}) = 0$  otherwise. The latter guarantees the HR impenetrability condition, which forces the distance between consecutive particles to exceed  $a$ . Here,  $S_N$  is the symmetric group over  $\{1, \dots, N\}$ , and  $\mathbf{x} = (x_1, \dots, x_N)$  represents the ordered particle coordinates. Since  $\Psi(\mathbf{x})$  vanishes for  $x_{j+1} - x_j < a$ , the

energy is fully kinetic and reads as  $E = \sum_j \hbar^2 k_j^2 / 2m$ . To determine the coefficients  $A_{\mathcal{P}}$  and quasi-momenta  $k_j$ , we use the continuity condition, requiring the wavefunction to vanish at the HR boundaries, i.e.  $\Psi(x_1, \dots, x_j, x_j + a, \dots, x_N) = 0$ , for all  $j \in \{1, \dots, N\}$ . It yields the relations  $A_{(j,j+1)\mathcal{P}} = -e^{i\Theta(k_{\mathcal{P}j+1} - k_{\mathcal{P}j})} A_{\mathcal{P}}$ , for any  $j$ , where  $\Theta(k) \equiv ka$  is a phase shift of two-body scattering. These equations are solved by

$$A_{\mathcal{P}} = (-1)^{\mathcal{P}} A \exp \left[ -i \sum_j k_{\mathcal{P}j} (j-1)a \right]. \quad (3)$$

Periodic boundary conditions then yield

$$k_j L = 2\pi I_j + \sum_{\ell \neq j} \Theta(k_j - k_\ell) \quad (4)$$

for any  $j \in \{1, \dots, N\}$ , where the quantum numbers  $I_j$  take integer or half-integer values for odd or even  $N$ , respectively. The set of  $N$  quantum numbers  $\{I_j\}$  characterizes the eigenstate uniquely. Equation (4) is the standard BA equation, where the phase shift  $\Theta(k)$  is determined by the interaction model  $V(x)$ . In general, it is solved numerically but, for HRs, the linear form of  $\Theta(k)$  allows for an explicit solution, and we find

$$k_j = \frac{2\pi I_j - 2\pi I a / L}{L - Na}, \quad I = \sum_j I_j. \quad (5)$$

Note that the dimensionless quantities  $k_j a$  and  $E a^2 m / \hbar^2$  are universal functions of  $a/L$  and  $N$ .

The TG gas, which describes infinitely repulsive, point-like particles, is found in the limit  $a \rightarrow 0$ , and we recover the known solution  $k_j^{\text{TG}} = 2\pi I_j / L$  [46]. In particular, note that the second term in the numerator vanishes. Previous treatment of the HR model relied on the idea that since the BA wavefunction must vanish for  $x_{j+1} - x_j < a$ , one can shift the coordinates as  $x'_j = x_j - (j-1)a$  and exclude the volume occupied by the rods [44]. This maps the HR model onto a modified Tonks-Girardeau (MTG) where the length is rescaled as  $L \rightarrow L - Na$ , and the quasimomenta become  $k_j^{\text{MTG}} = 2\pi I_j / (L - Na)$ . Comparing with our exact solution, Eq. (5), we find that this simplified approach correctly accounts for the rescaling of the effective length in the denominator but not for the shift  $-2\pi I a / L$  in the numerator. As we discuss below, this is sufficient for the ground state, which has  $I = 0$  but in general not for excited states.

The ground state is obtained by minimizing the energy. The quadratic form of the latter implies that the set of quasi-momenta  $k_j$  should be symmetric, and the total momentum  $K = \sum_j k_j = 2\pi I / L$  should vanish, i.e.  $K = 0$  and  $I = 0$ . The quantum numbers  $I_j$  take all integer or half-integer values in the set  $\{-\frac{N-1}{2}, \dots, \frac{N-1}{2}\}$ , hence creating a Fermi-like sea with maximum momentum  $k_{\text{max}} = \pi n / (1 - na)$ , where  $n = N/L$  is the particle density, in the thermodynamic limit. The energy per

particle is then

$$\frac{E_0}{N} \approx \frac{\hbar^2 \pi^2 n^2}{6m(1 - na)^2}. \quad (6)$$

Compared to the TG gas, the quasi-momenta are rescaled as  $k_j = k_j^{\text{TG}} / (1 - na)$  and the energy as  $E_0 = E_0^{\text{TG}} / (1 - na)^2$ . Note, however, that the zero-temperature chemical potential,  $\mu = \partial E_0 / \partial N$ , is rescaled differently, due to the dependence of the energy rescaling on the total particle number  $N$ . One finds

$$\mu = \frac{\hbar^2 \pi^2 n^2}{2m} \times \frac{1 - na/3}{(1 - na)^3}, \quad (7)$$

i.e.  $\mu = \mu^{\text{TG}} \times (1 - na/3)/(1 - na)^3$  with  $\mu^{\text{TG}} = \hbar^2 \pi^2 n^2 / 2m$ .

We now turn to excited states, for which in general  $I$  does not vanish. As usual in BA, elementary excitations are generated by changing one of the  $I_j$ 's, creating two distinct branches. Particle-type excitations (p, upper branch) are created by promoting a particle from the highest populated momentum  $k_N = k_{\text{max}}$  (Fermi level) to a larger value  $q > k_{\text{max}}$  while hole-type excitations (h, lower branch) involve moving a particle from within the range of populated momenta to just above the Fermi level. In both cases, although a single quantum number  $I_j$  is changed, all quasi-momenta are shifted since  $I$  turns from 0 to  $\neq 0$ , see Eq. (5). Although, the shift is infinitesimal in the thermodynamic limit, it plays a crucial role because it affects all the momenta and has a macroscopic impact (see below). This effect is absent in the MTG approach.

An equation for these collective excitation modes may be derived, in the thermodynamic limit, working along the lines of Ref. [47]. Adding a particle or a hole implies adding a quasi-momentum  $\pm q$  and shifting the other quasi-momenta so that  $k \rightarrow k + f_{\text{p,h}}(k)$ . The total momentum for a particle or hole excitation then reads as

$$p_{\text{p,h}} = \pm q + \int_{-k_{\text{max}}}^{k_{\text{max}}} J_{\text{p,h}}(k) dk \quad (8)$$

and the energy as

$$\varepsilon_{\text{p,h}}(p) = \pm \left( \frac{\hbar^2 q^2}{2m} - \mu \right) + \frac{\hbar^2}{2m} \int_{-k_{\text{max}}}^{k_{\text{max}}} 2k J_{\text{p,h}}(k) dk, \quad (9)$$

where the chemical potential  $\mu$  compensates the energy of adding or removing a particle and  $J_{\text{p,h}}(k) = \rho(k) f_{\text{p,h}}(k)$  is the density of quasi-momentum shift, where  $\rho(k)$  is the density of quasi-momenta in the ground state. The latter is determined using Eq. (4), which yields the self-consistent equation

$$2\pi J_{\text{p,h}}(k) = - \int_{-k_{\text{max}}}^{k_{\text{max}}} \mathcal{K}(k - k') J_{\text{p,h}}(k') dk' \mp [\pi + \Theta(q - k)], \quad (10)$$

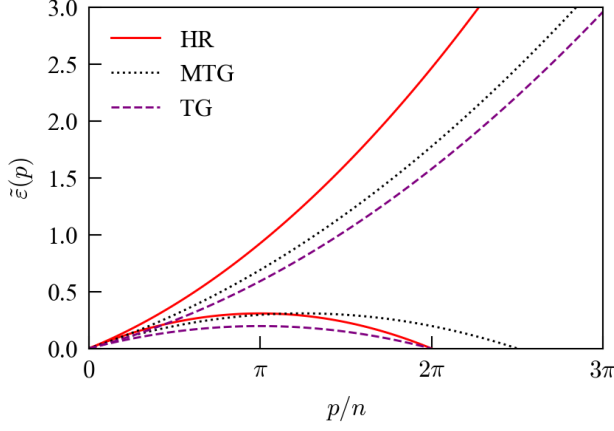


Figure 1. Dispersion relation of elementary particle (upper branch) and hole (lower branch) excitations for the HR (solid red line), TG (dashed purple line), and MTG (dotted black line) models at  $na = 0.2$ .

with  $\mathcal{K}(k) = d\Theta/dk$ . For HRs, the constant kernel,  $\mathcal{K}(k) = a$ , allows again for an analytical solution,

$$J_{p,h}(k) = \mp \frac{(1-na)(qa + \pi) - ka}{2\pi}. \quad (11)$$

Inserting this result in Eqs. (8) and (9), we find the analytical expressions

$$p_{p,h} = \pm(1-na)(q - k_{\max}), \quad (12)$$

and

$$\varepsilon_{p,h}(p) = \frac{\hbar^2 p(2\pi n \pm p)}{2m(1-na)^2}, \quad (13)$$

for the momentum and energy, respectively.

The dispersion relations of the elementary excitations for the HR, TG, and MTG models at  $na = 0.2$  are shown in Fig. 1. All models exhibit qualitatively similar behaviours but significant quantitative differences. The TG model is recovered from Eq. (13) for  $a \rightarrow 0$ , and we find, for HRs, the same energy rescaling  $\varepsilon_{p,h}(p) = \varepsilon_{p,h}^{\text{TG}}(p)/(1-na)^2$  as for the ground state. In particular, the hole branch has a momentum cut-off at the Fermi momentum  $p_F = \pi n$ , for both HR and TG models. In contrast, the MTG approach predicts an erroneous Fermi momentum at  $p = k_{\max} = p_F/(1-na)$ . This may be directly seen from Eq. (5) forcing  $I = 0$ , which yields  $p_{p,h}^{\text{MTG}} = \Delta k = \pm(q - k_{\max})$  and  $\varepsilon_{p,h}^{\text{MTG}}(p) = \pm \frac{\hbar^2}{2m}(q^2 - k_{\max}^2) = \pm \frac{\hbar^2 p}{2m} \left( \frac{2\pi n}{1-na} \pm p \right)$ . Hence the rearrangement of momenta in the HR model plays a crucial role. In fact, it simply appears as the momentum scaling factor  $(1-na)$  in Eq. (12), which allows to recover Eq. (13).

The rearrangement of quasi-momenta also affects the thermal fluctuations, which we consider now. This will allow to determine the thermodynamics of the model

and benchmark our analytic solution against QMC calculations. To write the thermodynamic Bethe ansatz (TBA) of the HR model, we work along the lines of Yang and Yang [48]. The approach is standard and we do not repeat it here. In brief, we define  $\rho(k)$  as the density of filled quasi-momenta and  $\rho_h(k)$  the density of holes, subject to the normalization condition

$$\rho(k) + \rho_h(k) = \frac{1}{2\pi} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{K}(k-k')\rho(k')dk'. \quad (14)$$

Minimizing the free energy written in terms of the dressed energy  $\epsilon(k) \equiv k_B T \ln[\rho_h(k)/\rho(k)]$ , one finds the self-consistent Yang-Yang equation

$$\epsilon(k) = -\mu + \frac{\hbar^2 k^2}{2m} + k_B T \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \mathcal{K}(k-k') \ln \left( 1 + e^{-\epsilon(k')/k_B T} \right). \quad (15)$$

This equation is solved numerically for  $\epsilon(k)$  using an iterative method, and, combining with Eq. (14), it yields  $\rho(k)$  and  $\rho_h(k)$ . For HRs, the kernel  $\mathcal{K}(k) = a$  simplifies the numerical complexity. The particle density  $n = \int_{-\infty}^{\infty} dk \rho(k)$  is found from Eq. (14), which reduces to  $\rho(k) + \rho_h(k) = (1-na)/2\pi$ , leading to

$$na = \frac{\chi_2}{1 + \chi_2}, \quad (16)$$

with  $\chi_2 = a \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{1 + e^{\epsilon(k)/k_B T}}$ . This formula may be used to compute the compressibility using the standard thermodynamic relation,

$$\kappa \equiv \left( \frac{\partial n}{\partial \mu} \right)_T = \frac{(1-na)^3 \chi_3}{k_B T a}, \quad (17)$$

with  $\chi_3 = a \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{\epsilon(k)/k_B T}}{(1 + e^{\epsilon(k)/k_B T})^2}$ .

To validate our TBA solution, we performed path-integral quantum Monte Carlo (QMC) simulations in continuous space. Our implementation works in the grand canonical ensemble with fixed temperature and chemical potential, and worm-algorithm updates [49]. Figure 2 shows (a) the density  $n$  and (b) the compressibility  $\kappa$  as a function of chemical potential for  $k_B T = 0.1\hbar^2/ma^2$ , using QMC for the HRs (purple disks) as well as TBA predictions for HRs [solid red lines, Eqs. (16) and (17)] and the TG model (dashed purple line, same equations for  $a \rightarrow 0$ ). Qualitatively similar results are found for various temperatures. The zero-temperature behavior, predicted by BA, is also shown [dashed black lines, from Eq. (7)]. QMC results show excellent agreement with TBA across all chemical potentials and finite temperatures, confirming the TBA predictions. Moreover, the scaling of QMC results for decreasing temperatures confirm the  $T = 0$  behavior predicted by BA, see inset of Fig. 2(a), from yellow to red curves. At zero temperature, the density exhibits a sharp edge,  $n \propto \sqrt{\mu}$ ,

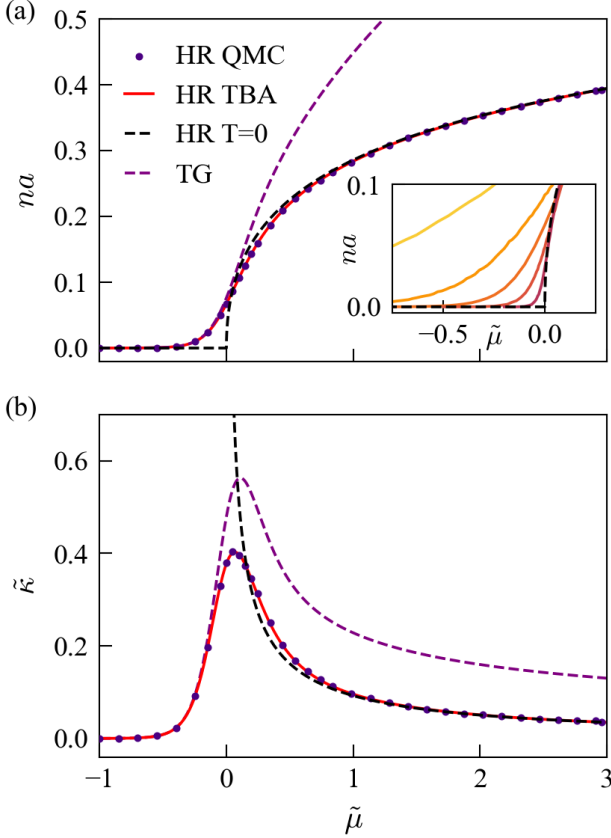


Figure 2. Thermodynamics of HRs. (a) Density and (b) compressibility,  $\tilde{\kappa} = \hbar^2 \kappa / ma$ , versus chemical potential,  $\tilde{\mu} = \mu ma^2 / \hbar^2$ , for the HR and TG models. For HRs, we show TBA predictions (solid red line) and QMC results (purple disks) at  $T = 0.1 \hbar^2 / ma^2 k_B$ , as well as BA predictions at  $T = 0$  (dashed black line). For TG gas, we show TBA predictions (dashed purple line). The inset shows QMC results for  $(k_B ma^2 / \hbar^2) T = 0.5, 0.2, 0.1, 0.05$ , and  $0.02$  (from yellow to red), alongside the zero-temperature prediction. Errorbars are smaller than the size of markers and width of lines.

consistent with Eq. (7) in the limit  $n \rightarrow 0$ . This yields a diverging compressibility  $\kappa \propto 1/\sqrt{\mu}$  around  $\mu \simeq 0$ . Finite-temperature effects smooth out this sharp edge of the density curve and eliminate the compressibility divergence. The latter is replaced at finite temperature by a maximum at finite chemical potential, which gets sharper as the temperature decreases. Comparing these results with those for the TG model [ $a = 0$  and  $K_{TG}(k) = 0$ ], we find that both models exhibit qualitatively similar behaviors but significant quantitative differences. As expected, the TG and HR models agree in the low-density limit  $na \ll 1$ . However, significant deviations occur as the chemical potential increases due to the finite-separation impenetrability of the HR model. Moreover, results not shown here indicate that these deviations are stronger when the temperature increases owing to the different excitation spectra of the HR and TG models. Note that

no simple relation between the HR and TG models exist because the rescaling of energies and chemical potentials are different and depend on density.

Finally, our results allow us to predict (up to numerical prefactors) the behavior of correlation functions assuming LL behavior. For instance, using mode expansion of the LL fields, the asymptotic one-body correlation function  $g_1(x) = \langle \psi^\dagger(x) \psi(0) \rangle / n$  may be written, for  $x \gg n^{-1}$ , as [50]

$$g_1(x) = \sum_{m=0} B_m \frac{\cos(2\pi n m x)}{|nd(x)|^{2m^2 K + 1/2K}}, \quad (18)$$

where  $K$  is the Luttinger parameter and  $d(x) = \frac{L_T}{\pi} \sinh\left(\frac{\pi x}{L_T}\right)$  is the thermal chord distance, with  $L_T = \hbar u / k_B T$  the thermal length and  $u$  the speed of sound. Similar formulas are found for higher-order correlation functions. For HR, the speed of sound may be computed from the exact dispersion relation (13),

$$u = \left. \frac{d\varepsilon}{\hbar dp} \right|_{p=0} = \frac{\pi \hbar n}{m(1-na)^2}, \quad (19)$$

and the LL parameter is given by [51]

$$K = \hbar p_F / mu = (1-na)^2. \quad (20)$$

The nonuniversal (model-dependent) coefficients  $B_m$  are, however, not determined by the theory.

Figure 3(a) shows the complete correlation functions  $g_1(x)$  as computed using QMC for  $na = 0.5$  and increasing temperatures (from purple to yellow). The results are consistent with Eq. (18). In particular, they exhibit oscillations with period  $1/n$  irrespective to temperature and decays that become stronger with increasing temperature, consistent with  $L_T \propto 1/T$ . For  $x \ll L_T$ , we find power-law decay reminiscent of the universal zero-temperature LL behavior, while for  $x \gg L_T$ , we have  $d(x) \sim \exp(\pi x / L_T)$  and exponential decay is found. Note that due to a relative decay factor  $|nd(x)|^{-2K}$ , the oscillations are suppressed with respect to the  $m = 0$  term in Eq. (18) at large distance and high temperature. More precisely, universal algebraic decay (up to oscillations) with respect to  $d(x)$  is expected. This is verified in the inset of Fig. 3(a), which shows  $g_1(x)$  versus  $d(x)$  in log-log scale for the same temperatures as in the main panel, and the dashed line denotes the leading term of Eq. (18), with  $K$  as predicted by Eq. (20) and  $B_0$  as a fitting parameter. We find all curves collapse (up to the oscillations), and agree with the LL predictions.

To further assess the LL behavior, we now proceed with fits of Eq. (18) up to the  $m = 1$  term to the QMC data for various values of  $na$ , with fitting parameters  $K$ ,  $L_T$ ,  $B_0$ , and  $B_1$ . We use a Levenberg-Marquardt least-square approach [52] in two steps. We first fit  $g_1(x)$  and obtain estimates for  $K$ ,  $B_0$ , and  $B_1$ . The thermal length  $L_T$  is, however, poorly estimated because thermal effects are only significant at large enough distance,  $x \gtrsim L_T$ , where



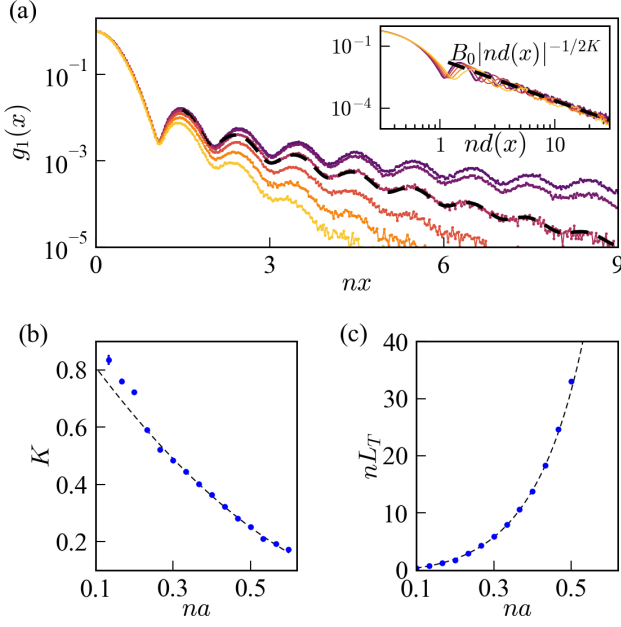


Figure 3. Correlations. (a) One-body correlation function  $g_1(x)$  for  $na = 0.5$  and various temperatures in semi-log scale. Solid lines are QMC results with temperatures  $(k_B m a^2 / \hbar^2) T = 0.1, 0.2, 0.4, 0.6, 0.8, 1.0$  from top (purple) to bottom (yellow), corresponding to  $nL_T = 31.4, 15.7, 7.86, 5.24, 3.93, 3.15$ , respectively. The dashed curve on top of the third result (red curve) shows a fit of Eq. (18) to the corresponding QMC result (see text). The inset shows the same data versus the thermal chord distance  $d(x)$  in a log-log scale, and the dashed line indicates the leading term of LL predictions, Eq. (18), where  $B_0$  is a fitted parameter and  $K$  is computed using Eq. (20). (b) and (c) Fit results of Eq. (18) to QMC results alongside LL predictions using Eqs. (19) and (20) for  $T = 0.1 \hbar^2 / m a^2 k_B$ . Errorbars are smaller than the markers.

the value of the correlation function is small and do not contribute much to the fit. To enhance the weight of the tails, we perform a second fit, now of  $\ln[g_1(x)]$ , using the previous estimates for  $K$ ,  $B_0$ , and  $B_1$  and fitting parameter  $L_T$  only. For all cases, we find excellent agreement between the fitted curves and the QMC data, see example in Fig. (3)(a) (fit: dashed black line, QMC data: solid red line). The results for the parameters  $K$  and  $L_T$  are shown in Figs. (3)(b) and (c), with errorbars estimated by the residual sum of squares. The speed of sound  $u$  can then be extracted knowing the temperature, here  $T = 0.1 \hbar^2 / m a^2 k_B$ . We find very good agreement with LL theory, except for a few points with  $na \lesssim 0.2$ , where the agreement is worse, but still fair with deviations less than 13%. The larger deviation at small values of  $na$  is due to the fact that dilute density leads to weak oscillations, which are further suppressed exponen-

tially by finite temperature. Thus, the correlation function approaches a pure exponential decay proportional to  $\exp(-\pi x / 2 K L_T)$ . The latter depends on  $K L_T$ , rather than  $K$  and  $L_T$  independently, leading to the poor fitting of  $K$ . For  $na \gtrsim 0.5$ , the thermal length is large,  $nL_T \gtrsim 30$ , and thermal effects appear in a range of  $x$  is where the QMC data is dominated by statistical fluctuations. Then, we cannot fit for  $L_T$  but the fits for  $K$  still work well in this regime since they are not affected by the large value of  $L_T$ .

In conclusion, we presented an exact solution for the 1D HR model using BA. Our approach reveals significant corrections to the excitation spectrum and thermodynamic properties, which were previously overlooked by the excluded-volume approach. By comparing our BA solution with QMC calculations for thermodynamic properties and correlation functions, we found excellent agreement, hence validating our theoretical framework. Our findings highlight the universal LL behavior of the HR model across a wide range of parameters, as evidenced by correlation functions. Furthermore, we have shown that the parameters of the Luttinger model  $K$  and  $u$  can be extracted from observation of correlation functions. More precisely, we obtain the thermal length  $L_T = \hbar u / k_B T$ , from which the speed of sound  $u$  can be deduced given the temperature  $T$ . Conversely, the same approach can be used to infer the temperature from the analytical value of the speed of sound. This makes the approach useful for thermometry in experiments.

Rydberg atom systems [53, 54] offer a promising platform to implement accurate approximations of the HR model we have discussed in this paper. The bare van der Waals interactions already provides a sharp decay of interactions as the sixth power of the interatomic distance  $r$ , showing hard core behavior of correlation functions on lattices [55]. This can be further improved using Rydberg dressing of the ground state, with effective interactions showing a plateau with a sharp  $1/r^6$  [56–59] decay or even sharper via the electromagnetically induced transparency [60, 61], and control of the width and height of the plateau controlled via the intensity and detuning of the coupling lasers. The excitation spectrum could then be measured using for instance quench spectroscopy [62–67] as recently implemented in Rydberg atom experiments [68].

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