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Boundary structures and holographic fluids in gravity

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Résumé

Cette thèse est dédiée à l'étude de certains aspects des espaces-temps dynamiques à bord. Une attention particulière est portée sur les bords asymptotiques comme le bord conforme d'AdS ou l'infini nul de l'espace plat. Le bord d'AdS est de genre temps et donc pseudo-Riemannien. Plus généralement nous considèrerons aussi les espaces asymptotiquement localement AdS. Dans ce cas la jauge de Fefferman-Graham nous permet d'établir que l'espace des solutions est paramétrisé par deux quantités de bord : une métrique est un tenseur énergie-impulsion. Nous considèrerons aussi la jauge de Bondi dans laquelle l'espace des solutions est paramétrisé différement, mais nous montrerons qu'elle est équivalente. Le formalisme covariant de l'espace des phases nous permettra de calculer les charges gravitationelles associées aux difféomorphismes résiduels et d'établir qu'elles correspondent à des charges de Noether du bord pour des conditions de bord de Dirichlet. De plus, nous montrerons qu'elles satisfont une algèbre conforme. Dans le cas où la métrique du bord est sourcée, la même charge de Noether n'est plus conservée et la non-conservation est sourcée par la courbure scalaire du bord. En conclusion de la partie AdS, nous considèrerons la limite plate de la jauge de Bondi, qui est non-triviale et non-singulière.

Nous verrons que l'infini nul de l'espace plat est lui décrit par une géométrie de Carroll. Cette dernière apparaît comme la limite ultra-relativiste, ou $c \to 0$, d'une géométrie pseudo-Riemannienne, qui dans notre cas correspond à la géométrie de bord d'AdS, établissant donc que la limite plate dans l'intérieur de l'espace-temps correspond à cette limite ultra-relativiste sur le bord. Nous verrons aussi comment les symétries de la gravité asymptotiquement plate se traduisent par des symétries globales de cette géométrie exotique de bord. Nous montrerons que le groupe BMS correspond simplement aux symmétries conformes de la géométrie de Carroll induite sur le bord des espaces asymptotiquement plat (pour des conditions de bord de Dirichlet bien particulières). Tout comme en AdS, l'espace des solution n'est pas paramétrisé seulement par la géométrie du bord mais aussi par un équivalent Carrollien du tenseur énergie-impulsion. Nous considèrerons aussi les charges gravitationelles BMS qui sont interpretées comme des charges de Noether Carrollienne du bord. De plus, quand la géométrie de Carroll du bord est sourcée, ces mêmes charges sont elles aussi non conservées.

Cette analyse de la structure de bord est d'une importance capitale pour la correspondence fluide/gravité car le fluide vit sur le bord. Dans ce contexte nous imposons des conditions d'intégrabilité sur le fluide du bord qui permettent une resommation de l'expansion aux dérivées en AdS. Ces conditions d'intégrabilité sont intriguantes car elles relient les termes dissipatifs du fluide à des quantités géométriques du bord construites à partir du tenseur de Cotton. Ce dernier permet d'établir si l'espace est asymptotiquement AdS ou seulement aymptotiquement localement AdS. La limite plate produit la notion de fluide Carrollien sur le bord, c'est à dire un fluide couplé à la géométrie de Carroll induite sur le bord de l'espace-temps. L'expansion hydrodynamique de ce fluide se traduit aussi par une expansion aux dérivées dans l'intérieur, ce qui donne une notion de correspondence fluide/gravité en espace plat. Nous verrons que nos conditions d'intégrabilité en AdS admettent une limite plate ce qui donne lieu à une resommation de l'expansion aux dérivées plate.

Un deuxième type de bord que nous étudirons est celui formé par l'horizon d'un trou noir. Ici, un autre genre de correspondence fluide/gravité existe : le paradigme des membranes. A l'aide des outils Carrollien développés pour l'étude des hypersurfaces nulles, nous revisitons ce concept et proposons une interprétation nouvelle des équations de Damour–Navier–Stokes en terme de lois de conservation ultra-relativistes. Nous montrerons que la géométrie induite sur l'horizon est Carrollienne et que la géométrie extrinsèque est décrite à partir d'éléments de courbure Carrolliens. Grâce à cette construction nous établirons que la limite proche-horizon correspond à une limite Carrollien et que les deux équations qui régissent la dynamique de la géométrie extrinsèque de l'horizon sont en fait des lois de conservation Carrolliennes.

1 Introduction

The most important progresses in the study of gravity in the recent years are certainly due to a better understanding of the role of boundaries for dynamical spacetimes. One of the main discoveries being that gravity, in some instances, is holographic [1, 2]. In the sense that the full theory of quantum gravity inside the bulk of the spacetime is expected to be dual to a field theory whose fundamental fields live on the boundary of the spacetime. The main realization of this property is the AdS/CFT correspondence where the spacetime is asymptotically AdS and the boundary theory is a relativistic conformal field theory. Quantum gravity would then be a quantum theory of boundary conditions. This resonates with the so called area-law of the black hole entropy [3], that states that the entropy of a black hole scales with its area rather than its volume. This was further motivated by the Ryu-Takayanagi formula [4] that states that the entanglement entropy associated with a subregion on the boundary, and therefore a subregion in the bulk (the entanglement wedge), is computed by the area of a surface that encloses the bulk subregion. The horizon area law is then only a particular case where one consider the entanglement entropy of the whole boundary of the asymptotically AdS spacetime. All these results suggest that gravity, even when the spacetime is not asymptotically AdS, could be described by the dynamics of its boundary. This is why studying the boundary structure of spacetimes that are asymptotically AdS or flat seems to be essential.

The role of asymptotic boundaries was also studied in classical gravity. This is where the so-called boundary conditions are imposed for the spacetime, which allow for the definition of consistent phase spaces. In classical gravity, it is often said that diffeomorphisms are pure gauge symmetries, this is not quite true when the spacetime possesses boundaries. In that case, some diffeomorphisms are promoted to real symmetries since they actually change the physical state, they are called asymptotic symmetries and have been widely studied [5] (see also [6] and [7] for recent reviews). Roughly speaking, this symmetries correspond to diffeomorphisms that are acting non trivially on the boundary of the spacetime. The asymptotic symmetry group in asymptotically flat gravity was derived by Bondi, Metzner and Sach [8, 9] and consists of an infinite-dimensional extension of the Poincaré group, dubbed the BMS group. The fact that the asymptotic symmetry group is bigger than the symmetries of the vacuum (here Minkowski) can come as a surprise but turns out to be often the case. A three-dimensional version of the BMS group exists also [10].

It is also possible to associate charges to these diffeomorphism through the covariant phase space formalism. In particular, it is thanks to this formalism that Iyer and Wald were able to show that the first law of black hole thermodynamics holds for any stationary black hole in any theory of gravity [11]. It is also using this formalism that Barnich and Troessaert were able to derive charges associated with the BMS symmetries in four dimensions and compute the algebra they satisfy [12]. This algebra is centrally extended but the central extension is field dependent and the interpretation of such a structure is still mysterious. As we said, there exists also a three-dimensional version of the BMS group and the charges were also computed in that case, giving rise to a central extension that is not field dependent anymore [13]. This central extension can be exploited to interpret the entropy of cosmological solutions in flat space in terms of a Cardy formula for BMS symmetry [14].

Interestingly, both in four and three dimensions, the BMS group can be shown to be isomorphic to the conformal isometries of an exotic geometry : a Carroll manifold [15]. In the case of asymptotically flat gravity, it is because the conformal boundary is null, it is the null infinity, and it inherits therefore a non pseudo-Riemannian geometry. Carroll geometry emerges when one takes the $c \to 0$ limit of a relativistic metric, it is the ultra-relativistic counterpart of the Newton-Cartan geometry that appears when the other limit ($c \to \infty$) is taken [16]. When doing so, the space-time equivalence is broken and the geometry is now composed of a spatial metric and a vector field that represents the time direction. On the null infinity, the spatial metric is simply the round metric on the celestial sphere and the time arrow is the null direction. The conformal isometries of this structure are in one-to-one correspondence with the BMS algebra. In this thesis, we show how the geometry induced on the null infinity of asymptotically flat spacetimes can always be interpreted in terms of Carroll geometry, going beyond the simple case of Minkowski. We will also show how the constraint equations of asymptotically flat gravity can be interpreted in terms of ultra-relativistic conservation laws.

The holographic correspondence in AdS has led people to study extensively asymptotically AdS spacetimes. A central result is that any asymptotically AdS metric can be written in a radial gauge called Fefferman–Graham gauge. In this gauge, there are two independent objects : the conformal boundary metric and a second boundary tensor. The latter is related to the extrinsic curvature of the conformal boundary and in the language of AdS/CFT it is the holographic energy-momentum tensor (see [17] for a review on all this). It is also possible to derive the asymptotic symmetry algebra in this gauge. In three dimensions, with the so-called Brown-Henneaux boundary condition [18] they are in one-to-one correspondence with the conformal symmetries of the two-dimensional boundary, i.e. the infinite-dimensional two copies of Virasoro. The gravitational charges realize then a centrally extended version of this conformal symmetry with the same central charge for left and right movers, related to the ratio of the AdS radius and the Newton constant. It is interesting to note that both in asymptotically flat and AdS, the asymptotic symmetries (for Dirichlet-like boundary conditions) are isomorphic to the conformal symmetries of the boundary geometry.

Part of this thesis will be devoted to the study of three-dimensional gravity with the most general boundary conditions. The advantage of working in three dimensions is that the computations are significantly simpler while the key concepts that we want to describe are present, i.e. the relation between the flat limit in the bulk and the Carrollian limit on the boundary. This part will be the occasion to show how the notions of Carroll structure and Carrollian momenta appear on the boundary of asymptotically flat spacetimes, they are the flat space surrogates for the AdS boundary metric and holographic energy-momentum tensor. We will see that they are nicely interpreted as coming from an ultra-relativistic limit that corresponds simply to the flat limit in the bulk. This will set the stage for the following section where the four-dimensional case is considered. To perform a consistent flat limit we describe the AdS solution space in Bondi gauge. After a reparametrization of the solution space, the latter is shown to be in one-to-one correspondence with a boundary metric and energy-momentum tensor. The same procedure is followed for the flat version of the Bondi gauge, where this time the solution space is parametrized by a Carroll structure on the null infinity and a Carrollian equivalent of the holographic energy-momentum tensor, that we dub Carrollian momenta. The gravitational charges are then interpreted in this language.

In AdS, the constraint equations associated with the conformal boundary can be written as the conservation of the holographic energy-momentum tensor. This is a key element of a spin off of gravitational holography : the fluid/gravity correspondence [19]. The main observation is that the dynamics of a relativistic fluid is also entirely contained in the conservation of its energy-momentum tensor. This establishes a relationship between an asymptotically AdS solution and a relativistic fluid living on its boundary. In fluid/gravity correspondence, one trades the usual Fefferman-Graham gauge for the Derivative Expansion, parametrized by the data of a boundary relativistic fluid. Then, assuming slow varying fields, one can map a bulk derivative expansion to the fluid's hydrodynamical expansion (see [20] for a review of the construction). A caveat of the Derivative Expansion is that it is an infinite expansion and one can wonder if it can be resummed, producing a closed line element in the gravity side and equivalently a closed energy-momentum tensor for the dual fluid. It is this aspect of fluid/gravity that we study in this thesis. In particular we give strong evidence in favor of such a resummation under mild conditions on the boundary fluid, dubbed integrability conditions, that translate into a restriction of the possible corresponding gravitational solutions. The advantage of the Derivative Expansion is that it is implemented in a coordinate system that admits a non-singular flat limit as opposed to the Fefferman-Graham gauge for which the flat limit is ill-defined. We show that the relationship between the bulk and the boundary persists in the limit, we trade the Einstein metric for a Ricci-flat metric in the bulk, while the relativistic fluid is replaced by a Carrollian fluid on the null infinity, i.e. a fluid coupled to the Carrollian geometry. We also develop a notion of hydrodynamical expansion for such a fluid. In the bulk it maps to a flat version of the Derivative Expansion, which provides the first example of fluid/gravity correspondence in flat space.

We conclude our journey with an application of the tools developed for the study of asymptotically flat gravity to another spacetime boundary : the horizon of a black hole. More precisely we revisit the membrane paradigm [21, 22, 23, 24, 25, 26] that relates the dynamics of the horizon to the Navier-Stokes equations. We propose a novel interpretation where the near-horizon limit is interpreted as an ultra-relativistic (or Carrollian) limit. We also show that the small mismatches between the Damour equation and the Navier-Stokes equation are due to the fact that the former is actually an ultra-relativistic conservation law rather than a Galilean one.

Comments of the author

This thesis is based on works I have published during the three years of my PhD [27, 28, 29, 30, 31, 32, 33] and on some ongoing works. The way things are presented here does not reflect the true chronology of the research I have conducted with my collaborators. I have started by studying the fluid/gravity correspondence, mainly in four dimensions, looking for these integrability conditions that would allow for a resummation of the Derivative Expansion in AdS. Wondering if this construction would make sense in flat space, we noticed that we could simply (simply only in principle) take the flat limit and that the corresponding limit for the fluid was a Carrollian limit. This observation led us to consider both sides of the non-relativistic limit of fluid dynamics, namely the Galilean and the Carrollian one. This is done in full generality in [27], where we derive the conservation equations for Galilean and Carrollian fluid on generic curved and time-dependent background by taking the corresponding limit of its relativistic counterpart. Having understood how to properly describe an ultra-relativistic fluid we could then consider the holographic one appearing on the boundary of the flat limit of the Derivative Expansion [28], setting a precise duality between a bulk Ricci flat solution and a Carrollian fluid. The integrability conditions also admit a flat limit such that the flat derivative expansion is also resummed.

The duality between Carrollian fluid and Ricci flat solutions led us to rewrite the constraint equations for asymptotically flat solution spaces in terms of conservation equations of a fluid living on the null infinity. These equations have a structure very similar to the constraint equations of the four-dimensional Bondi gauge. In AdS the constraint equations are written as the conservation of an energy-momentum tensor. One can wonder if such an object can be defined in flat space. In [29] we define the counterpart of the boundary energy-momentum tensor for asymptotically flat spacetimes, which abstractly could be seen as the variation of the bulk on-shell action w.r.t. the Carrollian geometry induced on the null infinity. As an example, we study the linearized Bondi gauge. It was shown in [15] that the symmetries of Bondi gauge, i.e. the BMS group, map to conformal Carrollian symmetries on the null infinity. We extend this work to the bulk dynamics and show that the constraint equations of Bondi gauge can be interpreted as conservation laws of our Carrollian energy-momentum tensor. This is reminiscent of what happens in AdS : the boundary energy-momentum tensor is sourced by the boundary geometry and the on-shell dynamics of Fefferman-Graham gauge reduces to its conservation. Finally, we also map the BMS charges to an ultra-relativistic version of the Komar charge on the boundary.

All this work was done in four dimensions and a natural question is if this construction exists in three dimensions. In [30] we build a reconstruction formula that associate a three-dimensional bulk solution, AdS or flat, to a boundary fluid, relativistic or Carrollian. We also study the effect of hydrodynamic frame for two-dimensional fluids, the latter is mapped to a rotation of the Cartan frame on the boundary. Finally we consider the space of bulk solutions whose dual fluid are in a specific hydrodynamical frame, i.e. perfect or pure dissipative. The outcome is that even though the two spaces of fluids are identified, the corresponding bulk solution spaces differ by their charge algebra, giving a sensitivity of the bulk to the hydrodynamic frame.

The Carrollian interpretation of the null infinity is tied with the fact that it is a null hypersurface. From this observation, we had the idea to apply the Carrollian approach to the study of another null hypersurface of physical interest : the horizon of a black hole. In [31] we make use of the notion of a timelike stretched horizon and show that the near-horizon limit can be reinterpreted as an ultra-relativistic limit on this stretched horizon. We show that the null Raychauduri and the Damour-Navier-Stokes equations should be physically interpreted as ultra-relativistic conservation laws. This brings a novel understanding of the membrane paradigm and explains both the small mismatch between the Damour-Navier-Stokes and the actual Navier-Stokes equation, and the large (often overlooked) difference between the null Raychauduri equation and a conventional Galilean conservation of energy. We give also a Carrollian interpretation of the near-horizon symmetries and their associated charges.

In [32] we develop the connection between null hypersurfaces and Carrollian spacetimes with a more mathematical approach. We show that the induced geometry on a null hypersurface is precisely Carrollian. We also find the minimal required condition under which the conformal isometries of such a geometry belong to a BMS-like algebra, generalising the work of [15].

My last published work touches upon another aspect of flat gravity, namely flat holography. If we believe in an eventual holographic duality, our works suggests that the field theory should be conformal and Carrollian. Inspired by this observation and by the recent works on holographic entanglement, in [33] we explore the holographic entanglement in Minkowski spacetime. The authors of [34] had already proposed an equivalent of the Ryu-Takayanagi formula for 3d gravity, matching a bulk geodesic length to the entanglement entropy of a subregion in a dual BMS_3 -invariant field theory. We refine this prescription and further propose a generalisation to 4d. We also show that under some general assumptions on the putative dual, the first law of entanglement is equivalent to the gravitational equations of motion in the bulk, linearised around Minkowski spacetime.

Finally we would like to comment on the structure of this PhD thesis. A choice was made not to rewrite in full details all the papers and results but rather give a self contained introduction to the context and the highlights of my research. The reader will be guided with a selection of review articles when a broader vision could be helpful and of course a clear connection will be maintained between the presentation and the corresponding works on which it is based.

2 Minkowski spacetime and Carrollian geometry

In this section we give an introduction to the Minkowski spacetime, its conformal boundary and its symmetries. We show how the Poincaré group maps to a subset of the conformal symmetries of the geometry on the null infinity, the full set being actually isomorphic to the BMS group. We show that the geometry induced on the null infinity corresponds to the simplest Carroll manifold. When gravity is turned on, we expect the Carroll geometry induced on the null infinity of the asymptotically flat solution to be non trivial. This is why we introduce the reader to the notion of Carroll structure that we then exemplify with the null infinity of the Kerr black hole.

2.1 Minkowski : causal structure and symmetries

We start by introducing the simplest solution to Einstein equations with vanishing cosmological constant : the Minkowski spacetime. We will not provide an exhaustive study of the properties of this spacetime but we will give the necessary ingredients for what will follow next. The Minkowski spacetime has vanishing Riemann tensor

$$R^a_{bcd} = 0, \tag{1}$$

in that sense it is the most "flat" spacetime one could define. In Cartesian coordinates the metric is simply

$$ds^2 = -dt^2 + \delta_{ij}dx^i dx^j = \eta_{ab}dx^a dx^b, \tag{2}$$

where $a = 1, .., D$. Its isometry group is the Poincaré group $\mathrm{ISO}(1, D-1)$ and is realized through the coordinate transformations

$$x'^a = \Lambda^a_b x^b + p^a, \tag{3}$$

where Λ^a_b is a Lorentz boost and p^a is a spacetime translation. It is easy to check that this coordinate transformation leaves the metric invariant, it follows from the fact that ${}^t\Lambda\eta\Lambda = \eta$.

In Cartesian coordinates, the geodesics are simply straight lines and describe trajectories followed by free falling objects. To have a better understanding of the causal structure of this spacetime and how physical objects freely travel inside, it is useful to draw its Penrose diagram. First we introduce a new set of coordinates adapted to the study of Minkowski's asymptotic regions, we define the retarded time and the advanced time

$$\begin{aligned} u &= t - r, \\ v &= t + r, \end{aligned} \tag{4}$$

so that the metric becomes $ds^2 = -dudv + r^2 d\Omega_d^2$, where we have defined the metric of the d-sphere such that $d = D - 2$. This allows to define the various asymptotic regions of Minkowski :

— $i^+ = \{t \to +\infty$ at fixed $r\}$ = future timelike infinity,
— $i^- = \{t \to -\infty$ at fixed $r\}$ = past timelike infinity,
— $i^0 = \{r \to \infty$ at fixed $t\}$ = spacelike infinity,
— $\mathcal{I}^+ = \{v \to +\infty$ at fixed $u\}$ = future null infinity,
— $\mathcal{I}^- = \{u \to -\infty$ at fixed $v\}$ = past null infinity.

Trajectories are characterized by where they start and end (when fully extended). A timelike trajectory will start at i^- and end at i^+, a lightlike one will start at $\mathcal{I}^-$ and end at $\mathcal{I}^+$ and finally a spacelike one will start at i^0 and come back to i^0. There are subtleties when $D = 2$ because there is no angle and we can define right and left asymptotic regions but we will not discuss this degenerate case. The representation of these regions become very clear when we compactify the coordinates u and v, therefore we define ψ and ζ as

$$\begin{aligned} v &= \tan\frac{1}{2}(\psi + \zeta), \\ u &= \tan\frac{1}{2}(\psi - \zeta), \end{aligned} \tag{5}$$

so that the metric becomes

$$ds^2 = \Omega(\psi, \zeta)^2(-d\psi^2 + d\zeta^2) + r(\psi, \zeta)^2 d\Omega_d^2, \tag{6}$$

with $\Omega(\psi, \zeta)^{-2} = 4\cos^2\frac{1}{2}(\psi + \zeta)\cos^2\frac{1}{2}(\psi - \zeta)$. The new coordinates range over the half-diamond $\zeta \pm \psi < \pi$, $\zeta > 0$. It is now very easy to read the causal structure of the Minkowski spacetime reported in Fig. 1.

We come back to Minkowski written in the coordinates (u, r), we will see later that this coordinate system is adapted to the study of more generic asymptotically flat spacetimes, from now on we will refer to them as Bondi coordinates. We would like to understand how to write the Poincaré group in these coordinates, this will be useful for the interpretation of the BMS group. In the following we will focus on the 4-dimensional case and later we will give the same results for the 3-dimensional case.

To study the symmetries in this new coordinate system it is easier to consider the infinitesimal version of the problem, i.e. we are looking for vector fields satisfying

$$\mathcal{L}_\xi g_{ab} = 0, \tag{7}$$

for the metric written in Bondi coordinates. The solution is

$$\begin{aligned} \xi^u &= T(\theta, \phi) + \frac{u}{2} D_A Y^A, \\ \xi^r &= -\frac{(r+u)}{2} D_C Y^C + \frac{D_C D^C T}{2}, \\ \xi^A &= \left(1 + \frac{u}{r}\right) Y^A(\theta, \phi) - \frac{D^A T}{r}, \end{aligned} \tag{8}$$

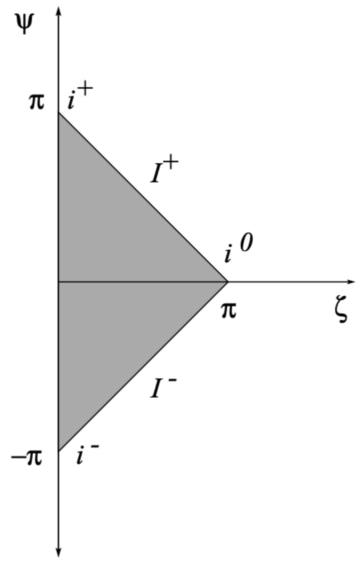

FIGURE 1: Penrose diagram of Minkowski, the asymptotic regions are now at finite distance. Each point in the shaded region is actually a d-sphere. Null geodesics are represented by 45^o lines.

where $x^A = \{\theta, \phi\}$ and D_A is the covariant derivative of the round two-sphere. The functions $T(\theta, \phi)$, $Y^\theta(\theta, \phi)$ and $Y^\phi(\theta, \phi)$ must satisfy certain constraints for the corresponding vector field ξ to be a Killing. Indeed, we know that the Poincaré group has 10 generators composed of 6 boosts and 4 translations and this should not change when we change coordinates. We will first spell out the conditions and then make the connection with our usual representation of the Poincaré algebra. The first set of conditions is that the vector $Y = Y^\theta \partial_\theta + Y^\phi \partial_\phi$ must be a global conformal Killing vector of the 2-sphere. There are 6 of them and they are known to satisfy an $\mathrm{SL}(2, \mathbb{C})$ algebra. The other set of conditions is that the function T, when expanded in spherical harmonics, must turn on only the first 4 harmonics

$$T(\theta, \phi) = \sum_{\ell=0,1} \sum_{m=-\ell}^{\ell} T_{\ell m} Y_{\ell m}(\theta, \phi), \quad \text{with} \quad T_{\ell -m} = (-1)^m T^*_{\ell m}. \tag{9}$$

All this is reassuring as we obtain the right amount of generators. We can go further and identify the generators of boosts and translations. Indeed let's recall that the conformal algebra on the 2-sphere is isomorphic to the Lorentz algebra : we write $Y = Y^{ab} M_{ab}$, with a and b antisymmetrized, $a = 0, .., 3$, such that the M_{ab}'s satisfy

$$[M_{ab}, M_{cd}] = \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}. \tag{10}$$

M_{ab} corresponds to the boost in the (x^a, x^b) plane. In spherical coordinates they become [1]

$$\begin{aligned}
M_{12} &= \partial_\phi, \quad M_{30} = \sin\theta \partial_\theta, \\
M_\pm &= \pm i M_{23} + M_{13}, \\
K_\pm &= \mp i M_{20} + M_{10}, \\
M_\pm &= e^{\pm i\phi} (\partial_\theta \pm i \cot\theta \partial_\phi), \\
K_\pm &= e^{\pm i\phi} (-\cos\theta \partial_\theta \mp i \sin^{-1}\theta \partial_\phi).
\end{aligned} \tag{11}$$

1. For example, to write the generator of a boost in the (t, z)-plane but in Bondi coordinates, one can just consider the vector field (8) and replace T by zero and Y by M_{30}.

It is also easy to find that the generator of a translation $\xi = p^a\partial_a$ maps in Bondi coordinates to the the vector (8) with $Y = 0$ and

$$T(\theta,\phi) = p^0 + p^1\sin\theta\cos\phi + p^2\sin\theta\sin\phi + p^3\cos\theta, \tag{12}$$

which indeed turns on only the first 4 harmonics of the sphere.

The bottom line of this little rewriting is that we have actually built a representation of the Poincaré group leaving on the null infinity, i.e. the asymptotic region $\mathcal{I}^+ = \{r \to +\infty \text{ at fixed } u\}$. The latter is parametrized by the retarded time u and the angular coordinates and its topology is $\mathbb{R}\times S^2$ (see Fig. 1). Indeed consider the $r \to \infty$ limit of the vector (8), it becomes

$$\xi_{T,Y} = \Big(T(\theta,\phi) + \frac{u}{2}D_AY^A\Big)\partial_u + Y^A\partial_A. \tag{13}$$

This is a vector leaving on $\mathcal{I}^+$, uniquely defined by the couple (T,Y) and we know that it is in one to one correspondence with the generators of the Poincaré algebra when we impose the two conditions stated above. We can compute their commutator

$$[\xi_{T_1,Y_1}, \xi_{T_2,Y_2}]_{\text{Lie}} = \xi_{T_{12},Y_{12}}, \tag{14}$$

where $T_{12} = Y_1^A\partial_AT_2 - Y_2^A\partial_AT_1 - \frac{1}{2}T_2D_AY_1^A + \frac{1}{2}T_1D_AY_2^A$ and $Y_{12}^A = Y_1^B\partial_BY_2^A - Y_2^B\partial_BY_1^A$. We now define the modes $\mathcal{M}_{\mu\nu}$ and $\mathcal{P}^\nu$ in the following way

$$\mathcal{P}^0 = \xi_{T=1,Y=0}, \quad \mathcal{P}^1 = \xi_{T=\sin\theta\cos\phi,Y=0}, \quad \mathcal{P}^2 = \xi_{T=\sin\theta\sin\phi,Y=0}, \quad \mathcal{P}^3 = \xi_{T=\cos\theta,Y=0}, \tag{15}$$

and $\mathcal{M}_{ab} = \xi_{T=0,Y=M_{ab}}$. One can finally check that the algebra (14) becomes the usual Poincaré algebra

$$\begin{aligned}
&[\mathcal{M}_{ab},\mathcal{M}_{cd}] = \eta_{ad}\mathcal{M}_{bc} + \eta_{bc}\mathcal{M}_{ad} - \eta_{ac}\mathcal{M}_{bd} - \eta_{bd}\mathcal{M}_{ac},\\
&[\mathcal{M}_{ab},\mathcal{P}_c] = \eta_{ac}\mathcal{P}_b - \eta_{bc}P_a,\\
&[\mathcal{P}_a,\mathcal{P}_b] = 0.
\end{aligned} \tag{16}$$

2.2 A first encounter with the BMS group

The Penrose diagram of Minkowski indicates that $\mathcal{I}^+$ is part of the conformal boundary of Minkowski, in that sense, the map we have found associates a "boundary" transformation to a "bulk" isometry. Now one could wonder if the Poincaré algebra can be obtained as a symmetry of the null infinity itself. In other words : is there a geometrical structure, defined intrinsically on the null infinity, whose symmetries are given by the vectors (14) ? For example, we know that the conformal group can be defined as the isometries of the AdS spacetime in one dimension higher or simply as the conformal symmetries of its boundary. This indicates that we are looking for a definition involving conformal symmetries. Indeed, defining

$$\vec{v} = \partial_u \quad \text{and} \quad \gamma = \gamma_{AB}dx^Adx^B, \tag{17}$$

where γ_{AB} is the round metric on the 2-sphere, it is easy to show that the vectors (14) satisfy

$$\mathcal{L}_{\xi_{T,Y}}\vec{v} = -\sigma\vec{v} \quad \text{and} \quad \mathcal{L}_{\xi_{T,Y}}\gamma = 2\sigma\gamma. \tag{18}$$

Taking the trace of the second equation gives $\sigma = \frac{1}{2}\nabla_A\xi^A$. This suggest that the Poincaré algebra is obtained as the conformal symmetries of the triple $(\mathcal{I}^+,\vec{v},\gamma)$. This is not quite the case, because there are actually an infinite amount of vectors that satisfy (18) and they are given by all the vectors (14) but where T is now *any* function on S^2. We conclude that the algebra satisfying the two conditions (18) is not the Poincaré algebra but the algebra of an infinite dimensional group. This group is the semidirect product of the conformal transformations of the 2-sphere (or $\mathrm{SL}(2,\mathbb{C})$) with the abelian group of *supertranslations* associated with the function T that can now turn any harmonics of the sphere on (it is a generalization of a spacetime translation). More formally, the group is

$$\mathrm{SL}(2,\mathbb{C}) \ltimes \mathcal{C}^\infty(S^2), \tag{19}$$

and can be represented by its action on $\mathcal{I}^+$. To do so it is easier to parametrize the sphere with complex coordinates, indeed consider the following change of coordinate

$$z = \cot\frac{\theta}{2}e^{i\phi}, \tag{20}$$

the round metric becomes

$$\gamma = d\theta^2 + \sin^2\theta d\phi^2 = \frac{4dzd\bar{z}}{(1+z\bar{z})^2}. \tag{21}$$

With this parametrization the group action becomes

$$\begin{aligned} u' &= K(z,\bar{z})(u + T(z,\bar{z})), \\ z' &= \frac{az+b}{cz+d}, \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}), \end{aligned} \tag{22}$$

and $K(z,\bar{z})$ is the conformal factor that the sphere metric receives under the transformation

$$\frac{4dz'd\bar{z}'}{(1+z'\bar{z}')^2} = K(z,\bar{z})^2\frac{4dzd\bar{z}}{(1+z\bar{z})^2}, \quad \text{with} \quad K(z,\bar{z}) = \frac{1+z\bar{z}}{(az+b)(\bar{a}\bar{z}+\bar{b})+(cz+d)(\bar{c}\bar{z}+\bar{d})}. \tag{23}$$

The reason we are spending time describing this group, which is a priori not related to pure Minkowski, is because it is actually the symmetry group of asymptotically flat gravity, the so-called Bondi–Metzner–Sachs (BMS) group. Indeed when gravity is turned on, one can define a proper phase space for solutions that are asymptotically flat, i.e. that look like Minkowski far from the source. This is achieved by imposing boundary conditions at infinity on the metric. When doing so one can show that it is possible to associate non trivial charges to a particular subset of the diffeomorphisms : the ones that have a non trivial action on the conformal infinity and they define the *Asymptotic Symmetry Group*. We will make all these concepts more precise later in this work.

The initial goal was to find a definition of the Poincaré group that would be intrinsic to the "boundary" of Minkowski, exactly like in AdS where the isometries are also the conformal symmetries of the boundary. The natural geometrical structure to consider on $\mathcal{I}^+$ seems to be this couple $(\vec{v},\gamma)$, where $\vec{v}$ points towards the null direction and γ is a metric on the base. Here the round metric on the celestial sphere. The reader should bear in mind that $\mathcal{I}^+$ is a null hypersurface, therefore the induced metric is degenerate and does not carry any information about the null directions. When looking for the conformal symmetries of this new structure we find a bigger group than Poincaré, this suggests that Poincaré arises when considering a stronger structure on $\mathcal{I}^+$. [2] But this was not useless because, surprisingly, we ended up defining the asymptotic symmetry group of gravity in flat space.

The triple $(\mathcal{I}^+,\vec{v},\gamma)$, with the property $\gamma(\vec{v}) = 0$, is a well-known geometrical structure called a *Carroll* manifold (in its weak version), see [16], it arises when considering the ultra-relativistic limit a pseudo-Riemannian structure, i.e. the $c \to 0$ limit of a relativistic metric. Interestingly, it sets a connection between the symmetries of asymptotically flat gravity and the conformal symmetries of an ultra-relativistic spacetime in one dimension less.

2.3 Ultra-relativistic limit and Carrollian geometry

We would like to make this connection a little more precise. Consider a $(d+1)$-dimensional relativistic spacetime (or equivalently $(D-1)$-dimensional) endowed with a pseudo-Riemannian metric,

$$ds^2 = (-\tau_\mu\tau_\nu + \lambda_{\mu\nu})dx^\mu dx^\nu, \tag{24}$$

2. The answer to this question seems to lie in the tractor formalism, see [35].

where $x^\mu = x^0, x^A$ and $\lambda_{0\mu} = 0$ such that the metric is indeed parametrized by $\frac{(d+1)(d+2)}{2}$ functions. We have in mind that the ultra-relativistic limit of this boundary metric should lead to the Carrollian geometry on the null infinity, even though our result is quite generic and does not apply only to the null infinity. Before taking the $c \to 0$ limit, we make the natural change of coordinates $x^0 = cu$ and the following assumptions

$$\begin{aligned} \tau_0(cu, x^A) &\underset{c\to 0}{\to} \Omega(u, x^A), \\ c^{-1}\tau_A(cu, x^A) &\underset{c\to 0}{\to} -b_A(u, x^A), \\ \lambda_{AB}(cu, x^A) &\underset{c\to 0}{\to} h_{AB}(u, x^A), \\ \xi^0(cu, x^A) &\underset{c\to 0}{\to} c\,\xi^u(u, x^A) \end{aligned} \tag{25}$$

Moreover, let ξ be a conformal Killing of our relativistic metric, it satisfies the equation

$$\mathcal{L}_\xi g_{\mu\nu} = 2\sigma g_{\mu\nu}, \tag{26}$$

we would like to take the $c \to 0$ limit of this equation. Making use of the scaling in c introduced for τ and γ we can compute the leading orders of each component of this equation to obtain its ultra-relativistic counterpart

$$\begin{aligned} &\xi^u\partial_u\Omega + \Omega\partial_u\xi^u + \xi^A\partial_A\Omega = \sigma\Omega, \\ &\partial_u\xi^A = 0, \\ &\xi^u\partial_u h_{AB} + \xi^C\partial_C h_{AB} + h_{AC}\partial_B\xi^C + h_{BC}\partial_A\xi^C = 2\sigma h_{AB}. \end{aligned} \tag{27}$$

These equations are not very enlightening, but with a little bit of algebra, it is easy to show that they are equivalent to

$$\mathcal{L}_\xi\vec{v} = -\sigma\vec{v} \quad \text{and} \quad \mathcal{L}_\xi h = 2\sigma h, \tag{28}$$

where $\vec{v} = \Omega^{-1}\partial_u$ and $h = 0\cdot du^2 + 0\cdot du dx^A + h_{AB}dx^A dx^B$, such that $h(\vec{v}) = 0$. Therefore taking the $c \to 0$ limit of the conformal Killing equations lead to the defining equations of the conformal symmetries of the Carroll manifold $(\vec{v}, h)$. From now on we will call such symmetries *conformal Carrollian Killing* (CCK). One should note that the scalings (25) we have imposed were important, indeed changing them could lead to another ultra-relativistic limit, we will see that this choice of scaling is relevant for the gravitational systems we want to study.

Two cases happen to be interesting for us right now, the first one is when the relativistic spacetime is $\mathbb{R}\times S^1$ endowed with the metric

$$ds^2 = -(dx^0)^2 + d\theta^2. \tag{29}$$

It corresponds to the choice $\Omega = 1$ and $h_{\theta\theta} = 1$. Taking the $c \to 0$ limit we obtain the corresponding CCK's

$$\xi_{T,Y} = (\partial_\theta Y u + T(\theta))\,\partial_u + Y(\theta)\partial_\theta, \tag{30}$$

where Y and T are any functions on the circle. This algebra corresponds to the CCK's of the Carroll manifold $(\vec{v} = \partial_u, h = d\theta^2)$. This is again an infinite-dimensional algebra, more precisely it is the Lie algebra of the group

$$\mathrm{Diff}(S^1) \ltimes \mathcal{C}^\infty(S^1). \tag{31}$$

This group acts on the coordinates u and θ in the following way

$$\begin{aligned} u' &= \partial_\theta f(u + T(\theta)), \\ \theta' &= f(\theta), \end{aligned} \tag{32}$$

where f is a diffeomorphism of the circle and T a function on the circle. The structure of this group remind us of course the structure of the BMS group (22), this is not a coincidence because this is actually the asymptotic symmetry group for flat space gravity in 3 dimensions, with a particular choice of Dirichlet boundary conditions (we will see later that the geometry induced on the null infinity in that case is exactly the Carroll geometry $(\partial_u, d\theta^2)$). It contains also the 3-dimensional Poincaré group as a subgroup but we will come back to this later when it will be properly defined in the covariant phase space formalism. From now on we will make the difference between the symmetries of flat space gravity in 3d and 4d by calling them BMS$_3$ and BMS$_4$.

The second case is when the relativistic spacetime is now $\mathbb{R} \times S^2$, endowed with the metric

$$ds^2 = -(dx^0)^2 + \gamma_{AB} dx^A dx^B, \tag{33}$$

where we remind that γ_{AB} is the round metric on S^2. It corresponds to the choice $\Omega = 1$ and $h_{AB} = \gamma_{AB}$. Again, taking the $c \to 0$ limit we obtain the corresponding CCK's

$$\xi_{T,Y} = \Big(\frac{u}{2} D_A Y^A + T(\theta, \phi)\Big) \partial_u + Y^A \partial_A, \tag{34}$$

where Y is a conformal Killing of the sphere and T any function on the sphere. We have obtained again the BMS$_4$ algebra. This is not a surprise because they correspond to the CCK's of the Carroll manifold $(\vec{v} = \partial_u, \gamma)$ as explained in the previous section. Something maybe less trivial is that we started with a 3-dimensional relativistic metric whose conformal algebra is finite-dimensional but taking the ultra-relativistic limit of the defining equations we obtain an infinite-dimensional algebra of CCK's. This is not inconsistent, the solution space of the zero-c limit of the defining equations (27) can be bigger that the zero-c limit of the defining equations' solution space.

2.4 Zero-c limit on the boundary as a flat limit in the bulk

Before turning our attention to the properties of ultra-relativistic geometry, it is worth noting how this Carrollian geometry induced on the null infinity of Minkowski can be seen as arising from the flat limit of an AdS spacetime's boundary geometry. The conformal boundary of AdS is timelike, therefore it is endowed with a relativistic metric. Indeed, consider AdS in the global coordinates

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{\ell^2}\right)} + r^2 \gamma_{AB} dx^A dx^B, \tag{35}$$

where ℓ is the radius of AdS. We change to Eddington-Finkelstein coordinates

$$u = t - \ell \arctan \frac{r}{\ell}, \tag{36}$$

such that the metric becomes

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) du^2 - 2du dr + r^2 \gamma_{AB} dx^A dx^B. \tag{37}$$

Taking the $\ell \to \infty$ limit of this metric we recover Minkowski in Bondi coordinates. Let's have a look at the effect of this limit on the conformal boundary, situated at $r \to \infty$. It's topology is $\mathbb{R} \times S^2$ and it is endowed with the metric

$$ds^2_\partial = -\frac{du^2}{\ell^2} + \gamma_{AB} dx^A dx^B. \tag{38}$$

This corresponds exactly to the second case described in the previous section with the identification $c = \ell^{-1}$. Taking the flat limit in the bulk leads to Minkowski, while the relativistic boundary of AdS is sent to the Carrollian geometry leaving on the null infinity. We will come back to this later.

2.5 More on Carrollian geometry

The structure we have encountered on $\mathcal{I}^+$ is called a Carroll manifold in its weak version. Its definition requires a triple $(\mathcal{M}, g, \vec{v})$, where $\mathcal{M}$ is a smooth $(d+1)$-dimensional manifold, endowed with a twice-symmetric covariant, positive, tensor field g, whose kernel is generated by the nowhere vanishing, complete vector field $\vec{v}$. In the case of the null infinity, the manifold is simply (a piece of) the conformal boundary of Minkowski, g is the metric on the celestial sphere and $\vec{v}$ is the null generator of $\mathcal{I}^+$. The metric g being degenerate, it does not provide an isomorphism between vectors and forms, its kernel is spanned by $\vec{v}$, the latter being the "time" direction, therefore g is a good metric only for spatial directions. There is also a strong definition which requires the addition of a torsionless connection ∇ that parallel-transports both g and $\vec{v}$. The simplest example of strong Carroll manifold is

$$\mathcal{M} = \mathbb{R}^{d+1}, \quad \vec{v} = \partial_t, \quad g = \delta_{AB} dx^A dx^B \quad \text{and} \quad \Gamma^\mu_{\nu\rho} = 0, \tag{39}$$

where $\mathcal{M}$ is parametrized by $x^\mu = t, x^A$. This is the "flat" Carroll manifold. One can look for its symmetries, i.e. the coordinate changes that leave $\vec{v}$, g and ∇ invariant. This results in the following group

$$\begin{aligned} t' &= t + \vec{\beta} \cdot \vec{x} + t_0, \\ \vec{x}' &= R\vec{x} + \vec{x}_0, \end{aligned} \tag{40}$$

where R is a spatial rotation. These coordinate changes define the Carroll group (introduced for the first time in [36]), they allow for spacetime translations, rotations in space and boosts in time only. Carrollian observers are dual to Galilean observers in the sense that for them space is absolute (up to rotations) rather than time. Actually, these coordinate changes can be obtained as the $c \to 0$ contraction of the Poincaré ones.

A big difference between the strong Carroll manifold and the Riemannian manifold is that the conditions on the connection do not uniquely fix it. We would like to study the space of allowed connections, but to do so we are going to assume that there exists an additional structure on our Carroll manifold : an Ehresmann connection. We will see later how this object naturally appears on the null infinity.

Let's call V the subbundle of $T\mathcal{M}$ described by the nowhere vanishing $\vec{v}$. An Ehresmann connection on $\mathcal{M}$ is a smooth subbundle H of $T\mathcal{M}$, called the horizontal bundle of the connection, which is complementary to V, in the sense that it defines a direct sum decomposition $T\mathcal{M} = H \oplus V$. In more detail, the horizontal bundle has the following properties.

— For each point $x \in \mathcal{M}$, H_x is a vector subspace of the tangent space $T_x\mathcal{M}$ to $\mathcal{M}$ at x, called the horizontal subspace of the connection at x.
— H_x depends smoothly on x.
— For each $x \in \mathcal{M}$, $H_x \cap V_x = \{0\}$.
— Any tangent vector in $T_x\mathcal{M}$ (for any $x \in \mathcal{M}$) is the sum of a horizontal and vertical component, so that $T_x\mathcal{M} = H_x + V_x$.

Throughout this thesis, a Carroll manifold equipped with an Ehresmann connection will be called a *Carroll structure*. This is the abstract definition, but more concretely it is equivalent to the knowledge of a projector p such that $\mathrm{Im}(p) = V$, $p^2 = p$. The horizontal subbundle is now simply $H = \mathrm{Ker}(p)$. It is easy to see that this projector should be written as

$$p = \tau \otimes \vec{v} \quad \text{such that} \quad \tau(\vec{v}) = \pm 1. \tag{41}$$

An Ehresmann connection is then specified by a one-form τ, satisfying $\tau(\vec{v}) = \pm 1$. We are going to make use of this one form to build connections on our Carroll manifold. The null infinity is a simple example, the vertical bundle is given by the span of ∂_u, while a basis for the horizontal bundle is simply $\{\partial_\theta, \partial_\phi\}$, we are going to see other examples where the splitting is less trivial. With the one-form τ we can define a pseudo-inverse metric $g^{\mu\nu}$ uniquely specified by the relations

$$\begin{aligned} g^{\mu\nu}\tau_\mu &= 0, \\ g^{\mu\nu} g_{\nu\rho} &= \delta^\mu_\rho + \tau_\rho v^\mu, \end{aligned} \tag{42}$$

where we have made the choice $\tau_\rho v^\rho = -1$. The set of relations satisfied by our fields is now

$$v^\mu \tau_\mu = -1, \quad g_{\mu\nu} v^\mu = 0, \quad g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu + \tau_\mu v^\rho \quad \text{and} \quad g^{\mu\nu}\tau_\nu = 0 \tag{43}$$

We consider now the connection ∇ on $T\mathcal{M}$ (we are now following [37, 38]), we have

$$\nabla_{\partial_\mu} \partial_\nu = \Gamma^\rho_{\mu\nu} \partial_\rho. \tag{44}$$

We consider first the conditions of parallel-transport of $\vec{v}$ and g. The metricity becomes

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\rho\sigma} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0. \tag{45}$$

by permuting the indices, summing and projecting on v^ν we obtain

$$(\mathcal{L}_{\vec{v}} g)_{\mu\rho} = v^\nu \left(T^\sigma_{\nu\mu} g_{\sigma\rho} + T^\sigma_{\nu\rho} g_{\sigma\mu} \right), \tag{46}$$

with

$$\begin{aligned} &T^\rho_{\mu\nu} = 2\Gamma^\rho_{[\mu\nu]}, \\ &(\mathcal{L}_{\vec{v}} g)_{\mu\rho} = v^\rho \left(\partial_\rho g_{\mu\nu} - \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu} \right). \end{aligned} \tag{47}$$

The tensor T is of course the torsion. The relation (46) implies

$$T^\mu_{\nu\sigma} = -\tau_{[\nu} (\mathcal{L}_{\vec{v}} g)_{\sigma]\rho} g^{\mu\rho} + 2X^\mu_{[\nu\sigma]}, \tag{48}$$

where $X^\mu_{[\nu\sigma]}$ satisfies

$$v^\nu X^\sigma_{[\nu\mu]} g_{\sigma\rho} + v^\nu X^\sigma_{[\nu\rho]} g_{\sigma\mu} = 0. \tag{49}$$

A useful relation is $v^\mu (\mathcal{L}_{\vec{v}} g)_{\mu\nu} = 0$. From (45) by permuting the indices, and summing but without projecting we obtain

$$\Gamma^\lambda_{\mu\rho} = -\Gamma^\sigma_{\mu\rho} v^\lambda \tau_\sigma + \frac{1}{2} g^{\nu\lambda} \left(\partial_\mu g_{\nu\rho} + \partial_\rho g_{\mu\nu} - \partial_\nu g_{\rho\mu} \right) + A^\lambda_{\mu\rho}, \tag{50}$$

with

$$A^\lambda_{\mu\rho} = \frac{1}{2} g^{\nu\lambda} \tau_\rho (\mathcal{L}_{\vec{v}} g)_{\mu\nu} + \frac{1}{2} g^{\nu\lambda} \left(X^\sigma_{[\mu\rho]} g_{\nu\sigma} - X^\sigma_{[\mu\nu]} g_{\sigma\rho} - X^\sigma_{[\rho\nu]} g_{\sigma\mu} \right). \tag{51}$$

We notice from (46) that the torsionless condition imposes immediately $\mathcal{L}_{\vec{v}} g = 0$, which also imposes $X^\sigma_{[\mu\nu]}$ to be zero using (48), so in the torsionless case, $A^\mu_{\nu\rho} = 0$. We would like now to impose the parallel-transport of $\vec{v}$. One can show that we already have

$$(\delta^\lambda_\rho + v^\lambda \tau_\rho)(\partial_\mu v^\rho + \Gamma^\rho_{\mu\nu} v^\nu) = 0. \tag{52}$$

Therefore the condition to parallel-transport $\vec{v}$ becomes

$$\tau_\rho (\partial_\mu v^\rho + \Gamma^\rho_{\mu\nu} v^\nu) = -(\partial_\mu \tau_\nu - \Gamma^\rho_{\mu\nu} \tau_\rho) v^\nu = 0. \tag{53}$$

This is solved by $\Gamma^\rho_{\mu\nu} \tau_\rho = \partial_\mu \tau_\nu + X_{\mu\nu}$ with $X_{\mu\nu}$ satisfying $v^\rho X_{\rho\nu} = 0$. Finally we obtain that the connection that parallel-transports $\vec{v}$ and g is written as

$$\begin{aligned} \Gamma^\lambda_{\mu\rho} = &-v^\lambda \partial_\mu \tau_\rho + \frac{1}{2} g^{\nu\lambda} \left(\partial_\mu g_{\nu\rho} + \partial_\rho g_{\mu\nu} - \partial_\nu g_{\rho\mu} \right) + \frac{1}{2} g^{\nu\lambda} \tau_\rho (\mathcal{L}_{\vec{v}} g)_{\mu\nu} \\ &- v^\lambda X_{\mu\rho} + \frac{1}{2} g^{\nu\lambda} Y_{\lambda\mu\rho}. \end{aligned} \tag{54}$$

with

$$Y_{\nu\mu\rho} = 2(X^\sigma_{[\mu\rho]} g_{\nu\sigma} + X^\sigma_{[\nu\mu]} g_{\rho\sigma} + X^\sigma_{[\rho\nu]} g_{\mu\sigma}). \tag{55}$$

We also have $X_{\mu\nu}$ and $Y_{\nu\mu\rho}$ satisfying $v^\nu X_{\mu\nu} = 0$, $v^\nu Y_{\nu\mu\rho} = v^\rho Y_{\nu\mu\rho} = 0$. Now computing the torsion we obtain

$$T^\lambda_{\mu\rho} = -2v^\lambda \partial_{[\mu}\tau_{\rho]} + g^{\nu\lambda}\tau_{[\rho}(\mathcal{L}_{\vec{v}} g)_{\mu]\nu} - 2v^\lambda X_{[\mu\rho]} + g^{\nu\lambda} Y_{\nu[\mu\rho]}. \tag{56}$$

We remember that the torsionless condition imposes $\mathcal{L}_{\vec{v}} g = 0$ and $X^\sigma_{[\mu\nu]} = 0$, so $Y_{\nu\mu\rho} = 0$ too. So the torsionless condition becomes, projected on τ_λ

$$X_{[\mu\rho]} = -\partial_{[\mu}\tau_{\rho]}, \tag{57}$$

the anti-symmetric part of $X_{\mu\nu}$ is fixed. And the condition $v^\rho X_{\mu\rho} = 0$ fixes the temporal part of $X_{(\mu\nu)}$:

$$v^\rho X_{(\mu\rho)} = v^\rho \partial_{[\mu}\tau_{\rho]}. \tag{58}$$

Finally we write the torsionless connection that parallel-transports $\vec{v}$ and g, that we denote ${}^*\Gamma^\lambda_{\mu\rho}$:

$${}^*\Gamma^\lambda_{\mu\rho} = -v^\lambda \partial_{(\mu}\tau_{\rho)} + \frac{1}{2} g^{\nu\lambda}\left(\partial_\mu g_{\nu\rho} + \partial_\rho g_{\mu\nu} - \partial_\nu g_{\rho\mu}\right) - v^\lambda X_{(\mu\rho)}. \tag{59}$$

We obtain that the knowledge of $\vec{v}$, τ and g fixes a torsionless, compatible connection up to a symmetric $(d+1)$-tensor.

2.6 Carroll structure on null surfaces and the null infinity of Kerr

The null infinity of Minkowski is an example of very simple Carroll manifold, quite trivial actually. It is interesting to study the null infinity of Kerr, where the geometry is less trivial as we are going to see. The metric, written in Eddington-Finkelstein coordinates, is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right)(du + a\sin^2\theta d\phi)^2 + 2(du + a\sin^2\theta d\phi)(dr + a\sin^2\theta d\phi) \\ &+ (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}, \tag{60}$$

where a is the angular momentum and m the mass in suitable units. In these coordinates, the null infinity $\mathcal{I}^+$ is defined to be the region $r \to \infty$ and the metric becomes singular when taking this limit. In order to resolve this region we have to regularize the metric by multiplying it by a conformal factor that vanishes on $\mathcal{I}^+$. We define the unphysical metric

$$d\tilde{s}^2 = r^{-2} ds^2. \tag{61}$$

We also perform the change of coordinate $\rho = r^{-1}$ so that the null infinity is now situated at $\rho = 0$ and becomes a generic null hypersurface w.r.t. the unphysical metric. The near $\mathcal{I}^+$ geometry becomes

$$d\tilde{s}^2 = -2(du + a\sin^2\theta d\phi)d\rho + d\theta^2 + \sin^2\theta d\phi^2 + \mathcal{O}(\rho^2). \tag{62}$$

The bulk metric, when evaluated at $\rho = 0$ (not the induced one) can be written as

$$d\tilde{s}^2_{\rho=0} = 2\tau_\mu dx^\mu d\rho + g_{\mu\nu} dx^\mu dx^\nu, \tag{63}$$

which allows to read the Carrollian geometry induced on $\mathcal{I}^+$

$$\begin{aligned} &\vec{v} = \partial_u, \\ &\tau = -(du + a\sin^2\theta d\phi), \\ &g = d\theta^2 + \sin^2\theta d\phi^2, \\ &g^{-1} = g^{AB}(b_A\partial_u + \partial_A)(b_B\partial_u + \partial_B), \end{aligned} \tag{64}$$

where we have defined $b = b_A dx^A = -a\sin^2\theta d\phi$. It is easy to check that this set of data satisfies the geometrical definition (43) of a Carroll manifold equipped with an Ehresmann connection τ.

The fact that a null hypersurface is equipped with a Carroll structure (Carroll manifold together with an Ehresmann connection) is actually quite general, it relies on the existence of a nowhere vanishing vector field, transverse to the surface. This is what we show in [32]. Indeed consider a null hypersurface $\mathcal{N}$ embedded in a spacetime that is one higher dimensional, $\mathcal{M}$. The normal to $\mathcal{N}$, that we denote $\vec{v}$, is null and therefore belongs to the tangent bundle of $\mathcal{N}$. This implies that the rank-1 normal bundle $N\mathcal{N}$ is a subbundle of $T\mathcal{N}$. Now suppose there exists a vector $\vec{n}$, nowhere vanishing on $\mathcal{N}$, who does not belong to $T\mathcal{N}$ and satisfies

$$g(\vec{n}, \vec{v}) = 1 \quad \text{and} \quad g(\vec{n}, \vec{n}) = 0, \tag{65}$$

where g here is the bulk metric. Then we can define a screening distribution $S(\mathcal{N})$ canonically isomorphic to $T\mathcal{N}/N\mathcal{N}$ such that $T\mathcal{N} = S(\mathcal{N}) \oplus N\mathcal{N}$. It consists of all the vectors $X \in T\mathcal{N}$ satisfying

$$g(\vec{n}, X) = 0 \tag{66}$$

The span of $\vec{n}$ defines a rank-1 subbundle of $T\mathcal{M}$ denoted $tr(T\mathcal{N})$ such that, for points on $\mathcal{N}$,

$$T\mathcal{M} = tr(T\mathcal{N}) \oplus N\mathcal{N} \oplus S(\mathcal{N}), \tag{67}$$

the $\oplus$'s being defined using the metric projector. Things are starting to become clear : the manifold $\mathcal{N}$ is now equipped with an induced degenerate metric, the normal $\vec{v}$ being in the kernel of the metric and the splitting $T\mathcal{N} = N\mathcal{N} \oplus S(\mathcal{N})$ will provide the Ehresmann connection.

Consider the local coordinates $x^a = \{\rho, u, x^A\}$ and we define $x^\mu = \{u, x^A\}$. We now make the assumption that $\mathcal{N}$ is defined through the equation $\rho = cst$, this implies $v = -d\rho$ (defined by acting on $\vec{v}$ with the metric). The normal $\vec{v}$ being null, this imposes $g^{\rho\rho} = 0$, so $\vec{v} = v^\mu \partial_\mu = -g^{\rho\mu}\partial_\mu$. Moreover the first equation of (65) imposes $n^\rho = -1$, so $\vec{n} = -\partial_\rho + n^\mu \partial_\mu$. The nullity of $\vec{n}$ imposes

$$g_{\rho\rho} = 2g_{\rho\mu}n^\mu - g_{\mu\nu}n^\mu n^\nu. \tag{68}$$

We define $\tau_\mu = g_{\rho\mu} - g_{\mu\nu}n^\nu$, to write the metric as

$$ds^2 = (2\tau_\mu n^\nu + g_{\mu\nu}n^\mu n^\nu)d\rho^2 + 2(\tau_\mu + g_{\mu\nu}n^\nu)d\rho dx^\mu + g_{\mu\nu}dx^\mu dx^\nu. \tag{69}$$

We also have

$$\begin{aligned} v^\mu g_{\mu\nu} &= -g^{\rho\mu}g_{\mu\nu} \underset{g^{\rho\rho}=0}{=} -g^{\rho a}g_{a\nu} = -\delta^\rho_\nu = 0, \\ v^\mu \tau_\mu &= -g^{\rho\mu}g_{\mu\rho} \underset{g^{rr}=0}{=} -g^{\rho a}g_{\rho a} = -1, \end{aligned} \tag{70}$$

which are exactly the consitutive relations of a Carroll structure. The metric duals to $\vec{v}$ and $\vec{n}$ are

$$v = -d\rho \quad \text{and} \quad n = -\tau_\mu(n^\mu d\rho + dx^\mu). \tag{71}$$

We obtain

$$ds^2 = n \otimes v + v \otimes n + g_{\mu\nu}(dx^\mu + n^\nu d\rho)(dx^\nu + n^\nu d\rho). \tag{72}$$

The inverse of this metric is

$$g^{\rho\rho} = 0, \quad g^{\mu\rho} = -v^\mu \tag{73}$$

and $g^{\mu\nu}$ such that $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma + \tau_\sigma v^\mu$. If we define $h = g_{\mu\nu}(dx^\mu + n^\nu d\rho)(dx^\nu + n^\nu d\rho)$ we obtain

$$ds^2_{\mathcal{N}} = n \otimes v + v \otimes n + h, \tag{74}$$

with $h(\vec{v}, .) = h(\vec{n}, .) = 0$, such that h is a metric on the screening distribution. The subscript $\mathcal{N}$ indicates that is is evaluated on $\mathcal{N}$, not induced. We now see how our Carrollian objects appear when there exists a nowhere vanishing transverse vector $\vec{n}$. We have recovered the vector $\vec{v}$ pointing in the null direction of $\mathcal{N}$, therefore belonging to the kernel of the induced metric $g_{\mu\nu}dx^\mu dx^\nu$, and the splitting $T\mathcal{N} = N\mathcal{N} \oplus S(\mathcal{N})$ defines the Ehresmann connection τ. There is also an extra vector n^μ induced on $\mathcal{N}$ but it will not play a crucial role in what follows (for more on this object, see [38]), it will be vanishing for Kerr.

The hypersurface being null, there is no canonical connection induced from the embedding, but there is a way around, indeed the transverse vector $\vec{n}$ allows for the definition of a connection $\nabla^{\mathcal{N}}$ on $\mathcal{N}$ though,

$$\nabla^{\mathcal{M}}_X Y = \nabla^{\mathcal{N}}_X Y + B(X,Y)\vec{n}, \quad \forall X, Y \in T\mathcal{N}. \tag{75}$$

The null hypersurface is therefore equipped with a *strong* Carroll structure.

We can now come back to the Kerr example. We can read from (62) that $n^\mu = 0$, so the nowhere vanishing transverse vector is simply $\vec{n} = -\partial_\rho$. The metric (63) is actually exaclty (74) for $n^\mu = 0$. This allows also to properly identify the Carroll structure on Kerr's null infinity (instead of guessing it as we did before). We recall it here

$$\begin{aligned}
&\vec{v} = \partial_u, \\
&\tau = -(du + a\sin^2\theta d\phi), \\
&g = d\theta^2 + \sin^2\theta d\phi^2 = \gamma_{AB}dx^A dx^B, \\
&g^{-1} = g^{AB}(b_A\partial_u + \partial_A)(b_B\partial_u + \partial_B),
\end{aligned} \tag{76}$$

where $b = -a\sin^2\theta d\phi$. We can also use the transverse vector to induce a connection on $\mathcal{I}^+$. Its Christoffels are simply given by the bulk Christoffel components $\Gamma^\mu_{\nu\rho}$ evaluated on $\mathcal{I}^+$. The only non-vanishing one are

$$\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta \quad \text{and} \quad \Gamma^\phi_{\theta\phi} = \cot\theta. \tag{77}$$

The only contribution comes from the sphere. We could now wonder if this coincides with one of the compatible, torsionless connections (59) that we have derived ealier. We evaluate ${}^*\Gamma^\mu_{\nu\sigma}$ on the Carroll structure associated with Kerr's null infinity (76), the only non-vanishing components are

$${}^*\Gamma^u_{AB} = -\partial_{(A}b_{B)} + b_C\gamma^C_{AB} - X_{(AB)} \quad \text{and} \quad {}^*\Gamma^A_{BC} = \gamma^A_{BC}, \tag{78}$$

where γ^A_{BC} are the sphere Christoffels. When replacing b_A by its value, the two first terms of ${}^*\Gamma^u_{AB}$ cancel each other, so that ${}^*\Gamma$ coincides with the induced connection provided

$$X_{(AB)} = 0. \tag{79}$$

We conclude that the induced connection (75) on the null infinity of Kerr is torsionless and compatible with the corresponding Carroll manifold.

3 Symmetries and charges in gravity

In the previous section we have mentioned the existence of symmetry groups associated with particular gravitational setups, such as asymptotically flat gravity in 3d or 4d. However, it could come as a surprise that such a thing exists. Indeed the only apparent symmetry of pure gravity is reparametrization invariance and unfortunately Noether's theorem (or at least the one we are taught in our first classes of mechanics) is of no help for this symmetry as one can show that the associated currrent vanishes on shell. One is therefore tempted to conclude that all diffeomorphisms are pure gauge and that they carry no physical information. This would be true if the spacetime was compact and had no boundary but we are going to see that when there is a boundary (in our case a conformal one), the notion of *surface charge* can then be defined, giving physical meaning to a particular class of diffeomorphisms i.e. the one acting non trivially on the boundary. See [6, 7] for complete descriptions of this subject.

Consider a classical system caracterized by its Lagrangian L, functional of its fields $\phi(x^a)$. For simplicity we consider only one field, the right indices will be easy to restaure when needed. The variation of the action w.r.t. the field is

$$\delta L = \frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial_a\partial\phi}\delta\partial_a\phi + ... = \frac{\delta L}{\delta\phi}\delta\phi + \partial_a\Theta^a. \tag{80}$$

To write the last equality we have simply used the Leibniz rule, thus defining a boundary term in the variation, Θ^a, called the presymplectic potential. It is a vector, but one can associate a corresponding $(D-1)$-form using Hodge duality

$$\boldsymbol{\Theta} = \Theta^a \sqrt{-g}\varepsilon_{ab_1...b_D} dx^{b_1} \wedge ... \wedge dx^{b_D}. \tag{81}$$

One can look at the various 1st order variations $\delta\phi$, $\delta\partial_a\phi$, etc, as a basis of the cotangent plane of the field configuration space. In that space, the presymplectic potential is a one-form, while in the total space (or bundle), whose basis is the spacetime and whose fibers are the field configurations, $\boldsymbol{\Theta}$ is a $(D-1,1)$-form. The field configuration space possesses a canonical exterior derivative δ which allows to define the presymplectic form

$$\boldsymbol{\omega} = \delta\boldsymbol{\Theta}. \tag{82}$$

It is a $(D-1,2)$-form in the total space. Now comes the fundamental theorem of the covariant phase space formalism. Contracting the presymplectic form with a gauge transformation $\delta_\xi\phi$, there exists a $(D-2,1)$-form $\mathbf{k}_\xi$ that satisfies the identity

$$\boldsymbol{\omega}(\phi, \delta\phi, \delta_\xi\phi) = d\mathbf{k}_\xi(\phi, \delta\phi), \tag{83}$$

where ϕ solves the equation of motion and $\delta\phi$ solves the linearized equations of motion, around the solution ϕ. One notice that the form $\mathbf{k}_\xi$ is defined up to a total spacetime derivative. It is this very form that will allow for the definition of charges associated with gauge transformations (in our case diffeomorphisms). We can integrate the $(D-2)$-form $\mathbf{k}_\xi$ on a boundary-less codimension 2 surface S to define

$$\not{\delta} Q_\xi = \int_S \mathbf{k}_\xi(\phi, \delta\phi). \tag{84}$$

It is a $(D-2,0)$-form. We expect this object to define the variation of a charge between the two solutions ϕ and $\phi + \delta\phi$. The scalar $\not{\delta} Q_\xi$ is a one-form in the field space, but nothing ensures that it is the exterior derivative of a functional Q_ξ, hence the notation $\not{\delta}$. If it is actually the variation of a functional, we say that the charges are *integrable*. It is now obvious that the charge Q_ξ is conserved on shell if $d\mathbf{k}_\xi(\phi, \delta\phi)$ vanishes on the codimension one surface drawn by the time evolution of S. Therefore we obtain that the charge Q_ξ is conserved if and only if

$$\boldsymbol{\omega}(\phi, \delta\phi, \delta_\xi\phi) = 0, \tag{85}$$

where ϕ and $\delta\phi$ are on shell. In gravity, the region where this equality holds is close to the conformal boundary. Indeed we are going to consider infinitesimal diffeomorphisms satisfying the equation $\delta_\xi\phi = 0$ but only in the region $r = \infty$, so that the associated charge will be conserved only when evaluated at infinity. These symmetries are called asymptotic symmetries. Note that we have now a procedure that, in principle, can associate a conserved charge to a larger class of diffeomorphisms than the restricted case of exact Killings.

The whole subtlety of this formalism now resides in finding a good phase space, such that the charges are well defined and that the true physical symmetries are properly counted. We now suppose that the field space is subject to boundary conditions, e.g. fall off conditions on the metric. We say that a diffeomorphism is allowed if it preserves the boundary conditions. Suppose, moreover, that the boundary conditions enforce the diffeomorphism to be an asymptotic Killing, i.e. it satisfies $\delta_\xi\phi = 0$ at $r = \infty$. Assuming the charge is finite and integrable, we can then define an associated conserved charge Q_ξ by choosing S to be in the asymptotic region (the celestial sphere for example). There are now two possible cases to distinguish : either the charge is non-zero, therefore the symmetry has a true physical meaning and is really acounting for non-equivalent solutions, or the charge vanishes and the corresponding symmetry is pure gauge (or trivial). This distinction allows for a proper definition of the asymptotic symmetry group

$$\text{Asymptotic symmetry group} = \frac{\text{Allowed diffeomorphisms}}{\text{Pure gauge transformation}}. \tag{86}$$

One notice that this definition makes sense only if the spacetime possesses a boundary. This boundary can be the asymptotic region in AdS or in flat space, but it can also be the horizon when the phase space describes black hole solutions.

Suppose the charges are integrable, we define the following bracket for the charges

$$\{Q_\chi, Q_\xi\} = \delta_\xi Q_\chi = \int_S \mathbf{k}_\chi(\phi, \delta_\xi \phi). \tag{87}$$

One can show that, on shell, this bracket defines a projective representation of the asymptotic symmetry algebra

$$\{Q_\chi, Q_\xi\} = Q_{[\chi,\xi]} + C_{\chi,\xi}(\phi), \tag{88}$$

up to a possible central extension C, that in principle can depend on the point ϕ in the field space. We are now going to study the case of 3-dimensional gravity, AdS and flat, where all these objects can be computed explicitly and where the flat limit is tractable. But before going further we would like to give explicit formulae for the quantities introduced in pure gravity :

$$\delta L = \frac{\sqrt{-g}}{16\pi G}\left(R^{ab} - \frac{1}{2}Rg^{ab} - \frac{1}{\ell^2}g^{ab}\right)\delta g_{ab} + \partial_a \Theta^a, \tag{89}$$

where Θ^a is given by

$$\Theta^a = \frac{\sqrt{-g}}{16\pi G}\left(\nabla_b \delta g^{ab} - 2\nabla^a \delta g^b_b\right). \tag{90}$$

From there, after a lengthy computation, one can compute the $(D-2)$-form $\mathbf{k}_\xi$, we report the result here

$$\mathbf{k}_\xi(g,\delta g) = \frac{\sqrt{-g}}{8\pi G}(d^{D-2}x)_{ab}\left(\xi^a \nabla_c \delta g^{bc} - \xi^a \nabla^b \delta g + \xi_c \nabla^b \delta g^{ac} + \frac{1}{2}\delta g \nabla^b \xi^a - \delta g^{cb}\nabla_c \xi^a\right), \tag{91}$$

where we have defined $\delta g = \delta g^a_a$ and have used Hodge duality to define the $(D-2)$-form

$$(d^{D-2}x)_{ab} = \frac{1}{2!(D-2)!}\varepsilon_{abc_2\ldots c_D} dx^{c_2} \wedge \ldots \wedge dx^{c_D}. \tag{92}$$

It is this formula that we are going to use to compute charges in three-dimensional pure gravity.

4 Three-dimensional AdS gravity and its flat limit

In 3 dimensions there is no propagation of the gravitational field strength, the Weyl tensor is identically zero. But it does not mean that the theory is trivial, it actually possesses many features in common with 4-dimensional gravity, such as the existence of black holes, particles, gravitational dressing. The main feature is that any solution is locally diffeomorphic to a homogeneous spacetime with the corresponding constant curvature. The physics will therefore be contained in the topological properties of the spacetime, for example, black holes in AdS_3 are obtained as quotient of the global homogeneous AdS_3 solution under a Killing symmetry (see [39] for a review of three-dimensional black holes). We would like now to define a proper phase space for 3-dimensional gravity, we will start with the AdS case. We will then take the flat limit in the bulk and show how it translates into an ultra-relativistic on the boundary.

4.1 Bondi gauge in AdS

We follow an algorithm that allows for the definition of a proper phase space : we firstly define the gauge-fixing conditions for the metric called *Bondi gauge*, then we derive its solution space and the variation of the latter under residual gauge diffeomorphisms. The Bondi gauge was also used for the study of AdS_3 in [40].

4.1.1 Definition

In the Bondi gauge, the metric is given by

$$ds^2 = \frac{V}{r} e^{2\beta} du^2 - 2e^{2\beta} du dr + r^2 e^{2\varphi}(d\phi - U du)^2, \tag{93}$$

with coordinates (u, r, ϕ). In this expression, V, β and U are functions of (u, r, ϕ), and φ is function of (u, ϕ). The three gauge-fixing conditions are

$$g_{rr} = 0, \quad g_{r\phi} = 0, \quad g_{\phi\phi} = r^2 e^{2\varphi}. \tag{94}$$

Note that $g_{\phi\phi} = r^2 e^{2\varphi}$ is the unique solution of the determinant condition

$$\partial_r \left(\frac{g_{\phi\phi}}{r^2} \right) = 0, \tag{95}$$

which can be generalized to define the Bondi gauge in higher dimensions.

The residual gauge diffeomorphisms ξ preserving the Bondi gauge fixing (94) have to satisfy the three conditions

$$\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{r\phi} = 0, \quad g^{\phi\phi} \mathcal{L}_\xi g_{\phi\phi} = 2\omega(u, \phi). \tag{96}$$

The explicit solution of these equations is given by

$$\begin{aligned} \xi^u &= f, & (97)\\ \xi^\phi &= Y - \partial_\phi f \, e^{-2\varphi} \int_r^{+\infty} \frac{\Delta r'}{r'^2} e^{2\beta}, & (98)\\ \xi^r &= -r[\partial_\phi \xi^\phi - \omega - U \partial_\phi f + \xi^\phi \partial_\phi \varphi + f \partial_u \varphi], & (99) \end{aligned}$$

where $f(u, \phi)$, $Y(u, \phi)$ and $\omega(u, \phi)$ are arbitrary functions of (u, ϕ).

4.1.2 Solution space

We discuss the most general solution space for three-dimensional general relativity in Bondi gauge. Interestingly, we do not have to impose any preliminary boundary condition here. This is in contrast with the procedure followed in the Fefferman–Graham gauge, where one has to impose fall-offs on the radial expansion of the metric. Therefore, in three dimensions, the gauge conditions (94) are to some extent stronger than those imposed to define the Fefferman–Graham gauge.

First we impose the Einstein equations leading to the metric radial constraints. Solving $G_{rr} - \frac{1}{\ell^2} g_{rr} = R_{rr} = 0$ gives

$$\beta = \beta_0(u, \phi). \tag{100}$$

The equation $G_{r\phi} - \frac{1}{\ell^2} g_{r\phi} = R_{r\phi} = 0$ leads to

$$U = U_0(u, \phi) + \frac{1}{r} 2 e^{2\beta_0} e^{-2\varphi} \partial_\phi \beta_0 - \frac{1}{r^2} e^{2\beta_0} e^{-2\varphi} N(u, \phi). \tag{101}$$

Eventually, $G_{ur} - \frac{1}{\ell^2} g_{ur} = 0$ gives

$$\frac{V}{r} = -\frac{r^2}{\ell^2} e^{2\beta_0} - 2r(\partial_u \varphi + D_\phi U_0) + M(u, \phi) + \frac{1}{r} 4 e^{2\beta_0} e^{-2\varphi} N \partial_\phi \beta_0 - \frac{1}{r^2} e^{2\beta_0} e^{-2\varphi} N^2, \tag{102}$$

where $D_\phi U_0 = \partial_\phi U_0 + \partial_\phi \varphi U_0$. Taking into account the previous results, the Einstein equation $G_{\phi\phi} - \frac{1}{\ell^2} g_{\phi\phi} = 0$ is automatically satisfied at all orders.

We now solve the Einstein equations to get the time evolution constraints on M and N. The equation $G_{u\phi} - \frac{1}{\ell^2} g_{u\phi} = 0$ returns

$$\begin{aligned} (\partial_u + \partial_u \varphi) N &= \left(\frac{1}{2} \partial_\phi + \partial_\phi \beta_0 \right) M - 2N \partial_\phi U_0 - U_0(\partial_\phi N + N \partial_\phi \varphi) \\ &\quad + 4 e^{2\beta_0 - 2\varphi} [2(\partial_\phi \beta_0)^3 - (\partial_\phi \varphi)(\partial_\phi \beta_0)^2 + (\partial_\phi \beta_0)(\partial_\phi^2 \beta_0)]. \qquad (103) \end{aligned}$$

Moreover, $G_{uu} - \frac{1}{\ell^2} g_{uu} = 0$ imposes

$$
\begin{aligned}
\partial_u M \;=\; & (-2\partial_u\varphi + 2\partial_u\beta_0 - 2\partial_\phi U_0 + U_0 2\partial_\phi\beta_0 - U_0 2\partial_\phi\varphi - U_0\partial_\phi)M \\
& + \frac{2}{\ell^2} e^{4\beta_0 - 2\varphi}[\partial_\phi N + N(4\partial_\phi\beta_0 - \partial_\phi\varphi)] \\
& - 2e^{2\beta_0 - 2\varphi}\{\partial_\phi U_0[8(\partial_\phi\beta_0)^2 - 4\partial_\phi\beta_0\partial_\phi\varphi + (\partial_\phi\varphi)^2 + 4\partial_\phi^2\beta_0 - 2\partial_\phi^2\varphi] - \partial_\phi^3 U_0 \\
& \qquad + U_0[\partial_\phi\beta_0(8\partial_\phi^2\beta_0 - 2\partial_\phi^2\varphi) + \partial_\phi\varphi(-2\partial_\phi^2\beta_0 + \partial_\phi^2\varphi) + 2\partial_\phi^3\beta_0 - \partial_\phi^3\varphi] \\
& \qquad + 2\partial_u\partial_\phi\beta_0(4\partial_\phi\beta_0 - \partial_\phi\varphi) + \partial_u\partial_\phi\varphi(-2\partial_\phi\beta_0 + \partial_\phi\varphi) \\
& \qquad + 2\partial_u\partial_\phi^2\beta_0 - \partial_u\partial_\phi^2\varphi\}.
\end{aligned} \tag{104}
$$

The solution space is thus characterized by five arbitrary functions of (u, ϕ), given by β_0, U_0, M, N, φ, with two dynamical constraints expressing the time evolution of M and N. The last two equations (103) and (104) are central. They are not very indicative but their interpretation is actually simple. Gravitationally speaking, they are constraint equations on the boundary of the spacetime that ensures that the bulk metric remains a solution when evolving inward radially. If the radial coordinate is seen as a Hamiltonian time, these equations are nothing but constraint equations on the $r = \infty$ leaf. Now another way to look at these equations is to interpret them holographically. We will not go through all the marvelous details of the holographic correspondence in AdS but rather try to give a flavor.

4.1.3 Holographic energy-momentum tensor

The strong statement of the duality is that the partition function of a gravitational system in AdS with boundary conditions is exactly equal to the generating functional of CFT correlation functions. The CFT can be thought of as leaving on the boundary of the gravitational spacetime and the operators, whose correlation functions are generated, are sourced by the boundary value of the bulk fields :

$$
Z^{AdS}_{grav}[\phi \underset{r\to\infty}{\to} \phi_0] = \left\langle \exp\left(i \int \phi_0 \mathcal{O} \right) \right\rangle_{CFT}. \tag{105}
$$

The operator $\mathcal{O}$ is said to be dual to the bulk field ϕ. Now if the boundary value of the metric is fixed to be $g^0_{\mu\nu}$, it sources its dual operator $T^{\mu\nu}$ which is nothing but the energy-momentum tensor of the CFT. We may use a saddle point approximation for the gravity partition function such that

$$
\exp\left(iS^{AdS}_{grav}[\tilde{\phi} \underset{r\to\infty}{\to} \phi_0] \right) = \left\langle \exp\left(i \int \phi_0 \mathcal{O} \right) \right\rangle_{CFT}, \tag{106}
$$

where S^{AdS}_{grav} is the classical gravitational action and $\tilde{\phi}$ a solution that asymptotes to ϕ_0 on the boundary. This tells us that the on-shell action of the bulk, with boundary condition ϕ_0, acts as a generating functional for correlation functions of an operator $\mathcal{O}$ in some CFT that lives in one dimension less. Of course a lot of things are put under the rack right now, let's just say that the CFT is expected to be strongly coupled while the classical limit in the bulk maps to a large central charge limit in the CFT.[3]

From there we can deduce a formula for the expectation value of the energy–momentum tensor of the boundary CFT

$$
\langle T^{\mu\nu} \rangle = \frac{-2}{\sqrt{g^0_{\mu\nu}}} \frac{\delta S^{AdS}_{grav}}{\delta g^0_{\mu\nu}}. \tag{107}
$$

3. The most famous example of duality is between a string theory in AdS$_5 \times S^5$ and a conformally invariant 4d Yang-Mills theory (with additional fields), see [1]. The Yang-Mills theory is parametrized by two constants, its coupling λ and the size of its gauge group $\mathrm{SU}(N)$, or equivalently its central charge. On the string theory side, we have the string coupling g_s and the ratio of the string tension with the AdS radius $\frac{\alpha'}{L^2}$. The string coupling controls the loop expansion for worlsheet diagrams, therefore taking a small g_s limit is like taking a classical limit of the string theory. The string tension is inversely proportional to the mass of the string modes, therefore asking $\frac{\alpha'}{L^2}$ to be small amounts to consider only the massless modes of the string. We end up with a classical gravitational theory in 10d that can be compactified to give the left hand side of (106) in AdS$_5$. Finally on the CFT side, the small g_s limit maps to a large N limit (large central charge) and the small α' limit maps to a strong coupling – or large λ – limit.

The action S_{grav}^{AdS} must be evaluated on a solution that asymptotes to $g^0_{\mu\nu}$ on the boundary. The bare action is usually infinite and a careful treatment of the boundary counter-terms that one has to add to regularize it is required, see [17]. This procedure is purely classical in the bulk but hands you the expectation value of an energy-momentum tensor in a strongly coupled, large central charge, CFT. Now changing the bulk solution amounts to changing the state on which the expectation value is taken. Two canonical examples are : pure AdS, which preserves the maximum of symmetries and is therefore dual to the vacuum and black hole geometries who are generically dual to thermal states such that the temperature matches the Hawking temperature.

We now come back to three-dimensional gravity. The constraint equations (103) and (104) are interpreted as the conservation of an energy-momentum tensor in the dual theory. A non trivial computation allows to extract the energy-momentum tensor and rewrite the constraint equations as

$$\nabla_\mu T^{\mu\nu} = 0. \tag{108}$$

The boundary metric – whose Levi-Civita connection appears in the conservation – is

$$g^0_{\mu\nu} = \begin{pmatrix} -\frac{e^{4\beta_0}}{\ell^2} + e^{2\varphi}U_0^2 & -e^{2\varphi}U_0 \\ -e^{2\varphi}U_0 & e^{2\varphi}, \end{pmatrix} \tag{109}$$

while the expression for $T^{\mu\nu}$ in terms of M, N, φ, β_0, U_0 is

$$\begin{aligned}
T_{tt} &= \frac{1}{16\pi G\ell} e^{-4\beta_0-2\varphi}\{4e^{8\beta_0}[2(\partial_\phi\beta_0)^2 - \partial_\phi\beta_0\partial_\phi\varphi + \partial_\phi^2\beta_0] + e^{4\beta_0+2\varphi}[e^{2\beta_0}(M-4NU_0) \\
&\quad -\ell^2((\partial_\phi U_0)^2 + U_0^2(-8\partial_\phi\beta_0\partial_\phi\varphi + (\partial_\phi\varphi)^2 + 4\partial_\phi^2\varphi) + (\partial_t\varphi)^2 \\
&\quad +2\partial_\phi U_0(U_0(-4\partial_\phi\beta_0 + 3\partial_\phi\varphi) + \partial_t\varphi) \\
&\quad +2U_0(2\partial_\phi^2 U_0 + (-4\partial_\phi\beta_0 + \partial_\phi\varphi)\partial_t\varphi + 2\partial_t\partial_\phi\varphi))] + e^{4\varphi}\ell^2 U_0^2[e^{2\beta_0}M + \ell^2((\partial_\phi U_0)^2 \\
&\quad +U_0^2(-4\partial_\phi\beta_0\partial_\phi\varphi + (\partial_\phi\varphi)^2 + 2\partial_\phi^2\varphi) + 2\partial_\phi\varphi\partial_t U_0 + \partial_t\varphi(-4\partial_t\beta_0 + \partial_t\varphi) \\
&\quad +2\partial_\phi U_0(2U_0(-\partial_\phi\beta_0 + \partial_\phi\varphi) \\
&\quad -2\partial_t\beta_0 + \partial_t\varphi) + 2U_0(\partial_\phi^2 U_0 - 2\partial_\phi\beta_0\partial_t\varphi + \partial_\phi\varphi(-2\partial_t\beta_0 + \partial_t\varphi) + 2\partial_t\partial_\phi\varphi) \\
&\quad +2(\partial_t\partial_\phi U_0 + \partial_t^2\varphi))]\}, \\
T_{t\phi} &= \frac{1}{16\pi G\ell} e^{-4\beta_0}\{2e^{6\beta_0}N - 2e^{4\beta_0}\ell^2[\partial_\phi U_0(2\partial_\phi\beta_0 - \partial_\phi\varphi) - \partial_\phi^2 U_0 + U_0(2\partial_\phi\beta_0\partial_\phi\varphi - \partial_\phi^2\varphi) \\
&\quad +2\partial_\phi\beta_0\partial_t\varphi - \partial_t\partial_\phi\varphi] + e^{2\varphi}\ell^2 U_0[-e^{2\beta_0}M - \ell^2((\partial_\phi U_0)^2 + U_0^2(-4\partial_\phi\beta_0\partial_\phi\varphi + (\partial_\phi\varphi)^2 + 2\partial_\phi^2\varphi) \\
&\quad +2\partial_\phi\varphi\partial_t U_0 + \partial_t\varphi(-4\partial_t\beta_0 + \partial_t\varphi) + 2\partial_\phi U_0(2U_0(-\partial_\phi\beta_0 + \partial_\phi\varphi) - 2\partial_t\beta_0 + \partial_t\varphi) \\
&\quad +2U_0(\partial_\phi^2 U_0 - 2\partial_\phi\beta_0\partial_t\varphi + \partial_\phi\varphi(-2\partial_t\beta_0 + \partial_t\varphi) + 2\partial_t\partial_\phi\varphi) + 2(\partial_t\partial_\phi U_0 + \partial_t^2\varphi))]\}, \\
T_{\phi\phi} &= \frac{1}{16\pi G\ell} e^{-4\beta_0+2\varphi}\{e^{2\beta_0}\ell^2 M + \ell^4[(\partial_\phi U_0)^2 + U_0^2(-4\partial_\phi\beta_0\partial_\phi\varphi + (\partial_\phi\varphi)^2 + 2\partial_\phi^2\varphi) + 2\partial_\phi\varphi\partial_t U_0 \\
&\quad +\partial_t\varphi(-4\partial_t\beta_0 + \partial_t\varphi) + 2\partial_\phi U_0(2U_0(-\partial_\phi\beta_0 + \partial_\phi\varphi) - 2\partial_t\beta_0 + \partial_t\varphi) \\
&\quad +2U_0(\partial_\phi^2 U_0 - 2\partial_\phi\beta_0\partial_t\varphi + \partial_\phi\varphi(-2\partial_t\beta_0 + \partial_t\varphi) + 2\partial_t\partial_\phi\varphi) + 2(\partial_t\partial_\phi U_0 + \partial_t^2\varphi)]\}.
\end{aligned} \tag{110}$$

Interestingly we have rercovered the whole solution space of the Fefferman-Graham gauge. Indeed the latter is characterized by a boundary metric and the extrinsic curvature of the boundary whose data is the same than the holographic energy-momentum tensor.

4.1.4 Variation of the solution space

As we explained earlier, some diffeomorphisms are expected to carry physical information, i.e. there exists non trivial associated charges. They correspond to the allowed diffeomorphisms (or residual diffeomorphisms) but evaluated on-shell. When doing so their action brings a solution to another solution. The residual gauge diffeomorphisms (97–99) evaluated on-shell are given by

$$
\begin{aligned}
\xi^u &= f, && (111)\\
\xi^\phi &= Y - \frac{1}{r}\partial_\phi f\, e^{2\beta_0 - 2\varphi}, && (112)\\
\xi^r &= -r[\partial_\phi Y - \omega - U_0 \partial_\phi f + Y\partial_\phi \varphi + f\partial_u \varphi] \\
&\quad + e^{2\beta_0-2\varphi}(\partial_\phi^2 f - \partial_\phi f \partial_\phi \varphi + 4\partial_\phi f \partial_\phi \beta_0) - \frac{1}{r} e^{2\beta_0 - 2\varphi}\partial_\phi f\, N. && (113)
\end{aligned}
$$

Under these residual gauge diffeomorphisms, the solution space transforms in a complicated way. The trick is to consider instead the two new objects we have introduced, g^0 and T. Using the expression for T in (110), one can check that its trace is completely determined by the boundary metric

$$T^\mu_\mu = \frac{\ell}{16\pi G} R, \tag{114}$$

where R is the scalar curvature of g^0. This actually tells us that the dual CFT is anomalous and one can read the conformal anomaly by matching the factor in front of R with $\frac{c}{24\pi}$

$$c = \frac{3\ell}{2G}. \tag{115}$$

This is the famous result of Brown and Henneaux, see [18]. We conclude that the boundary tensor T together with g^0 correspond to five functions of the boundary coordinates. We also recall that the solution space of Bondi is parametrized by M, N, φ, β_0, U_0, it is a perfect match.

We would like to compute the variation of the boundary metric and the energy-momentum tensor under the action of the residual gauge diffeomorphisms, but before that, we define

$$
\begin{aligned}
\xi_0^t &= f, \\
\xi_0^\phi &= Y, \\
\sigma &= \partial_\phi Y - \omega - U_0 \partial_\phi f + Y \partial_\phi \varphi + f \partial_t \varphi.
\end{aligned} \tag{116}
$$

This will simplify drastically the variations of the solution space. Finally, acting with ξ on the bulk metric, we deduce the variations of g^0 and T

$$
\begin{aligned}
\delta_\xi g^0_{\mu\nu} &= \mathcal{L}_{\xi_0} g^0_{\mu\nu} - 2\sigma g^0_{\mu\nu}, \\
\delta_\xi T_{\mu\nu} &= \mathcal{L}_{\xi_0} T_{\mu\nu} - \frac{\ell}{16\pi G}\left[\mathcal{L}_{\partial\sigma} g^0_{\mu\nu} - (g_0^{\rho\lambda}\mathcal{L}_{\partial\sigma} g^0_{\rho\lambda}) g^0_{\mu\nu}\right].
\end{aligned} \tag{117}
$$

These are the most general variations of the solution space in Bondi gauge. They are key ingredients in the computation of the asymptotic charge algebra.

One can go further by imposing boundary conditions on the gravity phase space, the most famous one is asking the boundary metric to be flat. It is like asking the spacetime of the dual field theory to be Minkowski

$$g^0_{\mu\nu} = \eta_{\mu\nu} \quad \Leftrightarrow \quad \varphi = 0, \quad \beta_0 = 0, \quad U_0 = 0. \tag{118}$$

This condition reduces the symmetries of the system, indeed asking the metric to be preserved by ξ boils down to

$$\mathcal{L}_{\xi_0}\eta_{\mu\nu} - 2\sigma \eta_{\mu\nu} = 0. \tag{119}$$

Taking the trace of this equation allows to express σ in terms of ξ_0. The symmetry algebra is then uniquely specified by the boundary vector ξ_0 that belongs to the conformal algebra according to (119) (it is the same equation than (26)). One can also check that the transformation of T under ξ_0 becomes exactly the usual transformation of the energy-momentum tensor in an anomalous CFT. In this context, one can compute the charges.

4.1.5 Virasoro Charges

With these last boundary conditions, the symmetry algebra reduces to conformal transformations of the 2d boundary. It is well know that this coincides with two copies of the Witt algebra. Suppose the boundary coordinate are x^+ and x^-, such that the boundary metric becomes

$$g^0 = 2dx^+dx^- = -\frac{dt^2}{\ell^2} + d\phi^2 = -(dx^0)^2 + (dx^1)^2. \tag{120}$$

The conformal symmetries in these coordinates are very simple, they correspond to holomorphic and anti-holomorphic changes of coordinates

$$x^+ \to f^+(x^+) \quad \text{and} \quad x^- \to f^-(x^-). \tag{121}$$

The corresponding Lie algebra is spanned by the vector fields

$$\xi_{Y^+,Y^-} = Y^+(x^+)\partial_+ + Y^-(x^-)\partial_-. \tag{122}$$

Now we can define the modes $\xi_n^+ = \xi_{e^{inx^+},0}$ and $\xi_n^- = \xi_{0,e^{inx^-}}$, they form a basis for the Lie algebra. One can compute their commutators and recover the usual conformal algebra in two dimensions, the non-zero commutators ae

$$i[\xi_m^+, \xi_n^+] = (m-n)\xi_{m+n}^+ \quad \text{and} \quad i[\xi_m^-, \xi_n^-] = (m-n)\xi_{m+n}^-. \tag{123}$$

Before computing the charge, we should make some comments on the holographic energy-momentum tensor , the trace condition (114) becomes

$$T_{+-} = 0, \tag{124}$$

while the conservation equation (108) becomes simply

$$\partial_- T_{++} = 0 \quad \text{and} \quad \partial_+ T_{--} = 0, \tag{125}$$

such that the functions T_{++} and T_{--} are respectively holomorphic and anti-holomorphic. We conclude that the holographic energy-momentum tensor satisfies all the properties that it has to satisfy in a 2d CFT.

Using the formalism introduced in Sec. 3 we can finally compute the charges associated to the corresponding bulk vector fields. The formula (91) for the charge 1-form gives, when integrated on the boundary spatial coordinate

$$\not\delta Q_\xi = \int d\phi (Y^+ \delta T_{++} + Y^- \delta T_{--}). \tag{126}$$

These charges are integrable, so we can write an absolute charge by integrating $\not\delta Q_\xi$ and subtracting the vacuum charge

$$Q_\xi = \int d\phi \left[Y^+ \left(T_{++} + \frac{\ell}{32\pi G} \right) + Y^- \left(T_{--} + \frac{\ell}{32\pi G} \right) \right]. \tag{127}$$

We can go further and compute the charge algebra. To do so we need the variations of T_{++} and T_{--} under a bulk diffeomorphism. Using (117) we find

$$\begin{aligned} \delta_\xi T_{++} &= Y^+\partial_+ T_{++} + 2T_{++}\partial_+ Y^+ - \frac{\ell}{16\pi G}\partial_+^3 Y^+, \\ \delta_\xi T_{--} &= Y^-\partial_- T_{--} + 2T_{--}\partial_- Y^- - \frac{\ell}{16\pi G}\partial_-^3 Y^-. \end{aligned} \tag{128}$$

The first two terms in the transformations are nothing but the homogeneous action of a conformal transformation on a CFT energy-momentum tensor while the last term is due to the anomaly (in the finite form of this transformation, it would coincide with the Schwarzian derivative). We can now compute the charge algebra for the bracket (88). We define the modes

$$\mathcal{Y}^+_n = \int d\phi \left(T_{++} + \frac{\ell}{32\pi G}\right) e^{inx^+} \quad \text{and} \quad \mathcal{Y}^-_n = \int d\phi \left(T_{--} + \frac{\ell}{32\pi G}\right) e^{inx^-}, \tag{129}$$

and find the algebra

$$\begin{aligned} i\{\mathcal{Y}^+_m, \mathcal{Y}^+_n\} &= (m-n)\mathcal{Y}^+_{m+n} + \frac{c}{12}(m^2-1)m\delta_{m+n,0}, \\ i\{\mathcal{Y}^-_m, \mathcal{Y}^-_n\} &= (m-n)\mathcal{Y}^-_{m+n} + \frac{c}{12}(m^2-1)m\delta_{m+n,0}. \end{aligned} \tag{130}$$

These are the usual two copies of a Virasoro algebra, where the central extension is the same for both right and left movers and equal to $\frac{3\ell}{2G}$.

Before going further, we would like to point out that the surface charges we have obtained can be written nicely in covariant way

$$Q_\xi = \int d\phi \, n_\mu J^\mu, \tag{131}$$

where we have defined the following objects

$$J^\mu = \mathcal{T}^\mu_\rho \xi^\rho_0, \quad n = \ell^{-1} dt \quad \text{and} \quad \mathcal{T}_{\mu\nu} = \begin{pmatrix} T_{++} + \frac{\ell}{32\pi G} & 0 \\ 0 & T_{--} + \frac{\ell}{32\pi G}. \end{pmatrix} \tag{132}$$

We recognize the Noether charge on the boundary. This is consistent with the holographic principle that sates that gauge symmetries in the bulk correspond to global symmetries on the boundary. It is a charge usually associated to a Killing symmetry or a conformal one. This charge is time-independent if the current J^μ is conserved. In our case it is conserved for three reasons : the energy-momentum is conserved, symmetric and the vector ξ_0 is a conformal symmetry. Indeed

$$\partial_\mu J^\mu = (\partial_\mu \mathcal{T}^\mu_\rho)\xi^\rho_0 + \mathcal{T}^\mu_\rho(\partial_\mu \xi^\rho_0). \tag{133}$$

The first term vanishes due to the conservation of $\mathcal{T}$, while the symmetry of $\mathcal{T}$ allows to replace the derivative of ξ_0 by its symmetric version

$$\partial_\mu J^\mu = \mathcal{T}^{\mu\rho}\partial_{(\mu}\xi^0_{\rho)} = \frac{1}{2}(\partial_\nu \xi^\nu_0)\mathcal{T}^\mu_\mu = 0, \tag{134}$$

where we have used the tracelessness of $\mathcal{T}$ for the last equality. We finally show that the charge is conserved

$$\partial_0 Q_\xi = \int d\phi \, \partial_0 J^0 = -\int d\phi \, \partial_\phi J^\phi = 0 + \text{corner term} \tag{135}$$

The conservation relies therefore on boundary conditions along the angular direction, for example it can be compact or if it is not, one can ask the current to be non zero only on a compact support.

4.1.6 Generic boundary metric

When the boundary is not flat, i.e. when g^0 is sourced one can also find a covariant formula for a conserved charge. Indeed suppose there exists a family of spacelike hypersurfaces Σ_λ on the boundary (here there are simply lines) and a covariantly conserved current J. We denote the normalized normal to Σ_λ by n^λ, then the conserved charge is given by

$$Q^\lambda_\xi = \int_{\Sigma_\lambda} \sqrt{h}\, n^\lambda_\mu J^\mu, \tag{136}$$

where h is the metric induced on Σ_λ. The conservation follows from Gauss theorem, indeed consider the difference between the value of the charge at λ_1 and λ_2, from Gauss law we obtain

$$Q^{\lambda_1}_\xi - Q^{\lambda_2}_\xi = \int_{\Sigma_{\lambda_1}} \sqrt{h}\, n^{\lambda_1}_\mu J^\mu + \int_{\Sigma_{\lambda_2}} \sqrt{h}\,(-n^{\lambda_2}_\mu) J^\mu = \int_V \sqrt{-g^0}\, \nabla_\mu J^\mu = 0, \tag{137}$$

where we have used the fact that the two line Σ_{λ_1} and Σ_{λ_1} are the boundary of a volume that we call V. We conclude that if we are able to build such a current when the boundary metric is sourced it will also define a conserved charge. Consider again the current associated to the vector field ξ_0

$$J^\mu = T^\mu_\rho \xi_0^\rho, \tag{138}$$

where T is the energy-momentum tensor for generic value of φ, β_0 and U_0, reported in Eq. (110). Now we suppose that we are on-shell and that we ask the boundary metric to be preserved. From (117) we deduce

$$\mathcal{L}_{\xi_0} g^0_{\mu\nu} - 2\sigma g^0_{\mu\nu} = 0. \tag{139}$$

Taking the trace of this equation fixes σ in terms of ξ_0, while ξ_0 is asked to be a conformal Killing of the boundary metric. One can wonder if the current J is then conserved :

$$\nabla_\mu J^\mu = (\nabla_\mu T^\mu_\rho)\xi_0^\rho + T^\mu_\rho(\nabla_\mu \xi_0^\rho) = \frac{1}{2}\nabla_\rho \xi_0^\rho \, T^\mu_\mu. \tag{140}$$

Again, we have used that T is conserved on-shell and that ξ_0 is a conformal Killing of g^0. The only problem this time is that, the boundary metric being non trivial, the second equation of motion (114) gives a non-zero value to the trace of T. On the boundary this is interpreted as a conformal anomaly and the current J is generically not conserved

$$\nabla_\mu J^\mu = \frac{\ell}{32\pi G}\nabla_\rho \xi_0^\rho \, R. \tag{141}$$

We conclude that when the boundary metric is not flat, the Noether charge is generically not conserved and the non conservation is sourced by the boundary scalar curvature, due to the conformal anomaly. Nevertheless, one should note that the Noether charge for exact Killings ξ_0 is conserved, since in that case $\nabla_\rho \xi_0^\rho = 0$. It would be interesting to study the surface charges with these boundary conditions. The answer should be found in [41], this is under investigation.

4.2 Flat limit

In this section we would like to take the flat limit, i.e. $\ell \to +\infty$, of our construction in AdS. We will insist on the boundary structure that emerges when taking this limit. Its interpretation will be that of a Carrollian geometry while the asymptotic symmetries will be shown to isomorphic to conformal symmetries of this geometry.

4.2.1 Solution space

The full solution space in Bondi gauge for vanishing cosmological constant can be readily obtained by taking the flat limit of the solution space obtained in section 4.1.2 for non-vanishing cosmological constant. In practice, we take $\ell \to \infty$ in the equations. The equation $G_{rr} = 0$ gives

$$\beta = \beta_0(u, \phi) \tag{142}$$

Solving $G_{r\phi} = 0$ leads to

$$U = U_0(u,\phi) + \frac{1}{r} 2 e^{2\beta_0} e^{-2\varphi} \partial_\phi \beta_0 - \frac{1}{r^2} e^{2\beta_0} e^{-2\varphi} N(u,\phi). \tag{143}$$

Solving $G_{ur} = 0$ gives

$$\frac{V}{r} = -2r(\partial_u \varphi + D_\phi U_0) + M(u,\phi) + \frac{1}{r} 4 e^{2\beta_0} e^{-2\varphi} N \partial_\phi \beta_0 - \frac{1}{r^2} e^{2\beta_0} e^{-2\varphi} N^2, \tag{144}$$

where $D_\phi U_0 = \partial_\phi U_0 + \partial_\phi \varphi U_0$. Taking into account the previous results, the Einstein equation $G_{\phi\phi} = 0$ is satisfied at all orders. Finally, we solve the Einstein equations giving the time evolution constraints on M and N. The equation $G_{u\phi} = 0$ gives

$$\begin{aligned}(\partial_u + \partial_u \varphi) N &= \left(\frac{1}{2}\partial_\phi + \partial_\phi \beta_0\right) M - 2N\partial_\phi U_0 - U_0(\partial_\phi N + N\partial_\phi \varphi) \\ &\quad + 4e^{2\beta_0} e^{-2\varphi}[2(\partial_\phi \beta_0)^3 - (\partial_\phi \varphi)(\partial_\phi \beta_0)^2 + (\partial_\phi \beta_0)(\partial_\phi^2 \beta_0)], \end{aligned} \tag{145}$$

whereas $G_{uu} = 0$ results in

$$\begin{aligned}\partial_u M &= (-2\partial_u \varphi + 2\partial_u \beta_0 - 2\partial_\phi U_0 + U_0 2\partial_\phi \beta_0 - U_0 2\partial_\phi \varphi - U_0 \partial_\phi) M \\ &\quad -2e^{2\beta_0} e^{-2\varphi}\{\partial_\phi U_0[8(\partial_\phi \beta_0)^2 + (\partial_\phi \varphi)^2 - 4\partial_\phi \beta_0 \partial_\phi \varphi + 4\partial_\phi^2 \beta_0 - 2\partial_\phi^2 \varphi] \\ &\quad -\partial_\phi^3 U_0 + U_0[\partial_\phi \beta_0(8\partial_\phi^2 \beta_0 - 2\partial_\phi^2 \varphi) + 2\partial_\phi^3 \beta_0 - \partial_\phi^3 \varphi + \partial_\phi \varphi(-2\partial_\phi^2 \beta_0 \\ &\quad +\partial_\phi^2 \varphi)] + 2\partial_u \partial_\phi \beta_0(4\partial_\phi \beta_0 - \partial_\phi \varphi) + \partial_u \partial_\phi \varphi(-2\partial_\phi \beta_0 + \partial_\phi \varphi) \\ &\quad +2\partial_u \partial_\phi^2 \beta_0 - \partial_u \partial_\phi^2 \varphi\}. \end{aligned} \tag{146}$$

These last two equations are obtained by taking the $\ell \to \infty$ limit of their AdS counterpart (103) and (104). The solution space is thus characterized by five arbitrary functions of (u, ϕ), given by β_0, U_0, M, N, φ, with two dynamical constraints given by the time evolution equations of M and N. It is exactly the same thing than in AdS, the only things that have changed are that $\frac{V}{r}$ has lost one term and the second conservation equation has been simplified. In practice, the flat limit is very simple to implement, but it changes a lot the interpretation of the boundary structure. Indeed, the boundary metric is now degenerate, due to the missing term in $\frac{V}{r}$. As expected the conformal boundary is a null hypersurface : it is the null infinity. Moreover the two conservation equations cannot be interpreted anymore as the conservation of an energy-momentum tensor. Of course, the answer resides in this limit we have taken and the fact that it maps to an ultra-relativistic limit on the boundary.

4.2.2 From the null infinity to the bulk

Instead of taking the flat limit, one can try to start from a Carroll structure on the boundary, here the null infinity, and try to build a corresponding bulk Ricci-flat solution. This construction is based on a paper with Luca Ciambelli, Marios Petropoulos and Romain Ruzziconi, soon to be published. We recall that a generic Carroll structure was defined in 2.5 in the following way

$$v^\mu \tau_\mu = -1, \quad g_{\mu\nu} v^\mu = 0, \quad g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu + \tau_\mu v^\rho \quad \text{and} \quad g^{\mu\nu}\tau_\nu = 0. \tag{147}$$

We recall that $x^\mu = u, \phi$. We introduce the vielbein $\tau^\star$ and the inverse vielbein $v_\star$ for the degenerate metrics $g_{\mu\nu}$ and $g^{\mu\nu}$, such that the constitutive relations become

$$\tau(v) = -1, \quad \tau^\star(v_\star) = 1, \quad \tau^\star(v) = 0 \quad \text{and} \quad \tau(v_\star) = 0. \tag{148}$$

The metric and its pseudo inverse are related to $\tau^\star$ and $v_\star$ via

$$g_{\mu\nu} = \tau^\star_\mu \tau^\star_\nu \quad \text{and} \quad g^{\mu\nu} = v^\mu_\star v^\nu_\star. \tag{149}$$

We define the Carrollian expansions θ and $\theta^\star$ via

$$d\tau^\star = \theta\, \tau^\star \wedge \tau \quad \text{and} \quad d\tau = \theta^\star\, \tau^\star \wedge \tau, \tag{150}$$

or with the Lie bracket of the vectors

$$[v, v_\star] = \theta^\star v - \theta v_\star. \tag{151}$$

We define the connection

$$A = \theta^\star \tau^\star - \theta \tau, \tag{152}$$

and its scalar curvature

$$s = \star dA = v(\theta^\star) - v_\star(\theta), \tag{153}$$

where we have defined the the $\star$ operator by its action on vectors and forms

$$(\star w)_\mu = \mathcal{D}\varepsilon_{\mu\nu} w^\nu, \quad (\star\omega)^\mu = -\mathcal{D}^{-1}\varepsilon^{\mu\nu}\omega_\nu, \quad \mathcal{D} = |\varepsilon^{\mu\nu}\tau_\mu\tau^\star_\nu|. \tag{154}$$

With these definitions we have

$$v_\star = \star\tau \quad \text{and} \quad v = \star\tau^\star. \tag{155}$$

Since we do not have a non-degenerate spacetime metric, the notion of Hodge duality has to be modified. Without the metric we loose the isomorphism between forms and vectors and the density $\sqrt{-g}$ that allows to define our usual notion of Hodge duality. The $\star$ operator maps vectors to forms and forms to vectors, while its density is built out of the two vielbeins of the Carroll structure.

This concerns the boundary geometry part. Now one would want to introduce the equivalent of a boundary energy-momentum tensor and write its conservation equations, in order to map them to the constraint equations of the flat Bondi gauge. Remember that in the relativistic case, the energy-momentum tensor had two degrees of freedom since it is symmetric and the trace was fixed by the scalar curvature. We suppose that it is the same in the flat case, that there exists two quantities $\epsilon(u,\phi)$ and $q(u,\phi)$ that are the Carrollian equivalent of an energy-momentum tensor. We call them *Carrollian momenta*. The quantities ϵ and q should be interpreted respectively as an energy density and a momentum flow.

We would like to write conservation for these quantities when they are coupled to the Carroll structure (148). For the moment we will just postulate them, later on we will explain how they can be recovered from an ultra-relativistic limit. The two conservation equations are

$$\begin{gathered}(v^\mu\partial_\mu + 2\theta)\epsilon + \frac{1}{4\pi G}(v^\mu_\star\partial_\mu + 2\theta^\star)s = 0,\\ (v^\mu_\star\partial_\mu + 2\theta^\star)\epsilon + (v^\mu\partial_\mu + 2\theta)q = 0.\end{gathered} \tag{156}$$

These equations are the Carrollian equivalent of the conservation of $T^{\mu\nu}$ on the boundary of AdS. The vector v belongs to the kernel of the spatial metric, therefore $v^\mu\partial_\mu$ should be interpreted as a time derivative, while $v^\mu_\star\partial_\mu$ is a spatial derivative. The first equation relates the time evolution of the energy density to the curvature of the Carroll structure and the second one related the gradient of energy density to the time evolution of the momentum flow. It is these two equations that are expected to be the same one as (145) and (146). One should remark how compact and enlightening they are compared to the Bondi ones.

In Bondi AdS, a bulk metric on-shell is entirely characterized by a boundary metric and an energy-momentum tensor that is conserved and satisfies the trace condition. We expect the same to happen in Bondi flat : the knowledge of a Carroll structure on the null infinity and the Carrollian momenta should define the bulk metric. This is materialized by the following reconstruction formula

$$ds^2 = 2\,\tau\otimes(dr + rA) + r^2\tau^\star\otimes\tau^\star + 8\pi G\,\tau\otimes(\epsilon\tau + q\tau^\star). \tag{157}$$

This automatically satisfies $R_{rr} = v^\mu R_{\mu r} = v^\mu_\star R_{\mu r} = 0$. However, the remaining components of the bulk Ricci tensor $v^\mu v^\nu R_{\mu\nu}$, $v^\mu_\star v^\nu_\star R_{\mu\nu}$ and $v^\mu_\star v^\nu R_{\mu\nu}$ vanish if and only if the Carrollian conservations (156) are satisfied. This is satisfying, we have found a reconstruction formula that associates a Ricci-flat metric to any Carroll structure together with Carrollian momenta satisfying our conservation equations.

This is not the end since we have not given the dictionary between the solution space of Bondi gauge and this reconstruction formula. Before doing so we should make a counting of degrees of freedom. In Bondi, the solution space was parametrized by five functions of the boundary coordinates : β_0, U_0, M, N and φ. In the reconstruction formula we have ϵ, q, τ and $\tau^\star$, which corresponds to six functions. All the other geometrical objects, v, $v_\star$, θ and $\theta^\star$ can be written in terms of τ and $\tau^\star$. We conclude that we need to impose a condition on the Carrollian data to really be in one-to-one correspondence with Bondi gauge. This condition can be deduced from the fact that there is no $(r\phi)$-components in Bondi gauge. Which corresponds to

$$\tau_\phi = 0 \tag{158}$$

We now write the dictionary between the Carrollian data and the Bondi ones

$$\tau = -e^{2\beta_0}du, \quad \tau^\star = e^{\varphi}(d\phi - U_0 du), \tag{159}$$

and the Carrollian momenta are

$$\begin{aligned} 8\pi G\epsilon &= e^{-2\beta_0}M + 4e^{-2\varphi}(\partial_\phi \beta_0)^2, \\ 4\pi G q &= -e^{-\varphi}N. \end{aligned} \tag{160}$$

This result is quite satisfying as it allows to understand the solution space of the flat Bondi gauge in terms of a robust boundary structure, exactly like in AdS. It also simplifies greatly the expressions for the bulk metric and the conservation equations.

The last thing to do is to write the asymptotic Killings in terms of our Carrollian data and their transformations. The asymptotic Killings of the flat Bondi gauge are the same ones than in the AdS case (111), (112) and (113). We make the same redefinition of the Killing data than we did in the AdS case

$$\begin{aligned} \xi_0^t &= f, \\ \xi_0^\phi &= Y, \\ \sigma &= \partial_\phi Y - \omega - U_0\partial_\phi f + Y\partial_\phi\varphi + f\partial_t\varphi. \end{aligned} \tag{161}$$

Moreover we define the vector

$$\xi_1 = \left(\mathcal{L}_{\xi_0}\tau(v_\star)\right) v_\star, \tag{162}$$

This rewriting therefore trades the functions f, Y and ω for three new functions : a vector ξ_0 and a scalar σ. We have introduced the vector ξ_1 to lighten the formula but it is entirely specified by our three new Killing pieces of data and the Bondi fields. The Killing (111), (112) and (113) becomes

$$\begin{aligned} \xi^\mu &= \xi_0^\mu + \frac{1}{r}\xi_1^\mu, \\ \xi^r &= -r\sigma + d^\dagger\xi_1 - \frac{4\pi G}{r}\, q\tau^\star(\xi_1). \end{aligned} \tag{163}$$

where we have defined the divergence operator associated to our Hodge : for a vector w the divergence is $d^\dagger w = \star d \star w = -(\partial_\mu + \partial_\mu \log \mathcal{D})w^\mu$. We also rewrite the transformations of M, N, β_0, U_0 and φ under the Killing as transformations of τ, $\tau^\star$, ϵ and q . The boundary geometry transforms as

$$\begin{aligned} \delta_\xi \tau^\star &= \mathcal{L}_{\xi_0}\tau^\star - \sigma\tau^\star, \\ \delta_\xi \tau &= h, \end{aligned} \tag{164}$$

where h is a one-form defined by [4]

$$h_u = \frac{\mathcal{D}}{\tau^\star_\phi}\left(\mathcal{L}_{\xi_0}\log\mathcal{D} - \mathcal{L}_{\xi_0}\tau^\star(v_\star) - \sigma\right), \quad h_\phi = 0. \tag{165}$$

While the variations of the Carrollian momenta are

$$\begin{aligned} \delta_\xi\epsilon &= \mathcal{L}_{\xi_0}\epsilon + 2\sigma\epsilon - \frac{1}{4\pi G}\left[\mathcal{L}_{\xi_1}A(v) - \theta\mathcal{L}_{\xi_1}\tau(v) + (v+\theta)(d^\dagger\xi_1)\right], \\ \delta_\xi q &= \mathcal{L}_{\xi_0}q + 2\sigma q + 2\tau^\star(\xi_1)\epsilon + \frac{1}{4\pi G}\left[\mathcal{L}_{\xi_1}A(v_\star) - \theta^\star\mathcal{L}_{\xi_1}\tau(v) + (v_\star + \theta^\star)(d^\dagger\xi_1)\right]. \end{aligned} \tag{166}$$

4. The function $\mathcal{D}$ is a density, therefore its Lie derivative has an additional divergence term : $\mathcal{L}_{\xi(0)}\log\mathcal{D} = \xi^a_{(0)}\partial_a\log\mathcal{D} + \partial_a\xi^a_{(0)}$.

It is clear form these transformations that ξ_0 correspond to a diffeomorphism on the boundary while σ is a Weyl transformation. The terms proportional to $\frac{1}{4\pi G}$ are the Carrollian equivalent of the anomalous part in the transformation of the energy-momentum tensor in AdS. From the transformation of $\tau^\star$ we can deduce the transformation of the boundary degenerate metric

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_{\xi_0} g_{\mu\nu} - 2\sigma g_{\mu\nu}, \tag{167}$$

which is exactly the same transformation as the one found for the boundary metric in AdS (139).

4.2.3 BMS charges

Finally, we can impose an equivalent of the Brown-Henneaux boundary condition, which consists in imposing the boundary geometry to be flat

$$\tau^\star = d\phi \quad \text{and} \quad \tau = du. \tag{168}$$

In terms of the Bondi data it corresponds to asking φ, U_0 and β_0 to vanish. This boundary condition must be preserved by the asymptotic Killings, the first equation of (165) tells us that ξ_0 must be a conformal transformation of $\tau^\star$, this imposes

$$\partial_\phi \xi_0^\phi = \sigma, \quad \partial_u \xi_0^\phi = 0. \tag{169}$$

While asking τ to be unchanged imposes $\partial_u \xi_0^u = \partial_\phi \xi_0^\phi$. The bulk Killings are therefore entirely specified by the following boundary vector

$$\xi_0 = (\partial_\phi Y(\phi) u + T(\phi))\partial_u + Y(\phi)\partial_\phi, \tag{170}$$

for any functions T and Y. We recover the BMS$_3$ algebra that we first described in 2.3. This algebra coincides with the conformal symmetries of the boundary Carroll manifold, i.e. the null infinity equipped with the vector $v = \partial_u$ and the degenerate metric $g = (\tau^\star)^2 = d\phi^2$. If the angular coordinate describes a circle, the corresponding group is

$$\mathrm{Diff}(S^1) \ltimes \mathcal{C}^\infty(S^1). \tag{171}$$

Now a subgroup of this group is $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathcal{T}$, where $\mathcal{T}$ corresponds to the translations in 3d, i.e. the functions T than turn on only the three first harmonics of the circle. While the $\mathrm{SL}(2,\mathbb{R})$ part corresponds to the two boosts and the rotation, i.e. the Lorentz group. At the level of the algebra, they also correspond to generators that turn on only the first three harmonics. In 3d, the asymptotic symmetry is also an infinite-dimensional extension of the Poincaré group. The main difference being that the Lorentz part of the group receives also an infinite extension, it is now the infinite dimensional Virasoro group. The corresponding transformations are called *superrotations*. The consequences of this symmetry are deep for three-dimensional asymptotically flat gravity and were extensively studied in [42].

It is satisfying to observe that both in AdS and in flat space, imposing on the phase space that the boundary geometry is trivial (Minkowski 2d in AdS and flat Carroll manifold in flat space) lead to an asymptotic symmetry algebra that is isomorphic to the conformal algebra of the corresponding boundary geometry. It turns out to be also the case in four dimensions when similar boundary conditions are imposed [43, 44]. The only difference is that the asymptotic symmetry group in AdS is finite-dimensional, it is the three-dimensional conformal group, while in flat space it is the BMS$_4$ group, which is infinite dimensional but corresponds indeed to the conformal symmetries of the Carroll manifold $v = \partial_u$ and $g = d\theta^2 + \sin^2\theta d\phi^2$. We should point out that an AdS version of the BMS group in 4d was found in [45].

With these conditions on the boundary geometry, the Carrollian conservation equations become very simple

$$\begin{aligned} \partial_u \epsilon &= 0, \\ \partial_\phi \epsilon + \partial_\phi q &= 0. \end{aligned} \tag{172}$$

The solution is $\epsilon(u,\phi) = \epsilon_0(\phi)$ and $q(u,\phi) = -\partial_\phi \epsilon_0(\phi) + q_0(\phi)$. Now one can also compute the gravitational charges associated to the BMS generators, this was done in [13] and using the dictionary established earlier in Eq. (160), we can write them in terms of the Carrollian data. With our boundary conditions they are integrable and their integrated version is

$$Q_\xi = \frac{1}{2}\int d\phi \left(T(\phi)\left(\epsilon_0(\phi) + \frac{1}{8\pi G}\right) - Y(\phi) q_0(\phi) \right). \tag{173}$$

We have already shifted ϵ_0 in order to absorb a trivial central charge. The conservation of this charge is obvious since it depends explicitly only on ϕ. We can do the same thing than in AdS and expand these charges into modes

$$P_n = \frac{1}{2}\int d\phi\, e^{in\phi}\left(\epsilon_0 + \frac{1}{8\pi G}\right), \quad J_n = -\frac{1}{2}\int d\phi\, e^{in\phi} q_0. \tag{174}$$

The transformations of ϵ_0 and q_0 under a BMS transformation are

$$\begin{aligned} \delta_\xi \epsilon_0 &= 2\epsilon_0 Y' + Y\epsilon' + \frac{Y'''}{4\pi G}, \\ \delta_\xi q_0 &= 2q_0 Y' + Y q_0' - 2T'\epsilon_0 - T\epsilon_0' + \frac{T'''}{4\pi G}. \end{aligned} \tag{175}$$

The third derivative terms in the transformations are going to produce a non-trivial central charge in the algebra. Indeed, we obtain

$$\begin{aligned} &i\{J_m, J_n\} = (m-n)J_{m+n}, \quad i\{J_m, P_n\} = (m-n)P_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \\ &i\{P_m, P_n\} = 0, \end{aligned} \tag{176}$$

with $c = {}^{3}\!/_{G}$. This central charge is the equivalent of the Brown-Henneaux central charge in asymptotically flat spacetimes.

4.2.4 Ultra-relativistic limit

Until now, we have studied the asymptotically flat case without referring to a flat limit. We would like now to understand the flat limit from a boundary perspective. We have postulated the Carrollian conservation equations, and the the Ricci-flat Einstein equations match indeed with these two equations for the corresponding metric ansatz (157). But we can also recover them through a flat limit. In AdS, they clearly correspond to the conservation of the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0. \tag{177}$$

Therefore it is this equation that we should take the limit of. The boundary data in AdS consist of a metric $g^0_{\mu\nu}$ and an energy-momentum tensor $T^{\mu\nu}$. Instead, we can consider a vielbeins and decompose the energy momentum tensor in this basis. Following my paper [30], we write the metric as

$$g^0 = \ell^2 \eta_{ij} \mathrm{u}^i \mathrm{u}^j, \tag{178}$$

where we define $\mathrm{u}^0 = \mathrm{u}$ and $\mathrm{u}^1 = \mathrm{u}^\star$, such that the metric, in coordinates becomes

$$g^0_{\mu\nu} = \ell^2(-u_\mu u_\nu + u^\star_\mu u^\star_\nu). \tag{179}$$

With the help of this Cartan basis, we can write the energy-momentum tensor in terms of three scalars, $\tilde{\varepsilon}$, $\tilde{\chi}$ and $\tilde{\tau}$. They correspond to all the independent components of a symmetric tensor in 2d. We define the energy-momentum tensor as follows

$$T_{\mu\nu} = \ell^2(\tilde{\varepsilon} u_\mu u_\nu + \tilde{\chi} u_\mu u^\star_\nu + \tilde{\chi} u^\star_\mu u_\nu + \tilde{\varepsilon} u^\star_\mu u^\star_\nu + \tilde{\tau} u^\star_\mu u^\star_\nu). \tag{180}$$

We know that in AdS, the trace of the energy momentum tensor is fixed by the curvature of the boundary geometry

$$R = 2\ell^2(\Theta^2 - \Theta_\star^2 + \vec{u}(\Theta) - \vec{u}_\star(\Theta_\star)), \tag{181}$$

where $\vec{u}$ and $\vec{u}_\star$ is a dual basis, satisfying

$$\mathrm{u}(\vec{u}) = -\frac{1}{\ell^2}, \quad \mathrm{u}^\star(\vec{u}_\star) = \frac{1}{\ell^2}, \quad \mathrm{u}(\vec{u}_\star) = 0, \quad \mathrm{u}^\star(\vec{u}) = 0. \tag{182}$$

We have also defined the corresponding expansions by computing the exterior derivatives of the Cartan basis

$$d\mathrm{u}^\star = \ell^2\Theta\, \mathrm{u}^\star \wedge \mathrm{u}, \quad d\mathrm{u} = \ell^2\Theta_\star\, \mathrm{u}^\star \wedge \mathrm{u}. \tag{183}$$

We conclude that the trace, which corresponds to τ must be

$$\tau = \frac{R}{8\pi G}. \tag{184}$$

We are left with the two parameters $\tilde{\varepsilon}$ and $\tilde{\chi}$. Finding their expressions to recover the Carrollian conservation equations (156) is a little bit tricky so we will just spell out the result. We parametrize them in the following way

$$\begin{aligned} \tilde{\varepsilon} &= \varepsilon + \frac{\ell^2}{8\pi G}(\vec{u}(\Theta) + \vec{u}_\star(\Theta_\star)) - \frac{R}{16\pi G}, \\ \tilde{\chi} &= \chi - \frac{\ell^2}{4\pi G}\vec{u}_\star(\Theta), \end{aligned} \tag{185}$$

This is just a reparametrization of the two independent components of the energy–momentum tensor. Now the main point of this section is that if we impose particular scalings for u, $\mathrm{u}^\star$, ε and χ (consistent with the their scaling when written in terms of AdS Bondi data) and we take the $\ell \to \infty$ limit of (177), we recover the Carrollian conservations (156). The scalings are

$$\begin{aligned} \tau &= \lim_{\ell\to\infty} \ell^2 \mathrm{u}, \\ \tau^\star &= \lim_{\ell\to\infty} \ell \mathrm{u}^\star, \\ \epsilon &= \lim_{\ell\to\infty} \varepsilon, \\ q &= \lim_{\ell\to\infty} \ell\chi. \end{aligned} \tag{186}$$

With these scalings, the boundary metric becomes degenerate in the limit

$$g^0 = \ell^2(-\mathrm{u}^2 + (\mathrm{u}^\star)^2) \underset{\ell\to\infty}{\to} (\tau^\star)^2, \tag{187}$$

and coincides exactly with the Carrollian degenerate boundary (149). Exactly like for the flat Bondi gauge, all the new pieces of data, ε, χ, u and $\mathrm{u}^\star$ can be matched with the five functions parametrizing the solution space of AdS Bondi while the bulk metric can be rewritten in terms of these new objects. To achieve a true bijection between the solution spaces, it is also necessary to fix the ϕ-component of u at zero.

4.2.5 Carrollian charges

Earlier we have obtained the gravitational charges for the boundary conditions $\tau^\star = d\phi$ and $\tau = du$. We notice that they can be written as

$$Q_\xi = \frac{1}{2}\int d\phi \left(\xi_0^u \epsilon - \xi_0^\phi q\right), \tag{188}$$

in terms of the Carrollian momenta and the boundary vector field that uniquely specifies the asymptotic Killing in that case. This looks very similar to the formula (131). Indeed, consider the following energy-momentum tensor

$$T^{\mu}_{\nu} = \frac{1}{2}\begin{pmatrix} \epsilon & q \\ 0 & \epsilon \end{pmatrix}. \tag{189}$$

The charges becomes

$$Q_{\xi} = \int d\phi\, \xi_0^{\mu} T^0_{\mu}, \tag{190}$$

which looks like a Noether charge again. The only difference is that the vector field is now a Carrollian conformal Killing, so the true reason for this charge to be conserved has changed. This is what we would like to understand now. It will be instructive to consider curved Carroll structure to study in more details the notion of Carrollian charge, but doing it in full generality is slightly heavy so we are going to consider a specific (but natural) class of Carroll structure. This analysis is based on my paper [29].

Consider the following parametrization for the boundary geometry

$$\tau = -\Omega du + b d\phi, \quad \tau^{\star} = \sqrt{a} d\phi, \tag{191}$$

where Ω, b and a all depend on both u and ϕ. The corresponding vectors are

$$v = \Omega^{-1}\partial_u, \quad v_{\star} = \frac{1}{\sqrt{a}}(\partial_{\phi} + \frac{b}{\Omega}\partial_u) \equiv \frac{1}{\sqrt{a}}\hat{\partial}_{\phi}. \tag{192}$$

One can check that they satisfy the defining relations (148). This Carroll structure cannot be captured by the Bondi case since in the Bondi gauge we have $\tau_{\phi} = 0$. The degenerate metric is purely spatial

$$(\tau^{\star})^2 = a d\phi^2, \tag{193}$$

and can be used to raise and lower spatial indexes :

$$w_{\phi} = a w^{\phi}, \quad w^{\phi} = a^{-1} w_{\phi}. \tag{194}$$

With this parametrization, the Carrollian conservation equations becomes

$$\begin{aligned} (\Omega^{-1}\partial_u + 2\theta)\epsilon + \frac{1}{4\pi G}(\hat{\partial}_{\phi} + 2\sqrt{a}\theta^{\star})(\sqrt{a}s) = 0, \\ (\hat{\partial}_{\phi} + 2\sqrt{a}\theta^{\star})\epsilon + (\Omega^{-1}\partial_u + \theta)(\sqrt{a}\, q) = 0, \end{aligned} \tag{195}$$

where the values for the expansions and the scalar curvature are (these objects were defined in Sec. 4.2.2)

$$\begin{aligned} \theta &= \Omega^{-1}\partial_u \log\sqrt{a}, \\ \sqrt{a}\theta^{\star} &= \Omega^{-1}(\partial_{\phi}\Omega + \partial_u b), \\ \sqrt{a}s &= \Omega^{-1}\partial_u(\sqrt{a}\theta^{\star}) - \theta\sqrt{a}\theta^{\star} - \hat{\partial}_{\phi}\theta. \end{aligned} \tag{196}$$

All these objects measure the "curvature" of the Carroll structure. The first one is interpreted as an expansion of the spatial metric. The second one and third one are harder to interpret physically. We recall that s is the scalar curvature associated with the Carrollian connection A introduced in Eq. (152). The only thing to understand is that there are two classes of objects in these Carrollian conservation equations. The one characterizing the geometry : Ω, b, a, θ, $\theta^{\star}$, s and the Carrollian momenta ϵ and q. Now we observe that s sources the time evolution of ϵ in the first equation. When considering the flat geometry, this term vanishes, we can then interpret (195) as the conservation of the tensor (189) and the corresponding charges are conserved. We are going to show that when $s \neq 0$, the ultra-relativistic equivalent of the Noether charge is not conserved anymore.

We would like to build an equivalent of the covariant charge (136) in the Carrollian case. The simplest thing to do is to compute its ultra-relativistic limit using the scalings we have found in the previous section. This is what we do and we find

$$Q_\xi = \frac{1}{2}\int d\phi\sqrt{a}\left((\Omega\xi_0^u - 2b\xi_0^\phi)\epsilon - \xi_0^\phi\sqrt{a}q\right) + \frac{1}{8\pi G}\int d\phi\sqrt{a}(\Omega\xi_0^u - 2b\xi_0^\phi)b\sqrt{a}s. \tag{197}$$

The first thing to notice is that for a flat Carroll structure, which corresponds to the values $\Omega = 1$, $a = 1$ and $b = 0$, the first term coincides with the BMS charge (188), while the second term vanishes. The conservation (or not) of this charge is far from obvious since all the fields depend both on u and ϕ. In AdS, things were simple thanks to the covariance of the formulae.

We would like to know if this charge is conserved or not when ξ_0 is a conformal Killing of the Carroll manifold $g = ad\phi^2$, $v = \Omega^{-1}\partial_u$ and when the conservation equations are satisfied, i.e. when we are on-shell (exactly like in AdS where we assumed that the vector field was a conformal Killing of the boundary metric and the energy-momentum tensor was conserved). The computation is easy but lengthy. Asking ξ_0 to be a conformal Killing imposes three differential equations

$$\begin{aligned}
&\xi_0^u\partial_u\Omega + \xi_0^\phi\partial_\phi\Omega + \Omega\partial_u\xi_0^u = \sigma\Omega,\\
&\partial_u\xi_0^\phi = 0,\\
&\xi_0^u\partial_u a + \xi_0^\phi\partial_\phi a + 2a\partial_\phi\xi_0^\phi = 2\sigma a
\end{aligned} \tag{198}$$

Using these equations, the conservation (195) and many integrations by part, we finally obtain

$$\partial_u Q_\xi \propto \int d\phi\sqrt{a}\left(\hat{\partial}_\phi(\Omega\xi_0^u - b\xi_0^\phi) - \sqrt{a}\theta^\star(\Omega\xi_0^u - b\xi_0^\phi)\right)\sqrt{a}\,s. \tag{199}$$

We observe that the only thing that is responsible for the non-conservation is this function s that we interpret as a scalar curvature of the Carroll structure in 4.2.2. It is exactly like in AdS where the covariant Noether charge was also not conserved and the scalar curvature R was responsible. It would be interesting to investigate the relationship between the Noether charges for a sourced Carroll structure and the gravitational charges associated with the corresponding bulk metric (157).

4.2.6 The parameter τ_ϕ

We have shown that if we want to recover the entire solution space of Bondi gauge in flat space, it is sufficient to consider a boundary Carroll structure with an Ehresmann connection satisfying $\tau_\phi = 0$. A natural question to ask is : what happens when $\tau_\phi \neq 0$? If we believe that the solution space of Bondi gauge contains all possible solutions of three-dimensional Ricci-flat gravity then it should mean that τ_ϕ is not a real parameter : it is pure gauge and can always be removed by a diffeomorphism that does not affect the charges. If not then this parameter will carry physical meaning and holding it fixed but non zero could define a family of non equivalent phase spaces for three-dimensional gravity. In our future publication, we show that all the solution spaces defined by different values of τ_ϕ can be reached by acting with a diffeomorphism on the Bondi solution space. Future directions of research include finding a proper gauge fixing procedure in three dimensions in which this parameter arises naturally, computing the asymptotic symmetries and derive the corresponding charges.

5 Relativistic and Carrollian conformal hydrodynamics

This section is an introduction to the relativistic dynamics of conformal fluid and its ultra-relativistic limit. In both cases, we describe the building blocks necessary to the description of a fluid and the geometry to which it couples. The latter is simply a pseudo-Riemannian manifold in the relativistic case, while the ultra-relativistic fluid couples to a Carrollian geometry readily obtained as the $c \to 0$ limit of its relativistic counterpart. The Carrollian fluid dynamics was derived in full generality (together with its Galilean counterpart), in any dimension, in my paper [27]. Here we study only the three-dimensional case that will be of central importance when we describe the flat version of fluid/gravity correspondence in bulk-four dimensions.

5.1 Relativistic hydrodynamics

In hydrodynamics the physics is not described with the usual Lagrangian picture. The action and its equations of motion are traded for local conservation laws that are supposed to capture all the essential dynamics. Quantities such as the energy density, the momentum flow, the stress, the number of particle, the electric charges, etc, can be subject to these conservation laws. The minimum of ingredients for a fluid with no particular symmetry are the energy density $\varepsilon(x)$, the density of pressure $p(x)$ and the fluid velocity, which is a time-like congruence satisfying $u_\mu u^\mu = -c^2$. The energy-momentum tensor is then written as

$$T_{\mu\nu} = (\varepsilon + p)\frac{u_\mu u_\nu}{c^2} + p g_{\mu\nu} + \tau_{\mu\nu} + \frac{u_\mu q_\nu}{c^2} + \frac{u_\nu q_\mu}{c^2}, \tag{200}$$

where $\tau_{\mu\nu}$ and q_μ are respectively the viscous stress tensor and the heat current. They are purely transverse

$$u^\mu \tau_{\mu\nu} = 0, \quad u^\mu q_\mu = 0, \quad q_\nu = -\varepsilon u_\nu - u^\mu T_{\mu\nu}. \tag{201}$$

Now for a given normalized time-like congruence, one can always decompose a symmetric tensor as we just did. The main difference here is that $\tau_{\mu\nu}$ and q_μ are not going to be considered as independent variables. They are going to be written in terms of spacetime derivatives of p, ε and the fluid velocity : they are the fluid's dissipative terms. Doing so, the dynamics is then entirely capture by the conservation of the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0, \tag{202}$$

and an equation of state $p(T)$ which relates the pressure and the temperature.

We are going to study holographic fluids, i.e. fluids whose energy-momentum tensor coincide with the holographic energy-momentum tensor of the corresponding AdS solution, therefore it will be sufficient to consider conformal fluids. For the latter, the equation of state relates the energy density and the pressure in a very simple way

$$\varepsilon = 2p. \tag{203}$$

Moreover, the stress tensor is traceless τ^μ_μ. These two equations follow simply from the fact that the energy-momentum tensor must be traceless for a conformal fluid

$$T^\mu_\mu = 0. \tag{204}$$

We should comment on the fact that this is true only for even bulk dimension. In odd number of dimensions there is a conformal anomaly that should be taken into account.

The order of expansion in hydrodynamics is the number of derivatives of the fields, it is assumed that every quantities is slowly varying. In principle, at each order in perturbation, the tensors built out of derivatives of ε and u^μ can be classified but it becomes rapidly tedious. As an example we can give the most generic first order fluid. It is characterized by three quantities : η and ζ that are the shear and bulk viscosities, and κ the thermal conductivity. They appear in the dissipative tensors in the following way

$$\tau_{(1)\mu\nu} = -2\eta\sigma_{\mu\nu} - \zeta h_{\mu\nu}\Theta, \tag{205}$$

$$q_{(1)\mu} = -\kappa h_\mu{}^\nu \left(\partial_\nu T + \frac{T}{c^2}\, a_\nu\right), \tag{206}$$

where $h_{\mu\nu}$ is the projector onto the space transverse to the velocity field :

$$h_{\mu\nu} = \frac{u_\mu u_\nu}{k^2} + g_{\mu\nu}, \tag{207}$$

and [5]

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \tag{208}$$

$$\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)} + \tfrac{1}{c^2} u_{(\mu} a_{\nu)} - \tfrac{1}{2}\Theta h_{\mu\nu}, \tag{209}$$

$$\omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]} + \tfrac{1}{c^2} u_{[\mu} a_{\nu]}, \tag{210}$$

are the acceleration (transverse), the expansion, the shear and the vorticity (both rank-two tensors are transverse and traceless).

It is customary to introduce the vorticity two-form

$$\omega = \frac{1}{2}\omega_{\mu\nu}\,\mathrm{dx}^\mu \wedge \mathrm{dx}^\nu = \frac{1}{2}\left(\mathrm{du} + \frac{1}{c^2}\mathrm{u}\wedge\mathrm{a}\right), \tag{211}$$

as well as the Hodge–Poincaré dual of this form, which is proportional to u (we are in $2+1$ dimensions) :

$$c\gamma \mathsf{u} = \star\omega \quad \Leftrightarrow \quad c\gamma u_\mu = \frac{1}{2}\eta_{\mu\nu\sigma}\omega^{\nu\sigma}, \tag{212}$$

where $\eta_{\mu\nu\sigma} = \sqrt{-g}\epsilon_{\mu\nu\sigma}$. In this expression γ is a scalar, that can also be expressed as

$$\gamma^2 = \frac{1}{2c^4}\omega_{\mu\nu}\omega^{\mu\nu}. \tag{213}$$

In three spacetime dimensions and in the presence of a vector field, one naturally defines a fully antisymmetric two-index tensor as

$$\eta_{\mu\nu} = -\frac{u^\rho}{c}\eta_{\rho\mu\nu}, \tag{214}$$

obeying

$$\eta_{\mu\sigma}\eta_\nu{}^\sigma = h_{\mu\nu}. \tag{215}$$

With this tensor the vorticity reads :

$$\omega_{\mu\nu} = c^2\gamma\eta_{\mu\nu}. \tag{216}$$

5.2 Weyl covariance, Weyl connection and the Cotton tensor

The fluid we are considering is conformal. This means that we assume that the fluid corresponds to the effective theory emerging from a CFT in three dimensions. This means that the whole description must be covariant under a Weyl rescaling. This will be our guideline when we write dynamical equations or dissipative tensors. We recall that a Weyl rescaling is a transformation of the metric

$$\mathrm{d}s^2 \to \frac{\mathrm{d}s^2}{\mathcal{B}^2}, \tag{217}$$

At the same time, u_μ is traded for $u_\mu/\mathcal{B}$ (velocity one-form), $\omega_{\mu\nu}$ for $\omega_{\mu\nu}/\mathcal{B}$ (vorticity two-form) and $T_{\mu\nu}$ for $\mathcal{B}T_{\mu\nu}$ see [20]. As a consequence, the pressure and energy density have weight 3, the heat-current q_μ weight 2, and the viscous stress tensor $\tau_{\mu\nu}$ weight 1.

Covariantization with respect to rescaling requires to introduce a Weyl connection one-form : [6]

$$\mathsf{A} = \frac{1}{c^2}\left(\mathsf{a} - \frac{\Theta}{2}\mathsf{u}\right), \tag{218}$$

5. Our conventions for (anti-) symmetrization are : $A_{(\mu\nu)} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)$ and $A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right)$.

6. The explicit form of A is obtained by demanding $\mathcal{D}_\mu u^\mu = 0$ and $u^\lambda \mathcal{D}_\lambda u_\mu = 0$.

which transforms as $\mathsf{A} \to \mathsf{A} - \mathrm{d}\log\mathcal{B}$. Ordinary covariant derivatives ∇ are thus traded for Weyl covariant ones $\mathcal{D} = \nabla + w\,\mathsf{A}$, w being the conformal weight of the tensor under consideration. We provide for concreteness the Weyl covariant derivative of a weight-w form v_μ :

$$\mathcal{D}_\nu v_\mu = \nabla_\nu v_\mu + (w+1)A_\nu v_\mu + A_\mu v_\nu - g_{\mu\nu}A^\rho v_\rho. \tag{219}$$

The Weyl covariant derivative is metric with effective torsion :

$$\begin{aligned} \mathcal{D}_\rho g_{\mu\nu} &= 0, && (220)\\ (\mathcal{D}_\mu\mathcal{D}_\nu - \mathcal{D}_\nu\mathcal{D}_\mu)\, f &= wfF_{\mu\nu}, && (221) \end{aligned}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{222}$$

is Weyl-invariant.

Commuting the Weyl-covariant derivatives acting on vectors, as usual one defines the Weyl covariant Riemann tensor

$$(\mathcal{D}_\mu\mathcal{D}_\nu - \mathcal{D}_\nu\mathcal{D}_\mu)\, V^\rho = \mathcal{R}^\rho{}_{\sigma\mu\nu}V^\sigma + wV^\rho F_{\mu\nu} \tag{223}$$

(V^ρ are weight-w) and the usual subsequent quantities. In three spacetime dimensions, the covariant Ricci (weight 0) and the scalar (weight 2) curvatures read :

$$\begin{aligned} \mathcal{R}_{\mu\nu} &= R_{\mu\nu} + \nabla_\nu A_\mu + A_\mu A_\nu + g_{\mu\nu}\left(\nabla_\lambda A^\lambda - A_\lambda A^\lambda\right) - F_{\mu\nu}, && (224)\\ \mathcal{R} &= R + 4\nabla_\mu A^\mu - 2A_\mu A^\mu. && (225) \end{aligned}$$

The Weyl-invariant Schouten tensor [7] is

$$\mathcal{S}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{4}\mathcal{R}g_{\mu\nu} = S_{\mu\nu} + \nabla_\nu A_\mu + A_\mu A_\nu - \frac{1}{2}A_\lambda A^\lambda g_{\mu\nu} - F_{\mu\nu}. \tag{226}$$

Other Weyl-covariant velocity-related quantities are

$$\begin{aligned} \mathcal{D}_\mu u_\nu &= \nabla_\mu u_\nu + \frac{1}{c^2}u_\mu a_\nu - \frac{\Theta}{2}h_{\mu\nu}\\ &= \sigma_{\mu\nu} + \omega_{\mu\nu}, && (227)\\ \mathcal{D}_\nu \omega^\nu{}_\mu &= \nabla_\nu\omega^\nu{}_\mu, && (228)\\ \mathcal{D}_\nu \eta^\nu{}_\mu &= 2\gamma u_\mu, && (229)\\ u^\lambda\mathcal{R}_{\lambda\mu} &= \mathcal{D}_\lambda\left(\sigma^\lambda{}_\mu - \omega^\lambda{}_\mu\right) - u^\lambda F_{\lambda\mu}, && (230) \end{aligned}$$

of weights -1, 1, 0 and 1 (the scalar vorticity γ has weight 1).

A geometrical object of central importance in what will follow is the Cotton tensor. It is generically a three-index tensor with mixed symmetries. In three dimensions, which is the case for our boundary geometry, the Cotton tensor can be dualized into a two-index, symmetric and traceless tensor. It is defined as

$$C_{\mu\nu} = \eta_\mu{}^{\rho\sigma}\mathcal{D}_\rho\left(\mathcal{S}_{\nu\sigma} + F_{\nu\sigma}\right) = \eta_\mu{}^{\rho\sigma}\nabla_\rho\left(R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}\right). \tag{231}$$

The Cotton tensor is Weyl-covariant of weight 1 (i.e. transforms as $C_{\mu\nu} \to \mathcal{B}\, C_{\mu\nu}$), and is *identically* conserved :

$$\mathcal{D}_\rho C^\rho{}_\nu = \nabla_\rho C^\rho{}_\nu = 0, \tag{232}$$

sharing thereby all properties of the energy–momentum tensor. Following (200) we can decompose the Cotton tensor into longitudinal, transverse and mixed components with respect to the fluid velocity u :

$$C_{\mu\nu} = \frac{3C}{2}\frac{u_\mu u_\nu}{c} + c\frac{C}{2}g_{\mu\nu} - \frac{c_{\mu\nu}}{c} + \frac{u_\mu c_\nu}{c} + \frac{u_\nu c_\mu}{c}. \tag{233}$$

7. The ordinary Schouten tensor in three spacetime dimensions is given by $R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}$.

Such a decomposition naturally defines the weight-3 *Cotton scalar density*

$$C = \frac{1}{c^3} C_{\mu\nu} u^\mu u^\nu, \tag{234}$$

as the longitudinal component. The symmetric and traceless *Cotton stress tensor* $c_{\mu\nu}$ and the *Cotton current* c_μ (weights 1 and 2, respectively) are purely transverse :

$$c_\mu{}^\mu = 0, \quad u^\mu c_{\mu\nu} = 0, \quad u^\mu c_\mu = 0, \tag{235}$$

and obey

$$c_{\mu\nu} = -c\, h^\rho{}_\mu h^\sigma{}_\nu C_{\rho\sigma} + \frac{Cc^2}{2} h_{\mu\nu}, \quad c_\nu = -C u_\nu - \frac{u^\mu C_{\mu\nu}}{c}. \tag{236}$$

One can use the definition (231) to further express the Cotton density, current and stress tensor as ordinary or Weyl derivatives of the curvature. We find

$$C = \frac{1}{c^2} u^\nu \eta^{\sigma\rho} \mathcal{D}_\rho \left(\mathcal{S}_{\nu\sigma} + F_{\nu\sigma} \right), \tag{237}$$

$$c_\nu = \eta^{\rho\sigma} \mathcal{D}_\rho \left(\mathcal{S}_{\nu\sigma} + F_{\nu\sigma} \right) - C u_\nu, \tag{238}$$

$$c_{\mu\nu} = -h^\lambda{}_\mu \left(k\eta_\nu{}^{\rho\sigma} - u_\nu \eta^{\rho\sigma} \right) \mathcal{D}_\rho \left(\mathcal{S}_{\lambda\sigma} + F_{\lambda\sigma} \right) + \frac{Cc^2}{2} h_{\mu\nu}. \tag{239}$$

5.3 Carrollian geometry

In order to properly describe an ultra-relativistic fluid, we need to understand the structure of the Carrollian geometry to which it couples. This will be done by studying the $c \to 0$ limit of the pseudo-Riemannian geometry described in the previous section.

We start with the relativistic metric, that we write it in the Randers-Papapetrou parametrization

$$ds^2_{\text{bdry}} = -c^2 (\Omega du - b_A dx^A)^2 + a_{AB} dx^A dx^B, \tag{240}$$

where Ω and b_A depend both on u and x^A (we recall that the bulk coordinates are $x^a = \{u, r, x^A\} = \{r, x^\mu\}$). We make also a choice of normalized fluid velocity

$$\mathsf{u} = \Omega^{-1} \partial_u, \tag{241}$$

the fluid is at rest. At this level of the discussion we make this choice for simplicity, a more generic case was consider in my paper [27]. It is clear that when we are going to take the ultra-relativistic limit, the resulting Carroll structure (defined in Sec. 2.5) will be composed of

$$g = a_{AB} dx^A dx^B, \quad v = \mathsf{u} = \Omega^{-1} \partial_u, \quad \tau = -(\Omega du - b_A dx^A). \tag{242}$$

The boundary metric is indeed degenerate, the Carrollian vector field, which coincides with the fluid velocity, belongs to its kernel and the Ehresmann connection is given by the second order of the relativistic metric. Ultimately we would like to write all the equations with a manifest splitting between the time coordinate u and the spatial ones x^A. This is not necessary but it allows for a simpler physical interpretation. This splitting breaks the full boundary covariance into a smaller one that we dub *Carrollian covariance*.

We define the Carrollian diffeomorphisms as

$$u' = u'(u, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \tag{243}$$

with Jacobian functions

$$J(u, \mathbf{x}) = \frac{\partial u'}{\partial u}, \quad j_A(u, \mathbf{x}) = \frac{\partial u'}{\partial x^A}, \quad J^A_B(\mathbf{x}) = \frac{\partial x^{A\prime}}{\partial x^B}. \tag{244}$$

Those are the diffeomorphisms adapted to the Carrollian geometry we have chosen since under such transformations, the fluid velocity remains proportional to ∂_u. Indeed,

$$a'_{AB} = a_{CD} J^{-1C}_{\ \ A} J^{-1D}_{\ \ B}, \quad b'_C = \left(b_A + \frac{\Omega}{J} j_A\right) J^{-1A}_{\ \ C}, \quad \Omega' = \frac{\Omega}{J}, \tag{245}$$

whereas the time and space derivatives become

$$\partial'_u = \frac{1}{J}\partial_u, \quad \partial'_B = J^{-1A}_{\ \ B}\left(\partial_A - \frac{j_A}{J}\partial_t\right). \tag{246}$$

We will show in a short while that the Carrollian fluid equations, i.e. the $k \to 0$ limit of the energy-momentum conservation, are precisely covariant under this particular set of diffeomorphisms. Expression (246) shows that the ordinary spatial derivative of a scalar function does not transform as a tensor. To overcome this issue, it is desirable to introduce a Carrollian derivative as

$$\hat{\partial}_A = \partial_A + \frac{b_A}{\Omega}\partial_u, \tag{247}$$

transforming as

$$\hat{\partial}'_A = J^{-1B}_{\ \ A}\hat{\partial}_B. \tag{248}$$

Acting on scalars this provides a tensor, whereas for any other tensor it must be covariantized by introducing a new connection for Carrollian geometry, called *Levi–Civita–Carroll* connection, whose coefficients are,[8]

$$\hat{\gamma}^A_{BC} = \frac{a^{AD}}{2}\left(\hat{\partial}_B a_{DC} + \hat{\partial}_C a_{DB} - \hat{\partial}_D a_{BC}\right) = \gamma^A_{BC} + c^A_{BC}. \tag{249}$$

The Levi–Civita–Carroll covariant derivative acts symbolically as $\hat{\boldsymbol{\nabla}} = \hat{\boldsymbol{\partial}} + \hat{\boldsymbol{\gamma}}$. It is metric and torsionless : $\hat{\nabla}_A a_{BC} = 0$, $\hat{t}^C_{\ AB} = 2\hat{\gamma}^C_{[AB]} = 0$. There is however an effective torsion, since the derivatives $\hat{\nabla}_A$ do not commute, even when acting of scalar functions Φ – where they are identical to $\hat{\partial}_A$:

$$[\hat{\nabla}_A, \hat{\nabla}_B]\Phi = \frac{2}{\Omega}\varpi_{AB}\partial_u\Phi. \tag{250}$$

Here ϖ_{AB} is a two-form identified as the Carrollian vorticity defined using the Carrollian acceleration one-form φ_A :

$$\varphi_A = \frac{1}{\Omega}\left(\partial_u b_A + \partial_A \Omega\right) = \partial_u \frac{b_A}{\Omega} + \hat{\partial}_A \log\Omega, \tag{251}$$

$$\varpi_{AB} = \partial_{[A} b_{B]} + b_{[A}\varphi_{B]} = \frac{\Omega}{2}\left(\hat{\partial}_A \frac{b_B}{\Omega} - \hat{\partial}_B \frac{b_A}{\Omega}\right). \tag{252}$$

We also define the Carrollian shear and expansion

$$\xi_{AB} = \frac{1}{2\Omega}\left(\partial_u a_{AB} - a_{AB}\partial_u \log\sqrt{a}\right), \tag{253}$$

$$\theta = \Omega^{-1}\partial_u \log\sqrt{a}. \tag{254}$$

Since the initial relativistic fluid is at rest, the flat limit of the various kinematical quantities such as the vorticity and the acceleration are purely geometric and originate from the choice Carroll structure. More precisely, the $k \to 0$ limit of the acceleration, the vorticity and the expansion of the fluid velocity are respectively φ_A, ϖ_{AB} and θ.

The time derivative transforms as in (246), and acting on any tensor under Carrollian diffeomorphisms, it provides another tensor. This ordinary time derivative has nonetheless an unsatisfactory feature : its action on the metric does not vanish. One is tempted therefore to set a new

8. We remind that the ordinary Christoffel symbols are $\gamma^A_{BC} = \frac{a^{AD}}{2}\left(\partial_B a_{DC} + \partial_C a_{DB} - \partial_D a_{BC}\right)$.

time derivative $\hat{\partial}_u$ such that $\hat{\partial}_u a_{AB} = 0$, while keeping the transformation rule under Carrollian diffeomorphisms : $\hat{\partial}'_u = \frac{1}{J}\hat{\partial}_u$. This is achieved by introducing a "temporal Carrollian connection"

$$\hat{\gamma}^A_{\ B} = \frac{1}{2\Omega} a^{AC}\partial_u a_{CB}, \tag{255}$$

which allows us to define the time covariant derivative on a vector field :

$$\frac{1}{\Omega}\hat{\partial}_u V^A = \frac{1}{\Omega}\partial_u V^A + \hat{\gamma}^A{}_B V^B, \tag{256}$$

while on a scalar the action is as the ordinary time derivative : $\hat{\partial}_u \Phi = \partial_u \Phi$. Leibniz rule allows extending the action of this derivative to any tensor. Calling $\hat{\gamma}^A{}_B$ a connection is actually misleading because it transforms as a genuine tensor under Carrollian diffeomorphisms : $\hat{\gamma}'^C{}_B = J^C_D J^{-1A}{}_B \hat{\gamma}^D{}_A$.

We can define the curvature associated with a connection, by computing the commutator of covariant derivatives acting on a vector field. We find

$$\left[\hat{\nabla}_C, \hat{\nabla}_D\right] V^A = \hat{r}^A{}_{BCD} V^B + \varpi_{CD}\frac{2}{\Omega}\partial_u V^A, \tag{257}$$

where

$$\hat{r}^A{}_{BCD} = \hat{\partial}_C \hat{\gamma}^A_{DB} - \hat{\partial}_D \hat{\gamma}^A_{CB} + \hat{\gamma}^A_{CE}\hat{\gamma}^E_{DB} - \hat{\gamma}^A_{CE}\hat{\gamma}^E_{CB} \tag{258}$$

is a genuine tensor under Carrollian diffeomorphisms, the Riemann–Carroll tensor.

As usual, the Ricci–Carroll tensor is

$$\hat{r}_{AB} = \hat{r}^C{}_{ACB}. \tag{259}$$

It is *not* symmetric in general ($\hat{r}_{AB} \neq \hat{r}_{BA}$) and carries four independent components :

$$\hat{r}_{AB} = \hat{s}_{AB} + \hat{K} a_{AB} + \hat{A}\eta_{AB}. \tag{260}$$

In this expression $\hat{s}_{AB}$ is symmetric and traceless, whereas [9]

$$\hat{K} = \frac{1}{2} a^{AB}\hat{r}_{AB} = \frac{1}{2}\hat{r}, \quad \hat{A} = \frac{1}{2}\eta^{AB}\hat{r}_{AB} = *\varpi\theta \tag{261}$$

are the scalar-electric and scalar-magnetic Gauss–Carroll curvatures, with

$$*\varpi = \frac{1}{2}\eta^{AB}\varpi_{AB}. \tag{262}$$

There are also curvature terms that we can define by computing the commutator of time and space-derivatives

$$\left[\frac{1}{\Omega}\hat{\partial}_u, \hat{\nabla}_A\right] V^A = -2\hat{r}_A V^A + \left(\theta\delta^B_A - \hat{\gamma}^B{}_A\right)\varphi_B V^A + \left(\varphi_A \frac{1}{\Omega}\hat{\partial}_u - \hat{\gamma}^B{}_A \hat{\nabla}_B\right) V^A. \tag{263}$$

A Carroll curvature one-form emerges thus as

$$\hat{r}_A = \frac{1}{2}\left(\hat{\nabla}_B \xi^B{}_A - \frac{1}{2}\hat{\partial}_A\theta\right). \tag{264}$$

The Ricci–Carroll curvature tensor $\hat{r}_{AB}$ and the Carroll curvature one-form $\hat{r}_A$ appear naturally in the $k \to 0$ limit of the the Ricci curvature of the AdS boundary $R_{\mu\nu}$. In the limit, the condition of shearlessness on the fluid velocity becomes simply

$$\xi_{AB} \propto \partial_u a_{AB} - a_{AB}\partial_u \log\sqrt{a} = 0. \tag{265}$$

9. We use $\eta_{AB} = \sqrt{a}\epsilon_{AB}$, which matches, in the zero-k limit, with the spatial components of the $\eta_{\mu\nu}$ introduced in (214). To avoid confusion we also quote that $\eta^{AC}\eta_{BC} = \delta^A_B$ and $\eta^{AB}\eta_{AB} = 2$.

This is the same as asking the spatial metric to be conformally flat, which is always true locally for a two-dimensional metric. Assuming this holds, one proves that the traceless and symmetric piece of the Ricci-Carroll tensor is zero,

$$\hat{s}_{AB} = 0. \tag{266}$$

The absence of shear will be imposed again in what follows since it plays a crucial role in the resummation of the flat derivative expansion.

5.4 The conformal Carrollian geometry

The boundary fluid in AdS was a conformal fluid. The main consequence being that every quantity or equation characterizing the fluid had to be Weyl covariant. This property persists when taking the flat limit. This because in both cases, the boundary, time-like of null, is only truly a conformal boundary, therefore we need to study Weyl covariance for the Carrollian geometry induced on the null infinity.

The action of Weyl transformations on the elements of the Carrollian geometry is inherited from (217) :

$$a_{AB} \to \frac{a_{AB}}{\mathcal{B}^2}, \quad b_A \to \frac{b_A}{\mathcal{B}}, \quad \Omega \to \frac{\Omega}{\mathcal{B}}, \tag{267}$$

where $\mathcal{B} = \mathcal{B}(u, \mathbf{x})$ is an arbitrary function. The Carrollian vorticity and shear transform covariantly : $\varpi_{AB} \to \frac{1}{\mathcal{B}}\varpi_{AB}$, $\xi_{AB} \to \frac{1}{\mathcal{B}}\xi_{AB}$. However, the Levi–Civita–Carroll covariant derivatives $\hat{\boldsymbol{\nabla}}$ and $\hat{\partial}_u$ defined previously for Carrollian geometry. They must be replaced with Weyl–Carroll covariant spatial and time derivatives built on the Carrollian acceleration φ_A and the Carrollian expansion θ, which transform as connections :

$$\varphi_A \to \varphi_A - \hat{\partial}_A \log \mathcal{B}, \quad \theta \to \mathcal{B}\theta - \frac{2}{\Omega}\partial_u \mathcal{B}. \tag{268}$$

In particular, these can be combined in [10]

$$\alpha_A = \varphi_A - \frac{\theta}{2} b_A, \tag{269}$$

transforming under Weyl rescaling as :

$$\alpha_A \to \alpha_A - \partial_A \log \mathcal{B}. \tag{270}$$

The Weyl–Carroll covariant derivatives $\hat{\mathscr{D}}_A$ and $\hat{\mathscr{D}}_u$ are defined according to the pattern (218), (219). They obey

$$\hat{\mathscr{D}}_B a_{CD} = 0, \quad \hat{\mathscr{D}}_u a_{CD} = 0. \tag{271}$$

For a weight-w scalar function Φ, or a weight-w vector V^A, i.e. scaling with $\mathcal{B}^w$ under (267), we introduce

$$\hat{\mathscr{D}}_B \Phi = \hat{\partial}_B \Phi + w\varphi_B \Phi, \quad \hat{\mathscr{D}}_B V^D = \hat{\nabla}_B V^D + (w-1)\varphi_B V^D + \varphi^D V_B - \delta^D_B V^A \varphi_A, \tag{272}$$

which leave the weight unaltered. Similarly, we define

$$\frac{1}{\Omega}\hat{\mathscr{D}}_u \Phi = \frac{1}{\Omega}\hat{\partial}_u \Phi + \frac{w}{2}\theta\Phi = \frac{1}{\Omega}\partial_u \Phi + \frac{w}{2}\theta\Phi, \tag{273}$$

and

$$\frac{1}{\Omega}\hat{\mathscr{D}}_u V^D = \frac{1}{\Omega}\hat{\partial}_u V^D + \frac{w-1}{2}\theta V^D = \frac{1}{\Omega}\partial_u V^D + \frac{w}{2}\theta V^D + \xi^D{}_A V^A, \tag{274}$$

10. Contrary to φ_A, α_A is not a Carrollian one-form, i.e. it does not transform covariantly under Carrollian diffeomorphisms (243).

where $\frac{1}{\Omega}\hat{\mathcal{D}}_u$ increases the weight by one unit. The action of $\hat{\mathcal{D}}_A$ and $\hat{\mathcal{D}}_u$ on any other tensor is obtained using the Leibniz rule.

The Weyl–Carroll connection is torsion-free because

$$\left[\hat{\mathcal{D}}_A, \hat{\mathcal{D}}_B\right]\Phi = \frac{2}{\Omega}\varpi_{AB}\hat{\mathcal{D}}_u\Phi + w\left(\varphi_{AB} - \varpi_{AB}\theta\right)\Phi \tag{275}$$

does not contain terms of the type $\hat{\mathcal{D}}_C\Phi$. Here $\varphi_{AB} = \hat{\partial}_A\varphi_B - \hat{\partial}_B\varphi_A$ is a Carrollian two-form, not conformal though. The commutator acting on a vector defines also curvature elements

$$\left[\hat{\mathcal{D}}_C, \hat{\mathcal{D}}_D\right]V^A = \left(\hat{\mathcal{R}}^A{}_{BCD} - 2\xi^A{}_B\varpi_{CD}\right)V^B + \varpi_{CD}\frac{2}{\Omega}\hat{\mathcal{D}}_uV^A + w\left(\varphi_{CD} - \varpi_{CD}\theta\right)V^A. \tag{276}$$

The combination $\varphi_{CD} - \varpi_{CD}\theta$ forms a weight-0 conformal two-form, whose dual $*\varphi - *\varpi\theta$ is conformal of weight 2 ($*\varpi$ is defined in (262) and similarly $*\varphi = \frac{1}{2}\eta^{AB}\varphi_{AB}$). Moreover

$$\begin{aligned}\hat{\mathcal{R}}^A{}_{BCD} \quad = \quad & \hat{r}^A{}_{BCD} - \delta^A_B\varphi_{CD} - a_{BC}\hat{\nabla}_D\varphi^A + a_{BD}\hat{\nabla}_C\varphi^A + \delta^A_C\hat{\nabla}_D\varphi_B - \delta^A_D\hat{\nabla}_C\varphi_C \\ & +\varphi^A\left(\varphi_C a_{BD} - \varphi_D a_{BC}\right) - \left(\delta^A_k a_{BD} - \delta^A_D a_{BC}\right)\varphi_E\varphi^E \\ & + \left(\delta^A_C\varphi_D - \delta^A_D\varphi_C\right)\varphi_B\end{aligned} \tag{277}$$

is the Riemann–Weyl–Carroll weight-0 tensor, from which we define

$$\hat{\mathcal{R}}_{AB} = \hat{\mathcal{R}}^C{}_{ACB} = \hat{r}_{AB} + a_{AB}\hat{\nabla}_C\varphi^C - \varphi_{AB}. \tag{278}$$

We also quote

$$\left[\frac{1}{\Omega}\hat{\mathcal{D}}_u, \hat{\mathcal{D}}_A\right]\Phi = w\hat{\mathcal{R}}_A\Phi - \xi^B{}_A\hat{\mathcal{D}}_B\Phi \tag{279}$$

and

$$\left[\frac{1}{\Omega}\hat{\mathcal{D}}_u, \hat{\mathcal{D}}_A\right]V^A = (w-2)\hat{\mathcal{R}}_AV^A - V^A\hat{\mathcal{D}}_B\xi^B{}_A - \xi^B{}_A\hat{\mathcal{D}}_BV^A, \tag{280}$$

with

$$\hat{\mathcal{R}}_A = \hat{r}_A + \frac{1}{\Omega}\hat{\partial}_u\varphi_A - \frac{1}{2}\hat{\nabla}_B\hat{\gamma}^B{}_A + \xi^B{}_A\varphi_B = \frac{1}{\Omega}\partial_u\varphi_A - \frac{1}{2}\left(\hat{\partial}_A + \varphi_A\right)\theta. \tag{281}$$

This is a Weyl-covariant weight-1 curvature one-form, where $\hat{r}_A$ is given in (264).

The Ricci–Weyl–Carroll tensor (278) is not symmetric in general : $\hat{\mathcal{R}}_{AB} \neq \hat{\mathcal{R}}_{BA}$. Using (259) we can recast it as

$$\hat{\mathcal{R}}_{AB} = \hat{s}_{AB} + \hat{\mathcal{K}}a_{AB} + \hat{\mathcal{A}}\eta_{AB}, \tag{282}$$

where we have introduced the Weyl-covariant scalar-electric and scalar-magnetic Gauss–Carroll curvatures

$$\hat{\mathcal{K}} = \frac{1}{2}a^{AB}\hat{\mathcal{R}}_{AB} = \hat{K} + \hat{\nabla}_C\varphi^C, \quad \hat{\mathcal{A}} = \frac{1}{2}\eta^{AB}\hat{\mathcal{R}}_{AB} = \hat{A} - *\varphi \tag{283}$$

both of weight 2.

All these definitions can seem heavy, but they will be necessary to simplify all the expressions when taking the $c \to 0$ limit of the conservation equations of the fluid's energy-momentum tensor (200).

5.5 The Carrollian fluid

Having described the conformal Carrollian geometry to which the ultra-relativistic fluid will couple, we would like now to obtain the conservation laws that control its dynamics. To do so we will simply take the $c \to 0$ limit of their relativistic counterpart . To compute this limit, we need to assume the c-dependence of the components of the enegy-momentum tensor, exactly like we did for the relativistic metric when we chose to write it as (240). We make the following choices for the relativistic quantities [11]

11. These choices will be relevant when we consider the holographic fluid.

— the energy density ε does not depend on c, neither the pressure since $\varepsilon = 2p$,
— the heat current has the following expansion : $q^A = Q^A + c^2\pi^A$,
— the stress tensor admits also an expansion : $\tau^{AB} = -c^{-2}\Sigma^{AB} - \Xi^{AB}$.

We recall that all the other components of the dissipative terms can be deduced from q^A and τ^{AB} since they are purely transverse (201). Using these scalings for the fluid's pieces of data, the choice of metric (240) and the fluid velocity $\mathrm{u} = \Omega^{-1}\partial_u$ we obtain the following structure for the zero-c limit of the relativistic conservation equations

$$\begin{aligned} \frac{c}{\Omega}\nabla_\mu T^\mu_0 &= \frac{1}{c^2}\mathcal{F} + \mathcal{E}, \\ \nabla_\mu T^{\mu A} &= \frac{1}{c^2}\mathcal{H}^A + \mathcal{G}^A. \end{aligned} \tag{284}$$

The zero-c limit gives rise two four equations : an energy conservation $\mathcal{E} = 0$, weight-4 Weyl-covariant

$$-\frac{1}{\Omega}\hat{\mathcal{D}}_u\varepsilon - \hat{\mathcal{D}}_A Q^A + \Xi^{AB}\xi_{AB} \quad = \quad 0, \tag{285}$$

a first constraint $\mathcal{F} = 0$, weight-4 also

$$\Sigma^{AB}\xi_{AB} \quad = \quad 0, \tag{286}$$

a momentum conservation $\mathcal{G}^A = 0$, weight-3

$$\frac{1}{2}\hat{\mathcal{D}}_B\varepsilon + 2Q^A\varpi_{AB} + \frac{1}{\Omega}\hat{\mathcal{D}}_u\pi_B - \hat{\mathcal{D}}_A\Xi^A{}_B + \pi_A\xi^A{}_B \quad = \quad 0, \tag{287}$$

and an additional constraint $\mathcal{H}_A = 0$, weight-3 also

$$\frac{1}{\Omega}\hat{\mathcal{D}}_u Q_A - \hat{\mathcal{D}}_B\Sigma^B_A + Q_B\xi^B_A \quad = \quad 0. \tag{288}$$

All these equations are manifestly Carrollian covariant and Weyl covariant. In these equations, ε, Q^A, π^A, Ξ^{AB} and Σ^{AB} are the Carrollian fluid data or Carrollian momenta, they are counterpart of the quantities ϵ and q (i.e. energy density and momentum flow) that we had introduced on the boundary of three-dimensional asymptotically flat spacetimes. The notion of fluid velocity has disappeared since before taking the limit the fluid was static. Finally it is these conservation equations that are going to describe the residual gravitational dynamics in the flat version of the fluid/gravity correspondence, as we are going to see later.

In my work [27], we consider the most generic case, i.e. without assuming the conformal state equation, in arbitrary dimension and where the initial fluid is not static, allowing for a Carrollian velocity field called β_A. We also study the dual limit, i.e. the Galilean one, with the same level of generality.

6 Fluid/Gravity correspondence

We now turn our attention to another aspect of gravity, the fluid/gravity correspondence. The idea that the dynamics of gravity has anything to do with hydrodynamics is not new and stems into the seminal works of T. Damour [21, 22, 23] who was the first one to notice a striking similarity between the constraint equations of gravity projected on the horizon of a black hole and the Navier Stokes equations. This correspondence was revived with the holographic correspondence by V. Hubeny, S. Minwalla and M. Rangamani [19] who establish a relation between the dynamics of Einstein equations with negative cosmological constant and the relativistic Navier-Stokes equations. In the latter, the fluid lives on the boundary of the asympotically AdS spacetime whereas in Damour's correspondence, the fluid lives on the horizon. Of course these two fluids are far from being the same but the geometric structure on which relies their appearance is quite similar : it is the existence of a boundary (or simply a special hypersurface).

When we consider the geometry in the vicinity of a codimension-one hypersurface, one can always gauge fix the metric locally and solve the transverse equations such that the only remaining Einstein equations are the constraint equations on this surface. This is what we have done for example in three dimensions : the dynamics of the solution space in Bondi gauge is dictated by constraint equations that live on the boundary (time-like in AdS, and null-like in flat space). In AdS, these constraint equations are written as the conservation of a relativistic energy-momentum tensor. The latter could very-well be the energy-momentum tensor of a relativistic fluid as it is in the fluid-gravity correspondence. In the case of a horizon, which is a null hypersurface, the constraint equations can also be interpreted as the conservation equations for a non-relativistic fluid but the matching is more subtle as we are going to see.

This section is based on my paper [28].

6.1 AdS Fluid/Gravity

This section will be devoted to the study of the Derivative Expansion which is a procedure that relates a solution of Einstein equations with negative cosmological constant to a relativistic fluid (see [20] for a review). In AdS, the dual fluid is relativistic, hence the introduction to relativistic hydrodynamics in the previous section. We are going to work in four dimensions of bulk, therefore the fluid will be three-dimensional. The hydrodynamical expansion maps onto the Derivative Expansion in the bulk, i.e. an expansion of the line element whose order is the number of derivatives of the fields. [12] It is generically an infinite expansion. We describe integrability conditions on the fluid such that the dual line element is resummed and Einstein.

6.1.1 The Derivative Expansion in AdS

Let $g_{\mu\nu}$, ε and u^μ be a boundary metric, an energy density and a vector field that do not depend on the boundary coordinates, all their derivatives vanish. The boundary velocity is normalized to $-\ell^{-2}$. Now consider the following bulk metric

$$ds^2 = 2\ell^2 u_\mu dx^\mu dr + r^2 g_{\mu\nu} dx^\mu dx^\nu + \ell^4 8\pi G \varepsilon \frac{u_\mu u_\nu}{r} dx^\mu dx^\nu. \tag{289}$$

One can check that this metric is a solution of Einstein equations with negative cosmological constant. The scalar curvature is proportional to ℓ^{-2}. This metric corresponds to a boosted black brane. One can compute the holographic energy-momentum tensor of this solution and the result is

$$T^{black\,brane}_{\mu\nu} = \frac{\varepsilon}{2}\left(3\frac{u_\mu u_\nu}{\ell^2} + g_{\mu\nu}\right). \tag{290}$$

We recognize the energy-momentum tensor of a perfect conformal fluid. It is trivially conserved since all the components of this tensor are constants. This amounts to say that, classically, the boosted black brane is dual to the simplest perfect fluid. Another thing to notice is that the AdS curvature plays again the role of an effective velocity of light on the boundary $c \leftrightarrow k$. Exactly like in three dimensions, we are going to make use of this property to interpret the boundary fluid when taking the flat limit. The idea of fluid gravity is to start from this simple solution and then modify it by asking the energy and the fluid velocity to depend on the boundary coordinates. This is achieved by making a Weyl redefinition of the metric and the fluid quantities together with a coordinate change

$$r \to \mathcal{B} r \tag{291}$$

The bulk spacetime is made out of an ensembles of black brane "tubes" all attached to the boundary. Assuming slow variation of the fluid quantities (i.e. the derivatives w.r.t. the boundary coordinates are small), one can use the resulting metric as an ansatz for the zeroth order in a

12. The term Derivative Expansion will equivalently refer to the concept or the actual line element it produces.

hydrodynamical expansion, i.e. in number of derivatives of the fluid quantities. The first order is then a perturbation of the bulk metric.

$$ds^2 = ds^2_{perfect} + ds^2_{first\,order} + O(2), \tag{292}$$

where $ds^2_{first\,order}$ contains only first derivatives of ε and u^μ. The quantities present in this term are dictated by Weyl covariance and the fact that the resulting tensor must be symmetric. The only things that is not constrained by symmetry are the coefficient in front of each tensor. This perturbation is then plugged in Einstein equations and linearized in number of derivatives which allows to extract all these coefficients that ultimately will characterize the fluid. Indeed after that, one can compute the corresponding holographic energy-momentum tensor and write it as

$$T^{\mu\nu} = T^{\mu\nu}_{perfect} + T^{\mu\nu}_{first\,order} + O(2), \tag{293}$$

where $T^{\mu\nu}_{first\,order}$ corresponds exactly to the first order hydrodynamical data presented in Sec. 5.1, except that now, the shear and bulk viscosities and the thermal conductivity are determined by Einstein equations. For example in our case we obtain [20]

$$\kappa = 0, \quad \zeta = 0, \quad \eta = \frac{\pi T^2}{9G}. \tag{294}$$

Here it is written in terms of the local temperature T. This can be done also for the pressure and the energy density using Stefan's law in three dimensions

$$p = \frac{4\pi^2 T^3}{27G}, \tag{295}$$

and the conformal equation of state $\varepsilon = 2p$.

This algorithm can be realized order by order to gravitationally fix all the possible dissipative coefficients of the dual fluid. The corresponding bulk metric is given by

$$\begin{aligned} \mathsf{d}s^2_{\mathsf{bulk}} \quad = \quad & 2\frac{\mathsf{u}}{k^2}(\mathsf{d}r + r\mathsf{A}) + r^2\mathsf{d}s^2_{\mathrm{bdry}} + \frac{\mathsf{S}}{k^4} \\ & + \frac{\mathsf{u}^2}{k^4 r^2}\left(1 - \frac{1}{2k^4 r^2}\omega_{\alpha\beta}\omega^{\alpha\beta}\right)\left(\frac{8\pi G T_{\lambda\mu}u^\lambda u^\mu}{k^2} r + \frac{C_{\lambda\mu}u^\lambda \eta^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^4}\right) \\ & + \text{terms with } \sigma,\, \sigma^2,\, \nabla\sigma,\, \ldots + \mathsf{O}\left(\mathcal{D}^4\mathsf{u}\right). \end{aligned} \tag{296}$$

In this expression S is a Weyl-invariant tensor :

$$\mathsf{S} = S_{\mu\nu}\mathsf{d}x^\mu\mathsf{d}x^\nu = -2\mathsf{u}\mathcal{D}_\nu\omega^\nu{}_\mu\mathsf{d}x^\mu - \omega_\mu{}^\lambda\omega_{\lambda\nu}\mathsf{d}x^\mu\mathsf{d}x^\nu - \mathsf{u}^2\frac{\mathcal{R}}{2}; \tag{297}$$

We have also defined $k = \ell^{-1}$. The bulk line line element is a priori an infinite expansion in terms of derivatives of the fluid variable. A question one can ask is if this expansion is resummable, which would provide an exact solution of Einstein equations. The answer is yes, but for a subclass of Einstein metrics : the algebraically special class. Moreover we are going to show that this procedure admits a non-trivial flat limit.

6.1.2 Resummation

To find an ansatz of potential resummation formula for the derivative expansion, we are going to make a crucial assumption and ask the fluid velocitiy to be *shearless*. This is asking that

$$\sigma_{\mu\nu} = 0, \tag{298}$$

where $\sigma_{\mu\nu}$ was defined in Eq. (209). This condition has the virtue to simplify drastically the derivative expansion since most of its terms involve the shear ad its derivatives. In other words, the number of tensors compatible with Weyl covariance is considerably reduced. The caveat is that the space of solutions that we are allowed to describe is reduced as we are going to show.

The resummation ansatz is obtained by making the simple change

$$1-\frac{\gamma^2}{r^2}\to\frac{r^2}{\rho^2},\tag{299}$$

with

$$\rho^2=r^2+\gamma^2.\tag{300}$$

The resummed expansion then reads

$$\mathsf{d}s^2_{\text{res.}}=2\frac{\mathsf{u}}{k^2}(\mathsf{d}r+r\mathsf{A})+r^2\mathsf{d}s^2_{\text{bdry}}+\frac{\mathsf{S}}{k^4}+\frac{\mathsf{u}^2}{k^4\rho^2}\left(8\pi G\varepsilon r+C\gamma\right),\tag{301}$$

which is indeed written in a closed form. This line element defines an exact Einstein space with $\Lambda=-3k^2$ under a set of conditions on the quantities introduced previously

— The congruence u must be shearless.

— The heat current of the boundary fluid introduced in (200) and (201) is identified with the transverse-dual of the Cotton current defined in (233) and (236) :

$$q_\mu=\frac{1}{8\pi G}\eta^\nu{}_\mu c_\nu=\frac{1}{8\pi G}\eta^\nu{}_\mu\eta^{\rho\sigma}\mathcal{D}_\rho\left(\mathcal{S}_{\nu\sigma}+F_{\nu\sigma}\right),\tag{302}$$

where we used (238) in the last expression.

— The viscous stress tensor of the boundary conformal fluid introduced in (200) is identified with the transverse-dual of the Cotton stress tensor defined in (233) and (236). Following the same pattern as for the heat current, we obtain :

$$\begin{aligned}\tau_{\mu\nu}&=-\tfrac{1}{8\pi Gk^2}\eta^\rho{}_\mu c_{\rho\nu}\\&=\tfrac{1}{8\pi Gk^2}\left(-\tfrac{1}{2}u^\lambda\eta_{\mu\nu}\eta^{\rho\sigma}+\eta^\lambda_{\ \mu}\left(k\eta_\nu{}^{\rho\sigma}-u_\nu\eta^{\rho\sigma}\right)\right)\mathcal{D}_\rho\left(\mathcal{S}_{\lambda\sigma}+F_{\lambda\sigma}\right),\end{aligned}\tag{303}$$

where we also used (239) in the last equality. The viscous stress tensor $\tau_{\mu\nu}$ is transverse symmetric and traceless because these are the properties of the Cotton stress tensor $c_{\mu\nu}$.

— The energy–momentum tensor defined in (200) with $p={\varepsilon}/{2}$, heat current as in (302) and viscous stress tensor as in (303) must be conserved. This is because part of Einstein equations are automatically satisfied by the line element (301) while the residual ones map onto the conservation equations of the fluid.

We should make a few comments here. Asking u to be shearless is not free of consequences. Indeed consider the congruence ∂_r in the bulk, it is obviously a null congruence. But one can show that it is also geodesic and shearless (this is a consequence of u being shearless). According to the Goldberg–Sachs theorem [46], the existence of such a congruence means that the spacetime is *algebraically special* (an vice versa), i.e. of Petrov type II, III, D, N or O. This classification refers to the number of principal null directions of the Weyl tensor. We will not go through its definition here, see App. A for an introduction or Chap. 4 of [46] for a complete description. For the reader not familiar with this classification we would like to remark that all famous black hole solutions fall into Petrov type D.

We should emphasize that the only quantities that are present in the expression (301) are the energy density ε, the boundary metric $g_{\mu\nu}$ and the fluid velocity u^μ. All the other quantities are built out of them. In particular the holographic fluid's dissipative tensors are dictated by the conformal curvature of the boundary metric since they are written in terms of the boundary Cotton tensor. These relations can be seen as self-duality conditions and are necessary for the line element to be Einstein. The only fluid quantity that is not fixed is the energy density which should be seen as the mass of the solution, while C characterize the NUT charge.

6.1.3 The Robinson Trautman family

Our result states that the line element (301) corresponds to an algebraically special Einstein solution under a set of assumptions on the fluid quantities. We will not prove it in full generality here but as an example, we will map our construction to a well-known family of solutions that have been extensively studied in the literature : the Robinson Trautman family.

This family of spacetimes correspond to algebraically special spacetimes whose degenerate principal null direction is geodesic, shearless and twistless, see Chap. 28 of [46]. We are going to recover it simply by imposing conditions on the boundary geometry and by finding a compatible shearless fluid velocity (see [47] and references therein).

Consider the following boundary metric

$$ds^2_{\text{bdry}} = -k^2 du^2 + \frac{2}{P(\zeta,\bar{\zeta})^2} d\zeta d\bar{\zeta}. \tag{304}$$

A shearless fluid velocity compatible with this metric is simply the velocity of a static fluid

$$u^\mu \partial_\mu = \partial_u. \tag{305}$$

We can compute the dissipative tensors for the fluid using the duality conditions

$$\mathsf{q} = -\frac{1}{16\pi G}\left(\partial_\zeta K \mathsf{d}\zeta + \partial_{\bar{\zeta}} K \mathsf{d}\bar{\zeta}\right), \tag{306}$$

$$\tau = \frac{1}{8\pi G k^2 P^2}\left(\partial_\zeta\left(P^2 \partial_u \partial_\zeta \log P\right) \mathsf{d}\zeta^2 + \partial_{\bar{\zeta}}\left(P^2 \partial_u \partial_{\bar{\zeta}} \log P\right) \mathsf{d}\bar{\zeta}^2\right), \tag{307}$$

where $K = 2P^2 \partial_{\bar{\zeta}} \partial_\zeta \log P$ is the Gaussian curvature of the spatial part of the boundary metric. With these data the conservation of the energy–momentum tensor enforces the absence of spatial dependence in $\varepsilon = 2p$, and leads to a single independent equation, the heat equation :

$$12M \partial_u \log P + \Delta K = 4\partial_t M, \tag{308}$$

where $\Delta = 2P^2 \partial_\zeta \partial_{\bar{\zeta}}$. This is exactly the Robinson–Trautman equation, here expressed in terms of $M(t) = 4\pi G \varepsilon(t)$. It is not over, indeed we have shown that the conservation of our fluid's energy-momentum tensor maps to the Robinson–Trautman equation, but the last thing to do is to plug these data, i.e. $\varepsilon = \frac{M(u)}{4\pi G}$ and $\mathsf{u} = \partial_u$ in the resummation formula. Doing so we obtain the line element

$$ds^2 = -\left[\Delta \log P - 2r\partial_u \log P - \frac{2M(u)}{r} + \frac{\Lambda r^2}{3}\right] du^2 - 2du dr + \frac{2r^2}{P^2} d\zeta d\bar{\zeta}. \tag{309}$$

This is exactly the line element of the AdS Robinson–Trautman family of solutions. One thing to notice is that the Robinson–Trautman equation does not depend on the cosmological constant.

Of course this does not produce exact solutions to Einstein equations, one still needs to solve this equation to obtain an Einstein metric. Exact solutions are, for example, the AdS Schwarzschild black hole, given by a constant M and the round metric on the celestial sphere : $P = 1 + \frac{\zeta\bar{\zeta}}{2}$. Or the C-metric, a type D solution of the Robinson Trautman equation which corresponds to an accelerating black hole (see Chap. 28 of [46]).

In terms of fluid gravity this is a non trivial result. Indeed we conclude that the fluid dual to the Robinson-Trautman family surprisingly satisfies the duality conditions between its dissipation terms and the boundary Cotton tensor. Moreover, we conclude that the hydrodynamic expansion of its energy-momentum tensor terminates at third order.

We conclude this section with comments on ongoing works. We have shown that an entire family of algebraically special spacetimes is recovered with our resumed version of the derivative expansion. In unpublished work, we have also shown that the whole Plebanski-Demianski family, i.e. all the type D solutions (which has an intersection with the Robinson–Trautman family by is not included) can be recovered by adding non-diagonal terms in the boundary metric and assuming the existence of two commuting Killing fields, with the same fluid velocity. Finally recently we have also shown that without assuming the existence of these two Killing fields, we can recover an AdS version of the twisting vacuum solutions described in Chap. 29 of [46], we expect this family to be the largest one that we can describe with the resummed derivative expansion.

6.2 The flat limit of fluid/gravity correspondence

An advantage of the Derivative Expansion is that it is written in a coordinate system à la Eddington-Finkelstein, therefore it admits a proper flat limit. We show that the relationship between the bulk and the boundary persists in the limit : we trade the Einstein metric for a Ricci-flat metric in the bulk, while the relativistic fluid is replaced by a Carrollian fluid on the null infinity. The flat Derivative Expansion admits also a resummation under a Carrollian equivalent of the aforementioned integrability conditions.

6.2.1 The Ricci-flat solution and its dual

We start by taking the zero-k limit of the relativistic conservation of the holographic fluid dual to our resummed derivative expansion, i.e. a fluid whose dissipative tensors are constrained by the Cotton tensor of the boundary geometry. The k-dependence of q_μ in (302) and $\tau_{\mu\nu}$ in (303) falls exactly in the case described in Sec. 5.5. Which means that the zero-k limit of the conservation equations corresponds exactly to the Carrollian conservation equations (285), (286), (287) and (288) that we had derived in the same section. Of course this is assuming the exact same parametrization of the fluid data and the boundary metric (and making the replacement $c \leftrightarrow k$). Actually for the holographic fluid, the two constraint equations are automatically satisfied and we are left with two equations that we report again here : the energy conservation

$$-\frac{1}{\Omega}\hat{\mathcal{D}}_u\varepsilon - \hat{\mathcal{D}}_A Q^A + \Xi^{AB}\xi_{AB} \quad = \quad 0, \tag{310}$$

and the momentum conservation

$$\frac{1}{2}\hat{\mathcal{D}}_B\varepsilon + 2Q^A\varpi_{AB} + \frac{1}{\Omega}\hat{\mathcal{D}}_u\pi_B - \hat{\mathcal{D}}_A\Xi^A{}_B + \pi_A\xi^A{}_B \quad = \quad 0. \tag{311}$$

In these equations, ε, Q^A, π^A and Ξ^{AB} are the holographic Carrollian fluid data or Carrollian momenta, they are counterpart of the quantities ϵ and q (i.e. energy density and momentum flow) that we had introduced for the boundary of three-dimensional asymptotically flat spacetimes. The notion of fluid velocity has disappeared since before taking the limit the fluid was chosen static (see Sec. 5.3). In [27], we also consider the case where the initial fluid is not static, allowing for a Carrollian velocity field called β_A. In principle this quantity could appear on the boundary when taking the flat limit but for simplicity we will not consider it here. Moreover the fluid velocity on the boundary of AdS was initially shearless, therefore the Carrollian metric must satisfy

$$\xi_{AB} = 0, \tag{312}$$

since the zero-k limit of the relativistic shear $\sigma_{\mu\nu}$, defined in Eq. (209), is $\sigma = \xi_{AB}\mathrm{d}x^A\mathrm{d}x^B$.

The dissipative tensors Q^A, π^A and Ξ^{AB}, respectively the two Carrollian heat currents and stress tensor are constrained by the boundary Carrollian geometry, exactly like in AdS. Their expressions are

$$\begin{aligned} Q_A &= -\frac{1}{16\pi G}\left(\hat{\mathcal{D}}_A\hat{\mathcal{K}} - \eta^B{}_A\hat{\mathcal{D}}_B\hat{\mathcal{A}} + 4*\varpi\eta^B{}_A\hat{\mathcal{R}}_B\right), \\ \pi_A &= \frac{3}{8\pi G}\eta^B_A\left(\eta^C_B\hat{\mathcal{D}}_C \star\varpi^2\right), \end{aligned} \tag{313}$$

and

$$\Xi_{AB} = \frac{1}{8\pi G}\left(\eta^D{}_A\hat{\mathcal{D}}_D\hat{\mathcal{D}}_B*\varpi + \frac{1}{2}\eta_{AB}\hat{\mathcal{D}}_D\hat{\mathcal{D}}^D*\varpi - a_{AB}\frac{1}{\Omega}\hat{\mathcal{D}}_u*\varpi^2\right). \tag{314}$$

On can show that the expressions for Q_A, π_A and Ξ^{AB} are actually dualized version of quantities that appear in the $k \to 0$ limit of the Cotton tensor (see [28]).

We are now going to give the flat limit of the bulk metric. This metric will be a Ricci-flat solution provided the Carrollian fluid equations (310) and (311) are satisfied.

The $k \to 0$ limit of the various three-dimensional Riemannian quantities give rise to all the corresponding Carrollian quantities :

$$\mathsf{u} = -k^2 \left(\Omega \mathrm{d}u - \boldsymbol{b}\right) \tag{315}$$

and

$$\begin{array}{rcl} \omega & = & \frac{k^2}{2}\varpi_{AB}\mathrm{d}x^A \wedge \mathrm{d}x^B, \\ \gamma & = & *\varpi, \\ \Theta & = & \theta, \\ \mathsf{a} & = & k^2\varphi_A \mathrm{d}x^A, \\ \mathsf{A} & = & \alpha_A\mathrm{d}x^A + \frac{\theta}{2}\Omega\mathrm{d}u, \\ \sigma & = & \xi_{AB}\mathrm{d}x^A\mathrm{d}x^B, \end{array} \tag{316}$$

where the left-hand-side quantities are Riemannian quantities introduced in Sec. 5.1 and 5.2, while the righ-hand-side quantities are defined in Sec. 5.3 and 5.4. We move now to second-derivative objects and collect the tensors relevant for the derivative expansion, following the same pattern (Riemannian vs. Carrollian) :

$$\mathcal{R} = \frac{1}{k^2}\xi_{AB}\xi^{AB} + 2\hat{\mathcal{K}} + 2k^2 * \varpi^2, \tag{317}$$

$$\omega_\mu{}^\lambda \omega_{\lambda\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = k^4 \varpi_A{}^D \varpi_{DB}\mathrm{d}x^A\mathrm{d}x^B, \tag{318}$$

$$\omega^{\mu\nu}\omega_{\mu\nu} = 2k^4 * \varpi^2, \tag{319}$$

$$\mathcal{D}_\nu \omega^\nu{}_\mu \mathrm{d}x^\mu = k^2 \hat{\mathcal{D}}_B \varpi^B{}_A \mathrm{d}x^A - 2k^4 * \varpi^2 \Omega \mathrm{d}u + 2k^4 * \varpi^2 \boldsymbol{b}. \tag{320}$$

Using (297) this leads to

$$\mathsf{S} = -\frac{k^2}{2}\left(\Omega\mathrm{d}u - \boldsymbol{b}\right)^2 \xi_{AB}\xi^{AB} + k^4 \boldsymbol{s} - 5k^6 \left(\Omega \mathrm{d}u - \boldsymbol{b}\right)^2 * \varpi^2 \tag{321}$$

with the Weyl-invariant tensor

$$\boldsymbol{s} = 2\left(\Omega\mathrm{d}u - \boldsymbol{b}\right)\mathrm{d}x^A \eta^B{}_A \hat{\mathcal{D}}_B * \varpi + *\varpi^2 a_{AB} dx^A dx^B - \hat{\mathcal{K}}\left(\Omega \mathrm{d}u - \boldsymbol{b}\right)^2. \tag{322}$$

In the derivative expansion (296), two explicit divergences appear at vanishing k. The first originates from the first term of S, which is the shear contribution to the Weyl-covariant scalar curvature $\mathcal{R}$ of the three–dimensional AdS boundary (Eq. (317)). The second divergence comes from the Cotton tensor and is also due to the shear. This is fortunate since we are considering the specific case of vanishing shear. Even though, when the shear is not vanishing, we expect all the other terms we have ignored when imposing $\sigma = 0$ to contribute again and certainly cancel out the divergences. Indeed it seems quite unnatural that the flat limit of the non resummed derivative expansion would require a shearless fluid velocity. Vanishing σ in the pseudo-Riemannian boundary implies vanishing ξ_{AB} in the Carrollian limit (see (316)), and in this case, the divergent terms in S and the Cotton are absent. We obtain the flat version of the resummed derivative expansion

$$\mathrm{d}s^2_{\text{res. flat}} = -2\left(\Omega\mathrm{d}u - \boldsymbol{b}\right)\left(\mathrm{d}r + r\boldsymbol{\alpha} + \frac{r\theta\Omega}{2}\mathrm{d}u\right) + r^2 g_{\text{bdry}} + \boldsymbol{s} + \frac{\left(\Omega\mathrm{d}u - \boldsymbol{b}\right)^2}{\rho^2}\left(8\pi G\varepsilon r + \hat{C} * \varpi\right), \tag{323}$$

where

$$\rho^2 = r^2 + *\varpi^2, \tag{324}$$

and

$$\hat{C} = \left(\hat{\mathcal{D}}_D\hat{\mathcal{D}}^D + 2\hat{\mathcal{K}}\right) * \varpi. \tag{325}$$

This last quantity is the zero-k limit of the Cotton density C introduced in Eq. (234).

Exactly like in AdS, this line element defines a Ricci-flat solution provided the conservation equations for the Carrollian fluid (310) and (311) are satisfied. Notice eventually that the Ricci-flat line element (323) inherits Weyl invariance from its relativistic ancestor. The set of transformations (267), (268) and (270), supplemented with $*\varpi \to \mathcal{B} * \varpi$, $\varepsilon \to \mathcal{B}^3\varepsilon$ and $\hat{C} \to \mathcal{B}^3\hat{C}$, can indeed be absorbed by setting $r \to \mathcal{B}r$ ($\boldsymbol{s}$ is Weyl invariant), resulting thus in the invariance of (323). In the relativistic case this invariance was due to the AdS conformal boundary. In the case at hand, this due to the fact the the null infinity is also a conformal boundary.

The class of vacuum solutions we can recover with this formula is again dictated by the existence of a null, geodesic and shearless congruence in the bulk. This null direction is simply ∂_r. Again, thanks to the Goldberg–Sachs theorem, this means that the solution is algebraically special.

In [28], we show that the flat version of the Robinson-Trautman family (i.e. all the twistless algebraically special vacuum solutions) can be recovered by setting the Ehresmann connection to $\tau = -du$, which is equivalent to $\Omega = 1$ and $b_A = 0$, together with a vanishing π_A. Moreover in unpublished work, we have also shown that for $b \neq 0$ and assuming the existence of two commuting Killings, we recover the full flat Plebanski-Demianski family. Finally if we do not assume the existence of the two Killings, the resummed derivative expansion actually covers the whole family of twisting vacuum solutions (see Chap. 29 of [46]), which should be the largest family we can describe.

7 The membrane paradigm

We conclude our study of the fluid/gravity correspondence by discussing an older relation between gravity and fluid dynamics : the membrane paradigm. In the membrane paradigm formalism [25], the black hole event horizon is seen as a two- dimensional membrane that lives and evolves in three-dimensional spacetime. This viewpoint was originally motivated by Damour's seminal observation that a generic black hole horizon is similar to a fluid bubble with finite values of electrical conductivity, shear and bulk viscosity [21, 22, 23]. It was moreover shown that the equations governing the evolution of the horizon take the familiar form of an Ohm's law, Joule heating law, and Navier-Stokes equation. The membrane paradigm developed by Thorne and Macdonald for the electromagnetic aspects, and by Price and Thorne for gravitational and mechanical aspects, combines Damour's results with the 3 + 1 formulation of general relativity, where one trades the true horizon for a 2+1-dimensional timelike surface located slightly outside it, called "stretched horizon" or "membrane". The laws of evolution of the stretched horizon then become boundary conditions on the physics of the external universe, hence making the membrane picture a convenient tool for astrophysical purposes. In order to derive the evolution equations of the membrane, a crucial step in [26] was to renormalize all physical quantities (energy density, pressure, etc) on the membrane, as they turned out to be divergently large as one approaches the real horizon. We will show that a better approach to this issue is to interpret the near-horizon limit as an ultra-relativistic limit for the stretched horizon, where the radial coordinate plays the role of a virtual speed of light. This ultra-relativistic limit has the same nature than the effect of the flat limit on the conformal boundary, it leads to Carrollian physics.

7.1 Constraint equations on the horizon

Exactly like in the conventional fluid/gravity correspondence, it is the constraint equations associated with a particular surface that are going to be interpreted as fluid conservation laws. The asymptotic boundary is now traded for a regular hypersurface (not so regular since it is a horizon) in the bulk. The null nature of the horizon will impose the fluid to be non-relativistic, in particular it will be Carrollian. Here, the ultra-relativistic limit can be understood as a near-horizon limit. Indeed, we are going to show that if one considers a time-like surface (or membrane), close to the horizon, Einstein equations will map onto the conservation of a relativistic energy-momentum tensor that belongs to the surface and corresponds to the extrinsic curvature of the membrane. This section is based on my paper [31] with Laura Donnay.

7.1.1 Horizon geometry and Carroll structure

We consider a D-dimensional spacetime whose coordinates are $x^a = (v, \rho, x^A)$, where v is the advanced time and ρ the radial coordinate. The surfaces of constant v and ρ are $(D-2)$-dimensional spheres $S_{v,r}$ and parametrized by x^A ($A = 3, \cdots, D$), the set of all these angular coordinates will be denoted $\mathbf{x}$. When we refer to spatial objects, it will be with respect to the angular coordinates. The constant v surfaces are null, and constant ρ are timelike. Finally, we assume the existence of a horizon $\mathcal{H}$ sitting at $\rho = 0$.

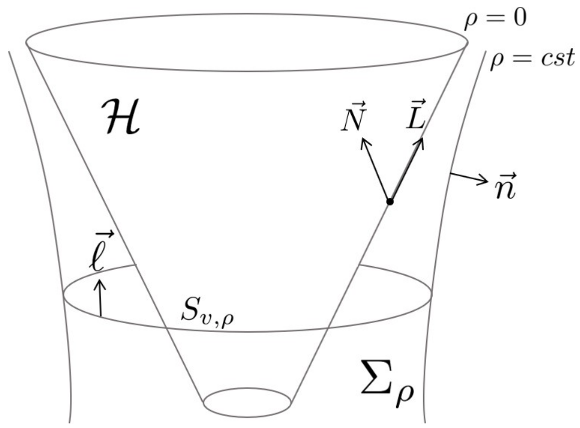

FIGURE 2: The horizon is a null hypersurface situated at $\rho = 0$ and Σ_ρ is a timelike constant ρ hypersurface near the horizon. We define also four vectors that are useful for our analysis, the null vector $\vec{L}$ is the normal to the horizon while $\vec{N}$ is transverse but also null. The spacelike vector $\vec{n}$ is the normal to Σ_ρ and the timelike vector $\vec{\ell}$ is the normal to a constant v section of Σ_ρ.

It is alway possible to find a coordinates system, usually called *null Gaussian coordinates*, such that the near-horizon geometry is given by [48]

$$ds^2 = -2\kappa\rho dv^2 + 2d\rho dv + 2\theta_A \rho dv dx^A + (G_{AB} + \lambda_{AB}\rho)dx^A dx^B + \mathcal{O}(\rho^2), \tag{326}$$

where κ, G_{AB}, λ_{AB}, θ_A in principle depend on the coordinates $\mathbf{x}$ and v. The spatial metric G_{AB} can be used to raise and lower spatial indexes.

There are now two types of geometrical objects we can define on $\mathcal{H}$: the first ones are intrinsic and the others extrinsic. In a Hamiltonian perspective, they are canonical conjugate of each other. Moreover, the canonical momenta satisfy constraint equations that are imposed by the gravitational dynamics [49, 50]. The induced geometry on $\mathcal{H}$ is degenerate and reads

$$ds^2_{\mathcal{H}} = 0 \cdot dv^2 + 0 \cdot dv dx^A + G_{AB} dx^A dx^B, \tag{327}$$

the intrinsic geometry being then entirely specified by the spatial metric in this gauge. The metric induced on $\mathcal{H}$ can be interpreted as a Carroll metric. We are going to see that actually all the elements of a Carroll structure appear on the horizon, simply because it is a null hypersurface.

We now perform a decomposition of the bulk metric adapted to the study of null hypersurfaces :

$$g_{ab} = q_{ab} + L_a N_b + N_a L_b, \tag{328}$$

where

$$\vec{L} = L^a \partial_a = \partial_v - \rho\theta^A \partial_A + \kappa\rho\partial_\rho \quad \text{and} \quad N = N_a dx^a = dv, \tag{329}$$

are respectively a null vector and a null form. They satisfy $N(\vec{L}) = 1$ and will allow us to define all the extrinsic curvature elements of $\mathcal{H}$. The vector $\vec{L}$ coincides with the normal to the horizon on $\mathcal{H}$, and has the particularity of being also tangent to the horizon. Besides, the vector $\vec{N} \equiv g^{-1}(N)$ is transverse to the horizon and together with $\vec{L}$ they define q_{ab}, the projector perpendicular to $\vec{L}$ and $\vec{N}$.

These two objects complete the Carroll structure (defined in Sec. 2.5), indeed, we can choose the Carroll vector field to be the null vector $\vec{L}$ evaluated on the horizon, which belong to the kernel of the induced metric $ds^2_{\mathcal{H}}$, while the Ehresmann connection is simply the one-form induced by $-N$ on the horizon. To summarize, the Carroll structure induced on the horizon is $(\mathcal{M}, g, \vec{v}, \tau)$ with

$$\begin{aligned} &\mathcal{M} = \{\rho = 0\} = \mathcal{H}, \\ &g = ds^2_{\mathcal{H}} = 0 \cdot dv^2 + 0 \cdot dv dx^A + G_{AB} dx^A dx^B, \\ &\vec{v} = \vec{L}_{\mathcal{H}} = \partial_v, \\ &\tau = -N_{\mathcal{H}} = dv. \end{aligned} \tag{330}$$

In the language of Sec. 5.3, it corresponds to

$$a_{AB} = G_{AB}, \quad b_A = 0, \quad \Omega = 1. \tag{331}$$

The simplicity of the Carroll structure is due to our choice of local coordinates around the horizon. Nevertheless, its existence does not rely on a particular choice of coordinate, the geometry of a null hypersurface always defines a Carroll structure (see [32]).

In his work [22, 23], T. Damour maps the black hole dynamics to the hydrodynamics of a fluid living on the horizon, and the vector $\vec{L}$ defines the fluid's velocity through $\vec{L}_{\mathcal{H}} = \partial_v + v^A \partial_A$. We have $v^A = 0$, as we have chosen comoving coordinates, *i.e.*, in Damour's interpretation the fluid would be at rest but on a dynamical surface [13].

The extrinsic geometry of the horizon is captured by a triple $(\Sigma^{AB}, \omega_A, \tilde{\kappa})$ where Σ^{AB} is the deformation tensor (or second fundamental form), ω_A is the twist field (Hajicek one-form) and $\tilde{\kappa}$ the surface gravity, defined as follows :

$$\Sigma_{AB} = \frac{1}{2} q^a_A q^b_B \mathcal{L}_{\vec{L}} q_{ab}, \quad \omega_A = q^a_A (N_b \mathcal{D}_a L^b) \quad \text{and} \quad L^b \mathcal{D}_b L^a = \tilde{\kappa} L^a, \tag{332}$$

where $\mathcal{L}$ denotes the Lie derivative, and $\mathcal{D}_a$ is the Levi-Civita associated with g_{ab}. Using the bulk metric (326), these quantities become on $\mathcal{H}$

$$\Sigma_{AB} = \frac{1}{2} \partial_v G_{AB}, \quad \omega_A = -\frac{1}{2} \theta_A \quad \text{and} \quad \tilde{\kappa} = \kappa. \tag{333}$$

We see that κ, the coefficient that appears in the bulk metric, really plays the role of the surface gravity and that θ_A, also appearing in the metric, is proportional to the twist. The deformation tensor gives rise to two new extrinsic objects : its trace and its traceless part, which are respectively the horizon expansion and the shear tensor :

$$\begin{aligned} &\Theta = G^{AB} \Sigma_{AB} = \partial_v \log \sqrt{G}, \\ &\sigma_{AB} = \frac{1}{2} \partial_v G_{AB} - \frac{\Theta}{D-2} G_{AB}, \end{aligned} \tag{334}$$

where $\sqrt{G}$ is the volume form of the spatial metric. The scalar expansion Θ measures the rate of variation of the surface element of the spatial section of $\mathcal{H}$. [14] It is possible to show, under the assumption that matter fields satisfy the null energy condition and that the null Raychaudhuri

13. As pointed out in [26], one can always set $v^A = 0$, namely the spatial coordinates x^A can always be taken to be comoving, except at caustics.

14. By definition, a non-expanding horizon has $\Theta = 0$.

equation (see next section) is satisfied, that Θ is positive everywhere on $\mathcal{H}$, which implies that the surface area of the horizon can only increase with time (see e.g. [51]).

In terms of Carrollian data, we recognize $\Theta = \theta$ the Carrollian expansion (254) and $\sigma_{AB} = \xi_{AB}$ the Carrollian shear (253). The other quantities θ_A and κ will appear in the definition of the Carrollian momenta, i.e. the energy density, pressure and dissipative terms for the Carrollian fluid.

7.1.2 Raychaudhuri and Damour equations

Those quantities being defined, we can deduce from Einstein equations two conservation laws (or constraint equations) that belong to $\mathcal{H}$: the null Raychaudhuri equation [52] and Damour equation [22, 23], which are respectively

$$L^a L^b R_{ab} = 0 \quad \text{and} \quad q^a_A L^b R_{ab} = 0; \tag{335}$$

they are thus given by projections of vacuum Einstein equations on the horizon. The first one is scalar and the second one is a vector equation w.r.t. the spatial section of $\mathcal{H}$. Using the near-horizon geometry (326), the null Raychaudhuri equation becomes

$$\partial_v \Theta - \kappa \Theta + \frac{\Theta^2}{D-2} + \sigma_{AB}\sigma^{AB} = 0, \tag{336}$$

where $\sigma^{AB} = G^{AC}G^{BD}\sigma_{CD}$. This equation describes how the expansion evolves along the null geodesic congruence $\vec{L}$ and is a key ingredient in the proofs of singularity theorems. Damour equation becomes

$$(\partial_v + \Theta)\,\theta_A + 2\nabla_A\left(\kappa + \frac{D-3}{D-2}\Theta\right) - 2\nabla_B \sigma^B_A = 0, \tag{337}$$

where ∇_A is the Levi-Civita connection associated with G. Damour has interpreted this last equation as a $(D-2)$-dimensional Navier-Stokes equation for a viscous fluid ; notice that the fluid velocity is not appearing here because we have chosen a comoving coordinate system as explained earlier. We will come back to this since the advent of Carrollian physics, and in particular our better understanding of Carrollian hydrodynamics, allows for a more accurate interpretation.

It is these two equations that we want to interpret as Carrollian fluid conservation equations. The Raychaudhuri and Damour equations being interpreted as ulta-relativistic conservation equations respectively for the energy and the momentum.

7.2 Through the looking glass

In order to motivate the interpretation of the constraint equations in terms of ultra-relativistic conservation laws it will be useful to introduce the notion of stretched horizon. It consists of a codimension-one hypersurface Σ_ρ of constant ρ very small, i.e. close to the horizon, see Fig. 2. The surface Σ_ρ is time-like and when $\rho \to 0$ it becomes null. In other words, for every $\rho > 0$, the surface describes a relativistic spacetime and when $\rho = 0$ it becomes Carrollian. Hence the interpretation of the near-horizon limit as an ultra-relativistic limit.

More precisely, consider the hypersurface Σ_ρ near $\rho = 0$. Its normal unit is given by (see Fig. 2)

$$n = \frac{d\rho}{\sqrt{2\kappa\rho}}, \tag{338}$$

and allows us to define the extrinsic curvature and the momentum conjugate to the induced metric :

$$T_{ab} = \frac{1}{8\pi G}(K p_{ab} - K_{ab}), \tag{339}$$

where $K^a_b = p^c_b \mathcal{D}_c n^a$ is the extrinsic curvature of Σ_ρ, $K = K^a_a$ its trace and $p_{ab} = g_{ab} - n_a n_b$ is the projector on the hypersurface perpendicular to n. The tensor T_{ab} is usually called the "membrane energy–momentum tensor" [25, 26, 53]. [15] Einstein equations ensure that it is conserved :

$$\bar{\nabla}_\mu T^{\mu\nu} = 0, \tag{340}$$

where the index μ refers to $\{v, \mathbf{x}\}$, and $\bar{\nabla}_\mu$ is the Levi-Civita connection associated with the induced metric on Σ_ρ. The membrane is then interpreted as a fluid whose equations of motion are given by this conservation law. One notices that (340) describes the dynamics of a *relativistic* fluid that lies in the $(D-1)$-dimensional spacetime given by the constant ρ hypersurface and equipped with the metric induced on Σ_ρ.

Now taking the $\rho \to 0$ limit has the same effect than taking an ultra-relativistic limit of these conservation laws, with an effective speed of light $c = \rho^2$, and the resulting equations are the Raychaudhuri and Damour equations. The first one corresponds to the zero-ρ limit of $\bar{\nabla}_\mu T^{\mu v}$ while the second one corresponds to the zero-ρ limit of $\bar{\nabla}_\mu T^\mu_A$. Indeed, it is easy to show that these two constraint equations can be written as

$$\begin{aligned}
&(\partial_v + \theta)\varepsilon + p\theta - \Xi^{AB}\xi_{AB} = 0, \\
&(\partial_v + \theta)\pi_A - \partial_A p - \nabla_B \Xi^B_A = 0.
\end{aligned} \tag{341}$$

This corresponds exactly to the Carrollian energy and momentum conservation (285) and (287). The difference being that now we are not in the conformal case : $\varepsilon \neq 2p$ (see [27] for a description of the non-conformal case). The geometry on which the Carrollian fluid lies is characterized by the Carrollian structure induced on the horizon

$$a_{AB} = G_{AB}, \quad b_A = 0, \quad \Omega = 1. \tag{342}$$

The corresponding first-derivative terms (see Eqs. (251), (252), (253) and (254)) are simple and match with intrinsic and extrinsic quantities of the horizon. The Carrollian acceleration and vorticity vanish

$$\varphi_A = \varpi_{AB} = 0, \tag{343}$$

while the expansion and the shear are

$$\begin{aligned}
&\theta = \Theta = \partial_v \log \sqrt{G}, \\
&\xi_{AB} = \sigma_{AB} = \frac{1}{2}\partial_v G_{AB} - \frac{\Theta}{D-2} G_{AB}.
\end{aligned} \tag{344}$$

Finally, the energy density, pressure and dissipative terms are also given in terms of horizon quantities,

$$\begin{aligned}
&\varepsilon = \Theta, \\
&p = -\left(\kappa + \frac{D-3}{D-2}\Theta\right), \\
&\Xi_{AB} = -\xi_{AB}, \\
&\pi_A = -\frac{1}{2}\theta_A.
\end{aligned} \tag{345}$$

The energy density is simply the expansion of the horizon, the pressure corresponds exactly to the "gravitational pressure" in [50]. The Carrollian stress is exactly the shear of the horizon while the heat current corresponds to the twist. We conclude that the Raychaudhuri and Damour equations map perfectly to Carrollian conservation laws and as we have shown it is not a coincidence.

15. In those papers, the approach is to study this membrane energy–momentum tensor for a small ρ and use it to define the fluid quantities like the energy density, the pressure, etc. The problem is that those quantities diverge when ρ is sent to zero. Their solution is to rescale them by hand to obtain finite quantities. We solve this problem by defining the Carrollian momenta that are already finite on the horizon and well suited for the ultra-relativistic interpretation.

We should now comment on Damour's interpretation. In his PhD thesis [22], Damour compares the equation with the Navier Stokes Stokes equation on curved background, which is just the Galilean version of the momentum conservation. If the local coordinate are chosen differently one can produce a vector $\vec{L}$ that is not aligned to ∂_v on the horizon, such that

$$\vec{L}_{\mathcal{H}} = \partial_v + v^A \partial_A. \tag{346}$$

With this choice, Darmour's equation becomes

$$\left(\partial_v + \widetilde{\Theta}\right)\omega_A + v^B \nabla_B \omega_A + \omega_B \nabla_A v^B - \nabla_A\left(\kappa + \frac{D-3}{D-2}\widetilde{\Theta}\right) + \nabla_B \widetilde{\sigma}_A^B = 0, \tag{347}$$

where $\widetilde{\Theta}$ and $\widetilde{\sigma}_{AB}$ are modified expansion and shear due to the change of $\vec{L}$:

$$\begin{aligned}
&D_{AB} \equiv \frac{1}{2}\left(\nabla_A v_B + \nabla_B v_A + \partial_v G_{AB}\right), \\
&\widetilde{\Theta} = G^{AB} D_{AB}, \\
&\widetilde{\sigma}_{AB} = D_{AB} - \frac{\widetilde{\Theta}}{D-2} G_{AB}.
\end{aligned} \tag{348}$$

This equation should be compared with the Galilean conservation of momentum on curved background. The four first terms correspond to a material derivative of θ_A on a background that depends both on space and time and for a fluid velocity v^A. The fifth term is interpreted as a derivative of the pressure while the two last term correspond to the derivative of the stress tensor.

The only caveat to this interpretation (and Damour comments on it) is that in the conservation of momentum, θ_A should correspond to a density of momentum, i.e. $\omega_A \sim \rho v_A$, where ρ is the density of matter. But this is obviously not the case here, the twist has no reason to be aligned with the vector field v^A. In the case studied above, v^A was vanishing whereas θ_A could take any value. Another issue is that if we really insist on keeping the density of momentum and the velocity v^A unrelated, then it should also appear in the conservation of energy, even when v^A vanishes. There should be a term of the type $\nabla_A \omega^A$ in the energy conservation. All these little issued are solved if one accept that the fluid should be interpreted as Carrollian. The twist ω_A (which corresponds to θ_A when v^A vanishes) can then be non-zero for a vanishing v^A since it is actually the Carrollian heat current π_A. The latter being absent of a Carrollian conservation of energy, which is consistent with the fact that θ^A does not appear in the Raychaudhuri equation.

8 Outlooks

This journey through various aspect of the role of boundaries in gravity has raised interesting directions of research that we would like to comment on. In three dimensions, AdS and flat, we have derived the most general solution space in Bondi gauge and the corresponding asymptotic Killings. The next step is to compute the corresponding charge and check whether they are finite/integrable or not and compute their algebra. Also in three dimensions, we have seen that the right way of parametrizing the solution space of three-dimensional asymptotically flat gravity seems to be through the notion of Carroll structure accompanied by a couple of Carrollian momenta. Nevertheless, a true gauge fixing procedure, such as the Bondi gauge, produces a solution space parametrized by five functions only, which is one less than what a Carroll structure (i.e. two one-forms) together with two momenta would produce. This is why me need to impose a condition on the form τ, more precisely $\tau_\phi = 0$. The role of this fixation deserves to be deepened. In particular, what happens when we ask τ_ϕ to be fixed but at a different value that zero, does this define another solution space that can be obtained from another gauge than the Bondi gauge. Does this change the asymptotic symmetry group and the charges ? Is there a large diffeomorphism that implements the change of value of τ_ϕ and is this diffeomorphism pure gauge or not ? All these questions deserve investigation.

In four dimensions, we have built a resummed version of the Derivative Expansion that is valid under the assumptions that the fluid velocity is shearless and that duality conditions between the boundary energy-momentum tensor and the Cotton tensor are satisfied. The solution space described by the resummed Derivative Expansion corresponds to *all* the twisting vacuum solutions, AdS and flat. The duality relations between the dissipative components of the energy-momentum tensor and the Cotton tensor seem to be related to self duality properties of the boundary value of the Weyl tensor. Indeed, when taking the $r \to \infty$ limit of the Weyl tensor (in a complex tetrad basis), the combination $T_{\mu\nu} + \frac{i}{8\pi G k} C_{\mu\nu}$ appears. Therefore our conditions can be translated into conditions on the boundary value of the bulk Weyl tensor. We would like to further this analysis.

An obvious follow-up question is the status of these integrability conditions in higher dimensions. Indeed the Derivative Expansion exists also in higher dimensions and we expect similar integrability conditions to exist and lead to a potential resummation. The boundary tensor $C_{\mu\nu}$ is actually the Cotton-York tensor, i.e. the Hodge dualized version of the true Cotton tensor who has three indexes. Therefore this object is exists only for three-dimensional boundaries. In higher dimensions it seems that we should consider the three-indexes one (actually a conformal version since the Cotton is not Weyl covariant in dimensions higher than three), or more precisely, a particular contraction of the latter that, together with the boundary energy-momentum tensor, could be used to write integrability conditions.

Another immediate extension of our work would be to consider additional fields in the bulk, such as a $\mathrm{U}(1)$ gauge field. The latter induces another gauge field on the boundary, that will source the conservation of the energy-momentum tensor, and a conserved current. In that case the bulk dynamics is dual to a magneto-hydrodynamical system on the boundary. In AdS the boundary system is relativistic and relativistic magneto-hydrodynamics is a well understood subject. Besides, the flat space limit calls for its Carrollian version. This direction seems rich and worth pursuing.

A last aspect, and probably the most ambitious, that we would like to comment on is the status of flat holography. Indeed a substantial part of this work was dedicated to the construction of a proper geometry to describe the boundary of generic flat spaces. One can wonder if this boundary can be the host of a field theory dual to asymptotically flat gravity. To our knowledge there is no example of exact duality (where both sides are known) between a theory of gravity in flat space and a field theory in one dimension less. But let's suppose that such a duality exists. If this is true, our analysis seems to show that it should be a conformal Carrollian field theory, and the flat limit from AdS holography could be interpreted as an ultra-relativistic limit at the level of the dual CFT. Even if we assume this is true, many conceptual questions remain unanswered : asymptotically flat spacetimes have two boundaries : the future and past null infinity, therefore should we describe the bulk gravity with two copies of Carrollian field theories with some coupling between them ? And most importantly, what would be the fundamental statement of the duality, i.e., which quantity in the bulk is related to which quantity on the boundary. The natural object to compute in the bulk is an S-matrix, but maybe not in the usual momentum basis. This is what the "Celestial Correlators" program is proposing [54], in that case, the field theory is expected to be of "Celestial" 2d CFT whose correlation functions correspond to the gravitational S-matrix. It would be interesting to see if, and under which conditions, this celestial CFT fulfills the more general expectations of conformal Carrollian field theory. Another problem is going beyond perturbation theory. In three dimensional flat gravity, one can show that the full classical state on a Cauchy slice is specified by boundary quantities, and this is true even in the full non-linear description, which indicates a potential holographic nature. In four dimensions, this property breaks down, one has to include bulk functions to describe a classical state on a Cauchy slice (see [55] for both cases). This seem like a bad news for flat holography in four dimensions. Maybe a restricted phase space should then be considered. Finally there is the problem of energy leak in flat space. The gravitational charges are not generically conserved in four dimensions, and this is due to a leak of energy through the null infinity. If we still believe in a dual field theory description, this field theory should then be coupled to a bath where the energy can be transmitted, which calls for additional structure in the boundary description.

A Petrov Classification

We provide an introduction to Petrov classification of the Weyl tensor's principal null directions (see [46] for a complete description). We introduce the complex null tetrad $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}}$ and write the metric as

$$ds^2 = -2\boldsymbol{kl} + 2\boldsymbol{m}\overline{\boldsymbol{m}}. \tag{349}$$

A principal null direction is a null vector field $\boldsymbol{k}$ satisfying

$$k_{[e}C_{a]bc[d}k_{f]}k^b k^c = 0, \tag{350}$$

and we would like to classify such vectors. Using all the symmetry properties of the Weyl tensor, one can show that its independent components can be described by five complex functions obtained by contracting C with the various basis forms

$$\begin{aligned}
\boldsymbol{\Psi}_0 &= C_{abcd}k^a m^b k^c m^d && (351)\\
\boldsymbol{\Psi}_1 &= C_{abcd}k^a l^b k^c m^d && (352)\\
\boldsymbol{\Psi}_2 &= C_{abcd}k^a m^b \overline{m}^c l^d && (353)\\
\boldsymbol{\Psi}_3 &= C_{abcd}k^a l^b \overline{m}^c l^d && (354)\\
\boldsymbol{\Psi}_4 &= C_{abcd}\overline{m}^a l^b \overline{m}^c l^d. && (355)
\end{aligned}$$

Moreover, one can show that if $\boldsymbol{k}$ is a p.n.d. then the scalar $\boldsymbol{\Psi}_0$ vanishes

$$k_{[e}C_{a]bc[d}k_{f]}k^b k^c = 0 \;\Leftrightarrow\; \boldsymbol{\Psi}_0 = C_{abcd}k^a m^b k^c m^d = 0. \tag{356}$$

After applying the most general null rotation controlled by the complex parameter E which keeps $\boldsymbol{l}$ fixed, equation (356) becomes

$$\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_0' - 4E\boldsymbol{\Psi}_1' + 6E^2\boldsymbol{\Psi}_2' - 4E^3\boldsymbol{\Psi}_3' + E^4\boldsymbol{\Psi}_4' = 0. \tag{357}$$

Since this expression is quartic in E, there are four complex roots, they can also be degenerate. The multiplicity of the solution of this equation will then be also the multiplicity of the principal null directions, which determine the Petrov class of the spacetime under consideration. The possibilities are

Petrov type	Multiplicity
I	$(1, 1, 1, 1)$
D	$(2, 2)$
II	$(2, 1, 1)$
III	$(3, 1)$
N	(4)

A Weyl tensor is said algebraically special (and consequently the spacetime will be said of algebraically special Petrov class) if it admits at least one degenerate principal null direction. That is, if it is of Petrov class D, II, III and N. A last class is the type O which corresponds to the conformally flat case, i.e. when the Weyl tensor vanishes.

Another way to obtain the multiplicity of the principal null directions is to look at the vanishing components of the Weyl tensor

$$\begin{aligned}
\boldsymbol{\Psi}_0 = 0,\ \boldsymbol{\Psi}_1 \neq 0, \dots \;&\Leftrightarrow \text{Multiplicity } 1 \;\Leftrightarrow \text{Petrov I} && (358)\\
\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_1 = 0,\ \boldsymbol{\Psi}_2 \neq 0, \dots \;&\Leftrightarrow \text{Multiplicity } 2 \;\Leftrightarrow \text{Petrov D, II} && (359)\\
\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_2 = 0,\ \boldsymbol{\Psi}_3 \neq 0, \dots \;&\Leftrightarrow \text{Multiplicity } 3 \;\Leftrightarrow \text{Petrov III} && (360)\\
\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_2 = \boldsymbol{\Psi}_3 = 0,\ \boldsymbol{\Psi}_4 \neq 0, \dots \;&\Leftrightarrow \text{Multiplicity } 4 \;\Leftrightarrow \text{Petrov N.} && (361)
\end{aligned}$$

We conclude with the Goldberg-Sachs theorem (1961) :

Goldberg-Sachs theorem : *a vacuum spacetime is algebraically special if and only if it admits a shear-free congruence of null geodesics.*

That is, if there exists a shearless, null and geodesic vector field, then the spacetime has Petrov class D, II, III, N or O.

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Annexes

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Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids

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Abstract

We provide the set of equations for non-relativistic fluid dynamics on arbitrary, possibly time-dependent spaces, in general coordinates. These equations are fully covariant under either local Galilean or local Carrollian transformations, and are obtained from standard relativistic hydrodynamics in the limit of infinite or vanishing velocity of light. All dissipative phenomena such as friction and heat conduction are included in our description. Part of our work consists in designing the appropriate coordinate frames for relativistic spacetimes, invariant under Galilean or Carrollian diffeomorphisms. The guide for the former is the dynamics of relativistic point particles, and leads to the Zermelo frame. For the latter, the relevant objects are relativistic instantonic space-filling branes in Randers–Papapetrou backgrounds. We apply our results for obtaining the general first-derivative-order Galilean fluid equations, in particular for incompressible fluids (Navier–Stokes equations) and further illustrate our findings with two applications: Galilean fluids in rotating frames or inflating surfaces and Carrollian conformal fluids on two-dimensional time-dependent geometries. The first is useful in atmospheric physics, while the dynamics emerging in the second is governed by the Robinson–Trautman equation, describing a Calabi flow on the surface, and known to appear when solving Einstein's equations for algebraically special Ricci-flat or Einstein spacetimes.

Contents

1 Introduction

Ordinary non-relativistic fluid dynamics is described in terms of a basic set of equations: continuity, energy conservation and momentum conservation (Euler equation). In most textbooks (as e.g. [1]) the fluid is observed from either inertial, or stationary rotating frames, using Cartesian or spherical/cylindrical coordinates. Although these set-ups are satisfactory for most practical purposes, they do not exhaust all possible situations because the equations at hand are not covariant under Galilean diffeomorphisms *i.e.* general coordinate transformations such as $t' = t'(t)$ and $\mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$. Most importantly, the geometry hosting the fluid is assumed to be three- or two-dimensional Euclidean space. This is a severe limitation, as we may want to study the fluid moving on a surface, which is neither flat nor static, and equipped with an arbitrary coordinate system.

Progress has been made over the last decades, sustained by the needs of the space programs or meteorology [2–7]. The most recent work [7] beautifully highlights the various contributions, and provides a covariant frame-independent formulation. Still, these authors do not address the issue of trading Euclidean space for an arbitrary curved and time-dependent geometry, and subsequent analyses have focused to the case of static surfaces (see *e.g.* [8]). Part of our work consists in filling this gap, and presenting the most general equations describing a non-relativistic viscous fluid moving on a space endowed with a spatial, time-dependent metric, and observed from an arbitrary frame. Each geometric object involved in this description has a well-defined transformation rule under Galilean diffeomorphisms, making the set of equations covariant.

In order to achieve the above program, we carefully analyze the infinite-light-velocity limit inside the relativistic fluid equations. Although standard (see §125 of [1] for the original presentation and [9] for a modern approach), this method has been only partially developed outside the realm of Minkowski spacetime (as *e.g.* in [10]). Hence, it has mostly led to non-relativistic fluids on plain Euclidean space in inertial frames. Choosing the form of a general spacetime metric such that it allows for a non-relativistic limit, enables us to reach our goal.

Considering the infinite-light-velocity limit in a relativistic framework suggests to study in parallel the alternative zero-light-velocity limit. This is actually ultra-relativistic, but we will keep on calling it non-relativistic as it decouples time and contracts the Poincaré group down to the Carroll group, as originally described in [11].

Carrollian physics has attracted some attention over the recent years [12, 13]. Although kinematically restricted – due to the vanishing velocity of light, the light-cone collapses to a line and no motion is allowed – the freedom of choosing a frame is as big as for Galilean physics though. In particular, the single particle has degenerate motion [14], but extended instantonic[1] objects do still exist and have non-trivial dynamics, making this framework rich and interesting. Following the pattern described above, we study the corresponding general set of equations for viscous fluids. The form of the spacetime metric appropriate for the limit at hand is of Randers–Papapetrou, slightly different from the one used in the former case, which is the Zermelo form.[2] The obtained equations are covariant under Carrollian coordinate transformations, $t' = t'(t, \mathbf{x})$ and $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$. In order to avoid any confusion, we will refer to the standard non-relativistic fluids as *Galilean*, whereas the latter will be called *Carrollian*.

Our motivation for the present work is twofold. On the one hand, as already mentioned, stands the need for a fully covariant formulation of Galilean fluid dynamics, on general spaces and from arbitrary frames, which might have useful physical applications. On the other hand, viscous Carrollian fluids were never studied and turn out to emerge in the context of asymptotically flat holography [16], in replacement of the relativistic fluids present in the usual fluid/gravity holographic correspondence of asymptotically anti-de Sitter space-

[1] In ordinary relativistic spacetime, we would call these objects tachyonic as they extend in space *i.e.* outside the local light-cone. Since the latter is everywhere degenerate in Carrollian spacetimes, instantonic is more illustrative.

[2] See [15] for an interesting discussion on Zermelo vs. Randers–Papapetrou forms.

times [17–20]. Performing this analysis in parallel is useful as both Galilean and Carrollian groups, and Zermelo and Randers–Papapetrou frames turn out to have intimate duality relationships.

We will start our exposition by designing the appropriate forms for relativistic spacetimes, hosting naturally the action of – *i.e.* being stable under – the two diffeomorphism groups that we want to survive in the infinite-c or zero-c limits, Secs. 2.1, 2.2. Local Galilean and Carrollian transformations are elegantly implemented in ordinary particle or instantonic space-filling brane dynamics, respectively. They are subsequently uplifted into Zermelo and Randers–Papapetrou metrics for the spacetime. The next step consists in studying ordinary viscous relativistic fluids on these environments and consider the infinite-c or zero-c limits in their equations. This is performed in Secs. 3.2 and 3.3, following a concise overview on relativistic fluids, Sec. 3.1. We find generalized continuity, energy-conservation and Euler equations for the usual Galilean fluids, as well as a set of two scalar (one for the energy) and two vector equations for the Carrollian ones. We analyze the covariance properties of the equations in both cases, and show that these transform as expected. Some examples are collected in Sec. 4: the Galilean fluid from a rotating frame or on an inflating surface, and the dynamics of a two-dimensional Carrollian viscous fluid. Further technical details, are provided in the appendix, where we introduce a new time connection for the Galilean geometry, and both temporal and spatial connections for the Carrollian and conformal-Carrollian geometry, together with their associated curvature tensors, allowing for a more elegant presentation of the corresponding covariant equations.

2 Galilean and Carrollian Poincaré uplifts

We present here the relativistic uplifts of Newton–Cartan and Carrollian non-relativistic structures. In these Lorentzian-signature spacetimes, respectively of the Zermelo and Randers–Papapetrou form, the Galilean and Carrollian diffeomorphisms are naturally realized, and the dynamics of free objects smoothly matches the ordinary Galilean and Carrollian dynamics, when the velocity of light becomes infinite or vanishes, respectively.

2.1 From Galileo Galilei …

Consider a free particle on an arbitrary d-dimensional space $\mathscr{S}$, endowed with a positive-definite metric

$$\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad i,j\ldots \in \{1,\ldots,d\}, \tag{2.1}$$

and observed from a frame with respect to which the locally inertial frame has velocity $\mathbf{w} = w^i\partial_i$. Its classical (as opposed to relativistic) dynamics is captured by the following

Lagrangian:

$$\mathcal{L}(\mathbf{v},\mathbf{x},t) = \frac{1}{2\Omega^2} a_{ij} \left(v^i - w^i\right) \left(v^j - w^j\right) \tag{2.2}$$

with action

$$S[\mathbf{x}] = \int_{\mathscr{C}} \mathrm{d}t \, \Omega \mathcal{L}(\mathbf{v},\mathbf{x},t). \tag{2.3}$$

In this expression:

- a_{ij} and w^i are general functions of $(t,\mathbf{x})$;[3]
- $v^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}$ are the usual components of the velocity $\mathbf{v} = v^i \partial_i$;
- $\mathcal{L}(\mathbf{v},\mathbf{x},t)$ appears as a Lagrangian density, with Lagrangian[4] $L(\mathbf{v},\mathbf{x},t) = \Omega \mathcal{L}(\mathbf{v},\mathbf{x},t)$.

Furthermore

- the Lagrange generalized momenta are (indices are lowered and raised with a_{ij} and its inverse)

$$p_i = \frac{\partial L}{\partial v^i} = \frac{1}{\Omega}(v_i - w_i), \tag{2.4}$$

- $H(\mathbf{p},\mathbf{x},t) = p_i v^i - L(\mathbf{v},\mathbf{x},t)$ is the Hamiltonian with Hamiltonian density $\mathcal{H} = \frac{1}{\Omega} H$:

$$\mathcal{H} = \frac{1}{2}\left(\mathbf{p}^2 + \frac{\mathbf{p}\cdot\mathbf{w}}{\Omega}\right). \tag{2.5}$$

The existence of an absolute Newtonian time requires Ω be a function of t only, the absolute time being thus $\int \mathrm{d}t \, \Omega(t)$. One should stress that keeping general $\Omega(t,\mathbf{x})$ does not spoil the consistency of the system (2.2), (2.3), but invalidates the interpretation of (2.1) as the spatial metric. Even though in practical situations we can set $\Omega = 1$, its rôle is important when dealing with general Galilean diffeomorphisms (see (2.11)–(2.15)), in the framework underlying the above dynamical system: the Newton–Cartan structures [21].[5]

We can compute the energy density expressing the Hamiltonian (2.5) in terms of the velocity:

$$\mathcal{H} = \frac{1}{2\Omega^2} a_{ij} \left(v^i + w^i\right) \left(v^j - w^j\right) = \frac{1}{2\Omega^2}\left(\mathbf{v}^2 - \mathbf{w}^2\right). \tag{2.6}$$

As usual $-\mathbf{w}^2/2\Omega^2$ plays the rôle of the potential for inertial forces. Using the energy theorem ($\mathrm{d}H/\mathrm{d}t = -\partial L/\partial t$) one finds

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = -\frac{1}{2\Omega^2}\left(v^i - w^i\right)\left(v^j - w^j\right)\partial_t a_{ij} + \frac{v_i - w_i}{\Omega}\partial_t \frac{w^i}{\Omega}. \tag{2.7}$$

[3]Here $\mathbf{x}$ stands for $\{x^1, \ldots, x^d\}$.

[4]Euler–Lagrange equations are $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial x^i}$.

[5]Some modern references on Newton–Cartan structure are *e.g.* [22–25].

The most canonical example of (2.2) is that of a massive particle moving in Euclidean space E_3 with Cartesian coordinates, and observed from a non-inertial frame:

$$a_{ij} = \delta_{ij}, \quad \Omega = 1, \quad \mathbf{w}(t, \mathbf{x}) = \mathbf{x} \times \boldsymbol{\omega}(t) - \mathbf{V}(t). \tag{2.8}$$

Here $\mathbf{V}(t)$ is the dragging velocity of the non-inertial frame, $\boldsymbol{\omega}(t)$ the angular velocity of its rotating axes, and $\mathbf{v} - \mathbf{w} = \mathbf{v} + \mathbf{V} + \boldsymbol{\omega} \times \mathbf{x}$ is the velocity as measured in the original inertial frame (Roberval's theorem).

The action (2.3) is invariant under general *Galilean diffeomorphisms i.e.* transformations

$$t' = t'(t) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(t, \mathbf{x}), \tag{2.9}$$

for which we define the following Jacobian functions:

$$J(t) = \frac{\partial t'}{\partial t}, \quad j^i(t, \mathbf{x}) = \frac{\partial x^{i\prime}}{\partial t}, \quad J^i_j(t, \mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j}. \tag{2.10}$$

The metric components transform as a tensor of $\mathscr{S}$:

$$a'_{ij} = a_{kl} J^{-1k}_{i} J^{-1l}_{j}, \tag{2.11}$$

the particle and frame velocities as gauge connections:

$$v'^k = \frac{1}{J}\left(J^k_i v^i + j^k\right), \tag{2.12}$$

$$w'^k = \frac{1}{J}\left(J^k_i w^i + j^k\right), \tag{2.13}$$

and the generalized momenta (2.4) as one-form components:

$$p'_i = p_k J^{-1k}_{i}; \tag{2.14}$$

Ω is just rescaled:

$$\Omega' = \frac{\Omega}{J}. \tag{2.15}$$

Since $J = J(t)$ and $\Omega = \Omega(t)$, Galilean transformations lead to $\Omega' = \Omega'(t')$, leaving invariant the absolute Newtonian time $\int \mathrm{d}t\, \Omega(t) = \int \mathrm{d}t'\, \Omega'(t')$. Observe also that $\frac{\mathbf{v}-\mathbf{w}}{\Omega}$ is a genuine vector of $\mathscr{S}$, which ensures the form-invariance of $\mathcal{L}$ and thus the covariance of the equations of motion.

There is a particular Newton–Cartan structure, which is invariant under the Galilean group: $\mathscr{S}$ is the Euclidean space E_d with Cartesian coordinates ($a_{ij} = \delta_{ij}$) and $\Omega = 1$, and the connection $\mathbf{w}$ is constant *i.e.* independent of $(t, \mathbf{x})$. This system describes the non-relativistic motion of a free particle in Euclidean space, observed from an inertial frame. The Galilean group acts as

$$\begin{cases} t' = t + t_0, \\ x'^k = R^k_i x^i + V^k t + x^k_0 \end{cases} \tag{2.16}$$

with all parameters being $(t, \mathbf{x})$-independent, and R^k_i the entries of an orthogonal matrix. The action of these transformations leave the Lagrangian and the equations of motion at hand *invariant*. In more general Newton–Cartan structures, the Galilean group acts in the tangent space equipped with a local orthonormal frame and it is no more a global symmetry.

The Galilean group is an infinite-c contraction of the Poincaré group. The latter acts locally in general $d+1$-dimensional pseudo-Riemannian manifolds $\mathscr{M}$. In order to recover the above Newton–Cartan structure and its class of diffeomorphisms (2.9) in the infinite-c limit, there is a natural choice for the form of the metric on $\mathscr{M}$:

$$\mathrm{d}s^2 = -\Omega^2 c^2 \mathrm{d}t^2 + a_{ij} \left(\mathrm{d}x^i - w^i \mathrm{d}t\right) \left(\mathrm{d}x^j - w^j \mathrm{d}t\right). \tag{2.17}$$

The form (2.17) is required for the functions Ω, a_{ij} and w^i to transform as in (2.11), (2.13) and (2.15) under a Galilean diffeomorphism (2.9). Actually, every metric is compatible with the gauge (2.17), provided a_{ij}, w^i and Ω, are free to depend on $x = (ct, \mathbf{x}) = \{x^\mu, \mu = 0, 1, \ldots, d\}$. The existence of a Galilean limit requires, however, Ω to depend on t only. Indeed, the proper time element for a physical observer is $\mathrm{d}\tau = \sqrt{-\mathrm{d}s^2/c^2}$. When c becomes infinite, $\lim\limits_{c\to\infty} \mathrm{d}\tau = \Omega \mathrm{d}t$ must coincide with the absolute Newtonian time, and this requires the absence of $\mathbf{x}$-dependence in Ω, as expected from our previous discussion on the dynamics of (2.3).

The spacetime Jacobian matrix associated with (2.9), reads (using (2.10)):

$$J^\mu_\nu(x) = \frac{\partial x^{\mu\prime}}{\partial x^\nu} \to \begin{pmatrix} J(t) & 0 \\ J^i(x) & J^i_j(x) \end{pmatrix} \quad \text{with} \quad J^i = \frac{j^i}{c}. \tag{2.18}$$

The metric form (2.17) is refered to as *Zermelo* (see [15]). A relativistic particle moving in (2.17) is described by the components of its velocity u, normalized as $\|\mathrm{u}\|^2 = -c^2$:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \Rightarrow u^0 = \gamma c,\ u^i = \gamma v^i, \tag{2.19}$$

where the Lorentz factor γ is defined as usual (although here, it depends also on the space-time coordinates):[6]

$$\gamma(t, \mathbf{x}, \mathbf{v}) = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\Omega\sqrt{1 - \left(\frac{\mathbf{v}-\mathbf{w}}{c\Omega}\right)^2}}. \tag{2.20}$$

[6]Expressions as $\mathbf{v}^2$ stand for $a_{ij}v^i v^j$, not to be confused with $\|\mathrm{u}\|^2 = g_{\mu\nu}u^\mu u^\nu$.

Under a Galilean diffeomorphism (2.18), the transformation of the components of u,

$$u'^0 = J u^0, \quad u'^i = J^i_k u^k + J^i u^0, \quad u'_0 = \frac{1}{J}\left(u_0 - u_j J^{-1j}{}_k J^k\right), \quad u'_i = u_k J^{-1k}{}_i, \tag{2.21}$$

induces a transformation on v^i, which matches precisely (2.12).

The dynamics of the relativistic free particle is described using *e.g.* the length of the world-line $\mathscr{C}$ as an action:

$$S[x] = \int_{\mathscr{C}} \mathrm{d}\tau = \int_{\mathscr{C}} \sqrt{-\frac{\mathrm{d}s^2}{c^2}}. \tag{2.22}$$

This is easily computed in the Zermelo environment (2.17), and expanded for large c:

$$\begin{aligned} S[x] &= \int_{\mathscr{C}} \mathrm{d}t\,\Omega \sqrt{1 - \frac{1}{c^2\Omega^2} a_{ij} \left(v^i - w^i\right)\left(v^j - w^j\right)} \\ &= \int_{\mathscr{C}} \mathrm{d}t\,\Omega \left(1 - \frac{1}{2c^2\Omega^2} a_{ij} \left(v^i - w^i\right)\left(v^j - w^j\right) + \mathrm{O}\left(1/c^4\right)\right). \end{aligned} \tag{2.23}$$

Hence, the dynamics (2.22), disregarding the first term in (2.23), which is a Galilean invariant, coincides in the infinite-c limit with the dynamics of the non-relativistic action displayed in (2.3). This shows that (2.17) is the natural relativistic spacetime uplift of a Galilean space $\mathscr{S}$ endowed with a Newton–Cartan structure.

2.2 . . . to Lewis Carroll

The Poincaré group admits another contraction at vanishing c [11]. Although this limit may sound degenerate as particle motion is frozen, it exhibits both an interesting dynamics and a rich mathematical structure.

A Euclidean space E_d with Cartesian coordinates, accompanied with a real time line t can be equipped with a structure alternative to Newton–Cartan's, known as Carrollian. This structure is left invariant by the Carrollian group acting as

$$\begin{cases} t' = t + B_i x^i + t_0, \\ x'^k = R^k_i x^i + x^k_0 \end{cases} \tag{2.24}$$

with all parameters being $(t, \mathbf{x})$-independent, and R^k_i the entries of an orthogonal matrix.

Invariant equations of motion can be considered for extended objects *i.e.* fields rather than particles. Indeed, at zero velocity of light, a particle cannot move in time but time can define an $\mathbf{x}$-dependent field. The scalar field $t(\mathbf{x})$ describes a d-brane, in other words a space-filling object in E_d, extended inside a portion of space $\mathscr{V} \subset E_d$.[7] Its invariant action can be

[7] Our guide in this section is symmetry, and our goal the adequate Poincaré uplift. The precise physical system and the nature of its dynamics are of secondary importance. Other systems with Carrollian symmetry may exist. It is interesting, though, to maintain a dual formulation for the two sides (Galilean and Carrollian), as for objects

e.g.

$$S[t] = \int_{\mathscr{V}} \mathrm{d}^d x \mathcal{L}(\boldsymbol{\partial} t) \tag{2.25}$$

with Lagrangian density

$$\mathcal{L}(\boldsymbol{\partial} t) = \frac{1}{2}\delta^{ij}\left(\partial_i t - b_i\right)\left(\partial_j t - b_j\right), \tag{2.26}$$

where b_i are constant parameters with inverse-velocity dimension, playing the rôle of a constant gauge-field background, and transforming by shift and rotation under (2.24): $b'_i = \left(b_j + B_j\right) R^{-1j}{}_i$.

More general Carrollian structures equip Riemannian manifolds $\mathscr{S}$ with metric (2.1) and time $t \in \mathbb{R}$. The Carrollian transformations (2.24) are realized locally, in the tangent space, and are no longer symmetries. The structure is covariant under *Carrollian diffeomorphisms*

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \tag{2.27}$$

with Jacobian functions

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J^i_j(\mathbf{x}) = \frac{\partial x^{i'}}{\partial x^j}. \tag{2.28}$$

The covariant action describing the Carrollian dynamics in the more general case at hand is[8]

$$S[t] = \int_{\mathscr{V} \subset \mathscr{S}} \mathrm{d}^d x \sqrt{a}\, \mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x}), \tag{2.29}$$

where a stands for the determinant of the matrix a_{ij} and $\mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x})$ is the Lagrangian density:

$$\mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x}) = \frac{1}{2} a^{ij}\left(\Omega \partial_i t - b_i\right)\left(\Omega \partial_j t - b_j\right). \tag{2.30}$$

Here the components of the metric, the scale factor Ω, and the components of the background gauge field $\boldsymbol{b} = b_i \mathrm{d}x^i$ depend all on $(t, \mathbf{x})$.

Under Carrollian diffeomorphisms, the metric transforms as in (2.11) *i.e.*

$$a'^{ij} = J^i_k J^j_l a^{kl}, \tag{2.31}$$

Ω is rescaled as in (2.15) – where everything now depends both on t and $\mathbf{x}$ – while the field gradients and the gauge connection obey respectively

$$\partial'_k t' = \left(J \partial_i t + j_i\right) J^{-1i}{}_{k'} \tag{2.32}$$

with dimension-one and codimension-one world-volumes.

[8]Notice that actions (2.25), (2.29) and (2.37) are all Euclidean-signature (instantonic) because of vanishing c.

and

$$b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1 i}_{\ \ k}. \tag{2.33}$$

Here

$$\beta_i = \Omega \partial_i t - b_i \tag{2.34}$$

transform as components of a one-form on $\mathscr{S}$, making the density Lagrangian form-invariant.

We will now uplift the above structure into a $d+1$-dimensional pseudo-Riemannian manifold $\mathscr{M}$, where the full Poincaré group is realized in the tangent space. Following the pattern used in the Galilean framework, Sec. 2.1, we can recover the general Carrollian structure and its class of diffeomorphisms (2.27) in the zero-c limit, starting from a metric on $\mathscr{M}$ of the form:

$$\mathrm{d}s^2 = -c^2 \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{2.35}$$

The form (2.35) is known as *Randers–Papapetrou*. It is universal, as every metric can be recast in this gauge. Here, it is required for the functions $\Omega(x)$, $a^{ij}(x)$ and $b_i(x)$ to transform as in (2.15), (2.31) and (2.33) under a Carrollian diffeomorphism (2.27) – again $x \equiv (x^0 = ct, \mathbf{x})$. The spacetime Jacobian matrix associated with transformations (2.27), reads (using (2.28)):

$$J^\mu_\nu(x) = \frac{\partial x^{\mu\prime}}{\partial x^\nu} \to \begin{pmatrix} J(x) & J_j(x) \\ 0 & J^i_j(\mathbf{x}) \end{pmatrix} \quad \text{with} \quad J_i = c j_i. \tag{2.36}$$

The Carrollian dynamics captured in the action (2.29) is the zero-c limit of a relativistic instantonic d-brane in a spacetime $\mathscr{M}$ with Randers–Papapetrou metric (2.35). As already mentioned (footnote 1), in this context instantonic means that the world-volume does not extend in time; it is a kind of codimension-one snap shot materialized in a space-like d-dimensional hypersurface $\mathscr{V}$, coordinated with $y^i, i = 1, \ldots, d$. Under these assumptions, the Dirac–Born–Infeld action reads:

$$S[h] = \int_{\mathscr{V}} \mathrm{d}^d y \sqrt{h}, \tag{2.37}$$

where h is the determinant of the induced metric matrix

$$h_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} \tag{2.38}$$

with $g_{\mu\nu}$ the background metric components.

For the Randers–Papapetrou environment displayed in (2.35), we find:

$$h_{ij} = \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} \left(a_{kl} - c^2 \left(\Omega \partial_k t - b_k\right) \left(\Omega \partial_l t - b_l\right)\right). \tag{2.39}$$

In this expression, $\partial_k t$ stands for ${}^{\partial t}/_{\partial x^k}$. Consequently, we implicitly assume that the functions $x^k = x^k(y^i)$ are invertible, which is equivalent to saying that one can choose a gauge where

$y^i = x^i$. This is what happens in practice. Indeed, one can readily compute the root of the determinant and its expansion in powers of c^2. Naming $\alpha_i^k = \frac{\partial x^k}{\partial y^i}$, we obtain:

$$\sqrt{h} = \det\alpha\,\sqrt{a}\left(1 - \frac{c^2}{2}a^{kl}\left(\Omega\partial_k t - b_k\right)\left(\Omega\partial_l t - b_l\right) + \mathrm{O}\left(c^4\right)\right). \tag{2.40}$$

Hence (2.37) becomes

$$S[h] = \int_{\mathscr{V}} \mathrm{d}^d x\,\sqrt{a}\left(1 - \frac{c^2}{2}a^{kl}\left(\Omega\partial_k t - b_k\right)\left(\Omega\partial_l t - b_l\right) + \mathrm{O}\left(c^4\right)\right). \tag{2.41}$$

Neglecting the first term, which is invariant under Carrollian diffeomorphisms (2.27), (2.28), in the zero-c limit, (2.41) describes the same dynamics as (2.29), (2.30). This result, in close analogy with the Galilean discussion in the previous section, shows that the form (2.35) is well-suited for the zero-c limit.

3 Fluid dynamics in the non-relativistic limits

The aim of the present chapter is to exhibit the general fluid equations in the Galilean and Carrollian structures. This is achieved starting from plain relativistic viscous-fluid dynamics in the appropriate background – Zermelo or Randers–Papapetrou – and analyzing the associated, infinite or vanishing light-velocity limit. By construction, the equations reached this way are covariant under the corresponding diffeomorphisms. We study here neutral fluids, moving freely *i.e.* subject only to pressure, friction forces and thermal conduction processes. We conclude with some comments on a duality relating the two limits under consideration.

3.1 Relativistic fluids

Free relativistic viscous fluids are described in terms of their energy–momentum tensor obeying the set of $d+1$ conservation equations

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.1}$$

The time component is the energy conservation, the other d spatial ones, momentum conservation, usually called *Euler* equations.

The energy–momentum tensor is made of a perfect-fluid piece and terms resulting from friction and thermal conduction. It reads:

$$T^{\mu\nu} = (\varepsilon + p)\frac{u^\mu u^\nu}{c^2} + p g^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\nu}{c^2} + \frac{u^\nu q^\mu}{c^2}, \tag{3.2}$$

and contains $d+2$ dynamical variables:

- energy per unit of proper volume (rest density) ε, and pressure p;
- d velocity-field components u^i (u^0 is determined by the normalization $\|\mathrm{u}\|^2 = -c^2$).

A local-equilibrium thermodynamic equation of state[9] $p = p(T)$ is therefore needed for completing the system. We also have the usual Gibbs–Duhem relation for the grand potential $-p = \varepsilon - Ts$ with $s = \partial p/\partial T$. The viscous stress tensor $\tau^{\mu\nu}$ and the heat current q^μ are purely transverse:

$$u^\mu q_\mu = 0, \quad u^\mu \tau_{\mu\nu} = 0, \quad u^\mu T_{\mu\nu} = -q_\nu - \varepsilon u_\nu, \quad \varepsilon = \frac{1}{c^2} T_{\mu\nu} u^\mu u^\nu. \tag{3.3}$$

Hence, they are expressed in terms of u^i and their spatial components q_i and τ_{ij}.

The quantities q_i and τ_{ij} capture the physical properties of the out of equilibrium state. They are usually expressed as expansions in temperature and velocity derivatives, the coefficients of which characterize the transport phenomena occurring in the fluid. The transport coefficients can be determined either from the underlying microscopic theory, or phenomenologically. In first-order hydrodynamics

$$\tau_{(1)\mu\nu} = -2\eta\sigma_{\mu\nu} - \zeta h_{\mu\nu}\Theta, \tag{3.4}$$

$$q_{(1)\mu} = -\kappa h_\mu{}^\nu \left(\partial_\nu T + \frac{T}{c^2} a_\nu\right), \tag{3.5}$$

where [10]

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \tag{3.6}$$

$$\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)} + \frac{1}{c^2} u_{(\mu} a_{\nu)} - \frac{1}{d}\Theta h_{\mu\nu}, \tag{3.7}$$

$$\omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]} + \frac{1}{c^2} u_{[\mu} a_{\nu]}, \tag{3.8}$$

are the acceleration, the expansion, the shear and the vorticity of the velocity field, with η, ζ the shear and bulk viscosities, and κ the thermal conductivity. In the above expressions, $h_{\mu\nu}$ is the projector onto the space transverse to the velocity field, and one similarly defines the longitudinal projector $U_{\mu\nu}$:

$$h_{\mu\nu} = \frac{u_\mu u_\nu}{c^2} + g_{\mu\nu}, \quad U_{\mu\nu} = -\frac{u_\mu u_\nu}{c^2}. \tag{3.9}$$

In three spacetime dimensions, the Hall viscosity appears as well in $\tau_{(1)\mu\nu}$:

$$-\zeta_{\mathrm{H}} \frac{u^\sigma}{c} \eta_{\sigma\lambda(\mu}\sigma_{\nu)\rho} g^{\lambda\rho}, \tag{3.10}$$

with $\eta_{\sigma\lambda\mu} = \sqrt{-g}\,\epsilon_{\sigma\lambda\mu}$.

[9] We omit here the chemical potential as we assume no independent conserved current.

[10] Our conventions for (anti-) symmetrization are $A_{(\mu\nu)} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)$ and $A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right)$.

In view of the subsequent steps of our analysis, an important question arises at this stage, which concerns the behaviour of q_i and τ_{ij} with respect to the velocity of light. Answering this question requires a microscopic understanding of the fluid *i.e.* a many-body (quantum-field-theory and statistical-mechanics) determination of the transport coefficients. In the absence of this knowledge, we may consider a large-c or small-c expansion of these quantities, in powers of c^2 – irrespective of the derivative expansion. In the same spirit, we could also work out similar expansions for each of the functions entering the metrics (2.17) or (2.35), as these possibly carry deep relativistic dynamics. The advantage of such an exhaustive analysis would be to set-up general conditions on a relativistic fluid and its spacetime environment for a large-c or a small-c regime to make sense. As a drawback, this approach would blur the universality of the equations we want to set. We will therefore adopt a more pragmatic attitude and assume that Ω, b_i, w^j and a_{ij} are c-independent. Regarding the viscous stress tensor τ_{ij}, we will assume the following behaviours:

$$\tau_{ij} = -\Sigma^{\mathrm{G}}{}_{ij} \tag{3.11}$$

or

$$\tau^{ij} = -\frac{\Sigma^{\mathrm{C}ij}}{c^2} - \Xi^{ij}. \tag{3.12}$$

The first is appropriate for the Galilean limit. It is standard and considered *e.g.* in [1], where $\Sigma^{\mathrm{G}}{}_{ij}$ is named σ'_{ij}. For the Carrollian dynamics, our choice is inspired by flat-spacetime holography (see [16]). Similarly, for the heat current, we will adopt

$$q_i = Q^{\mathrm{G}}{}_i, \tag{3.13}$$

$$q^i = Q^{\mathrm{C}i} + c^2\pi^i, \tag{3.14}$$

in Galilean and Carrollian dynamics, respectively. Although kinematically poorer – because at rest, Carrollian fluids carry a richer internal information than their Galilean pendants since both the heat current and the viscous tensor are doubled in the above ansatz. Observe the position of the spatial indices, different for the two cases under consideration. They are designed to be covariant under different classes of diffeomorphisms.

One should finally notice that, in writing the energy–momentum tensor (3.2), we have not made any assumption regarding the hydrodynamic frame, which is therefore left generic.[11] There are two reasons for this. The first is the absence of a conserved relativistic current, which makes hydrodynamic-frame conditions delicate. Further subtleties arise when studying the system in special limits such as the Galilean, where the relativistic arbitrariness for the velocity field is lost, due to the decoupling of mass and energy. This is the second reason.

[11] The freedom of choosing the hydrodynamic frame was raised in [1]. Modern discussions can be found in [9, 26, 27] (see also [28]).

3.2 Galilean fluid dynamics from Zermelo background

The essence of the classical limit

We will consider in the following the ordinary non-relativistic limit of fluid equations, formally reached at infinite c. The physical validity of this situation is based on two assumptions.

The first is kinematical: it presumes that the global velocity of the fluid with respect to the observer is small compared to c. This is easily implemented using the Zermelo form of the metric (2.17), where the control parameter for the validity of the classical limit is $\left|\frac{\mathbf{v}-\mathbf{w}}{c}\right|$. We find

$$\begin{cases} u^0 = \gamma c = \dfrac{c}{\Omega} + \mathrm{O}\left(1/c\right), \quad u_0 = -c\Omega + \mathrm{O}\left(1/c\right), \\ u^i = \gamma v^i = \dfrac{v^i}{\Omega} + \mathrm{O}\left(1/c^2\right), \quad u_i = \dfrac{v_i - w_i}{\Omega} + \mathrm{O}\left(1/c^2\right). \end{cases} \tag{3.15}$$

The second is microscopic. The internal particle motion should also be Galilean, in other words the energy density should be large compared to the pressure: $\varepsilon \gg p$. This sets restrictions on the equation of state, as not every equation of state is compatible with such a microscopic assumption.[12]

An important consequence of the microscopic assumption is the separation of mass and energy, now both independently conserved. It is customary to introduce the following:

- ϱ the usual mass per unit of volume (mass density);
- ϱ_0 the usual mass per unit of proper volume (rest-mass density);
- e the internal energy per unit of mass;
- h the enthalpy per unit of mass.

These local thermodynamic quantities are related as

$$\begin{cases} \varepsilon = \left(e + c^2\right) \varrho_0, \\ h = e + \frac{p}{\varrho}, \\ \varrho_0 = \dfrac{\varrho}{\Omega\gamma} = \varrho\sqrt{1 - \left(\frac{\mathbf{v}-\mathbf{w}}{c\Omega}\right)^2} \approx \varrho - \frac{\varrho}{2}\left(\frac{\mathbf{v}-\mathbf{w}}{c\Omega}\right)^2, \end{cases} \tag{3.16}$$

where we have used Eq. (2.20) for the Lorentz factor γ, and expanded it for small $\left|\frac{\mathbf{v}-\mathbf{w}}{c}\right|$.

[12] For example, the conformal equation of state, $\varepsilon = dp$ is not compatible with the non-relativistic limit at hand.

The structure of the equations

The fluid equations are the conservation (3.1) of the energy–momentum tensor (3.2), in the background (2.17). It is computationally wise to split these equations as:

$$\nabla_\mu T^{\mu 0} = 0, \quad \nabla_\mu T^{\mu}{}_{i} = 0. \tag{3.17}$$

Indeed, applying a Galilean diffeomorphism (2.9), (2.18), the time components up and space components down transform faithfully and irreducibly. On the divergence of the energy–momentum tensor we find:

$$\nabla'_\mu T'^{\mu 0} = J \nabla_\mu T^{\mu 0}, \quad \nabla'_\mu T'^{\mu}{}_{i} = J^{-1l}{}_{i} \nabla_\mu T^{\mu}{}_{l}. \tag{3.18}$$

Hence, the two sets of equations (3.17) do not mix[13] and have furthermore a d-dimensional covariant transformation, which is our goal for the Galilean fluid dynamics.

The expressions displayed so far are fully relativistic. The next step is to consider the large-c regime. In this regime, Eqs. (3.17) can be expanded in powers of $1/c$. This expansion must be performed with care as the time equation needs an extra c factor with respect to the next d spatial equations because it describes the evolution of energy, which is a momentum multiplied by c. We find:[14]

$$c \nabla_\mu T^{\mu 0} = c^2 \frac{\mathcal{C}}{\Omega} + \frac{\mathcal{E}}{\Omega} + \mathrm{O}\left(\frac{1}{c^2}\right), \tag{3.19}$$

$$\nabla_\mu T^{\mu}{}_{i} = \mathcal{M}_i + \mathrm{O}\left(\frac{1}{c^2}\right). \tag{3.20}$$

At infinite c this leads to $d+2$ equations (rather than $d+1$, since in the Galilean limit, mass and energy are separately conserved) for ϱ, e, p and v^i:

- continuity equation (mass conservation) $\mathcal{C} = 0$;
- energy conservation $\mathcal{E} = 0$;
- momentum conservation $\mathcal{M}_i = 0$;

this system is completed with the equation of state $p = p(e, \varrho)$.

It is important to stress that Galilean diffeomorphisms (2.9), (2.10) do not involve c, and consequently they do not mix the various terms in the expansions (3.19) and (3.20). All $d+2$

[13] They do mix for general diffeomorphisms though.

[14] Had we considered $\Omega = \Omega(t, \mathbf{x})$, the divergence $\nabla_\mu T^{\mu}{}_{i}$ would have exhibited an extra, dominant term in the large-c limit: $c^2 \partial_i \ln \Omega$. The spatial conservation equation, $\nabla_\mu T^{\mu}{}_{i} = 0$, would then automatically require the $\mathbf{x}$-independence for Ω. Notice also the rescaling by Ω in (3.19), which guarantees that $\mathcal{C}$ and $\mathcal{E}$ are invariants under Galilean diffeomorphisms, see (3.35).

fluid equations reached this way on general backgrounds[15] are guaranteed to be covariant under Galilean diffeomorphisms, and this was one motivation of our work.

The dissipative tensors in Zermelo background

Before displaying the advertised equations, we would like to elaborate on the two tensors which capture the deviation of the real fluid with respect to the perfect one: the heat current and the viscous stress tensor.

Orthogonality conditions (3.3) allow to express every component of these tensors in terms of q_i and τ_{ij}. We assume here the Zermelo form of the metric (2.17), and a fluid velocity field as in (2.19), (2.20). We find

$$q_0 = -\frac{v^i q_i}{c}, \quad q^0 = \frac{\left(v^i - w^i\right) q_i}{c\Omega^2}, \quad q^i = a^{ij} q_j + \frac{w^i \left(v^j - w^j\right) q_j}{c^2\Omega^2}. \tag{3.21}$$

Similarly, the components of the stress tensor are obtained from the τ_{ij}s. For example:

$$\tau_{00} = \frac{v^k v^l \tau_{kl}}{c^2}, \quad \tau_{0j} = -\frac{v^k \tau_{kj}}{c}, \quad \tau^0{}_j = -\frac{\left(v^k - w^k\right) \tau_{kj}}{c\Omega^2}, \quad \tau^{00} = \frac{\left(v^k - w^k\right)\left(v^l - w^l\right) \tau_{kl}}{c^2\Omega^4}, \dots \tag{3.22}$$

We now define $Q^{\mathrm{G}}{}_i = q_i$ as anticipated in (3.13), and

$$Q^{\mathrm{G}i} = a^{ij} Q^{\mathrm{G}}{}_j. \tag{3.23}$$

Similarly, calling for $\Sigma^{\mathrm{G}}{}_{ij}$ introduced in (3.11), we define

$$\Sigma^{\mathrm{G}}{}_i{}^j = \Sigma^{\mathrm{G}}{}_{ik} a^{kj}, \quad \Sigma^{\mathrm{G}ij} = a^{ik} \Sigma^{\mathrm{G}}{}_k{}^j. \tag{3.24}$$

Using the generic transformation rules of q_μ and $\tau_{\mu\nu}$ under spacetime diffeomorphisms, we find that $\boldsymbol{Q}^{\mathrm{G}}$ and $\boldsymbol{\Sigma}^{\mathrm{G}}$ introduced above, appearing as classical c-independent objects, transform as they should, namely as d-dimensional tensors under Galilean diffeomorphisms (2.9), (2.18):

$$Q^{\mathrm{G}\prime}{}_i = Q^{\mathrm{G}}{}_k J^{-1k}{}_i, \qquad Q^{\mathrm{G}\prime i} = J^i_k Q^{\mathrm{G}k}, \tag{3.25}$$
$$\Sigma^{\mathrm{G}\prime}{}_{ij} = J^{-1k}{}_i J^{-1l}{}_j \Sigma^{\mathrm{G}}{}_{kl}, \qquad \Sigma^{\mathrm{G}\prime}{}_i{}^j = J^{-1k}{}_i \Sigma^{\mathrm{G}}{}_k{}^l J^j_l, \quad \Sigma^{\mathrm{G}\prime ij} = \Sigma^{\mathrm{G}kl} J^i_k J^j_l. \tag{3.26}$$

Continuity and energy conservation

Using Eq. (3.2) for the energy–momentum tensor $T^{\mu\nu}$ with $g^{\mu\nu}$ and u^μ given in (2.17) and (2.19), using Eqs. (3.21), (3.23) for the heat current and (3.22), (3.24) for the stress tensor as

[15] We stress again that here, as for instance in [29, 30], Galilean fluids evolve on general, curved and time-dependent spaces $\mathscr{S}$, as opposed to other works on non-relativistic fluid dynamics (see *e.g.* [31]).

well as the definitions (3.16), we can perform the large-c expansion of the relativistic energy conservation equation (3.19). This requires the expansion of the Christoffel symbols, displayed in App. A.1.

We find the following at $\mathrm{O}(c^2)$:

$$\mathcal{C} = \frac{\partial_t \sqrt{a}\,\varrho}{\Omega\sqrt{a}} + \frac{1}{\Omega}\nabla_i \varrho v^i, \tag{3.27}$$

where a stands for the determinant of the d-dimensional metric $a_{ij}(t,\mathbf{x})$, and ∇_i is the Levi–Civita covariant derivative associated with $a_{ij}(t,\mathbf{x})$ and Christoffel symbols given in (A.9). The standard continuity equation $\mathcal{C} = 0$ is thus recovered. It is customary to decompose $\mathcal{C}$ in (3.27) as

$$\frac{\partial_t \sqrt{a}\,\varrho}{\Omega\sqrt{a}} + \frac{1}{\Omega}\nabla_i \varrho v^i = \frac{1}{\Omega}\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \varrho\theta^{\mathrm{G}}, \tag{3.28}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + v^i\nabla_i \tag{3.29}$$

is the *material derivative*, and

$$\theta^{\mathrm{G}} = \frac{1}{\Omega}\left(\partial_t \ln\sqrt{a} + \nabla_i v^i\right) \tag{3.30}$$

the *effective Galilean fluid expansion*. The latter combines the divergence of the fluid congruence with the logarithmic expansion of the volume form to produce a genuine scalar under Galilean diffeomorphisms (2.9), (2.10) (see Eqs. (2.15) and (A.17)). The material derivative (3.29), in the form $\frac{1}{\Omega}\frac{\mathrm{d}}{\mathrm{d}t}$, is also an "invariant" when acting on a scalar function. This is due to (2.12), (A.12) and (A.13). When acting on arbitrary tensors, it should be supplemented with the appropriate **w**-connection terms, as shown in the appendix, Eq. (A.24).

At the next $\mathrm{O}(c^0)$ order, we obtain:

$$\begin{aligned}
\mathcal{E} &= \frac{1}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}\,\varrho\left(e+\frac{1}{2}\left(\frac{\mathbf{v}-\mathbf{w}}{\Omega}\right)^2\right)\right) + \frac{1}{\Omega}\nabla_i\left(\varrho v^i\left(e+\frac{1}{2}\left(\frac{\mathbf{v}-\mathbf{w}}{\Omega}\right)^2\right)\right) \\
&\quad + \frac{1}{\Omega}\nabla_i\left(\left(v^j - w^j\right)\left(p\delta^i_j - \Sigma^{\mathrm{G}}{}_j{}^i\right)\right) + \nabla_i Q^{\mathrm{G}i} + \frac{1}{\Omega}\Pi^{\mathrm{G}ij}\left(\nabla_i w_j + \frac{1}{2}\partial_t a_{ij}\right) \qquad (3.31) \\
&= \frac{\varrho}{\Omega}\frac{\mathrm{d}}{\mathrm{d}t}\left(e+\frac{1}{2}\left(\frac{\mathbf{v}-\mathbf{w}}{\Omega}\right)^2\right) + \frac{1}{\Omega}\nabla_i\left(p\left(v^i - w^i\right)\right) + \nabla_i Q^{\mathrm{G}i} \\
&\quad - \frac{1}{\Omega}\nabla_i\left(\left(v^j - w^j\right)\Sigma^{\mathrm{G}}{}_j{}^i\right) + \frac{1}{\Omega}\Pi^{\mathrm{G}ij}\left(\nabla_i w_j + \frac{1}{2}\partial_t a_{ij}\right), \qquad (3.32)
\end{aligned}$$

where the alternative expression (3.32) is obtained from (3.31) using the continuity equation $\mathcal{C} = 0$. Here we introduced

$$\Pi^{\mathrm{G}ij} = \varrho\frac{\left(v^i - w^i\right)\left(v^j - w^j\right)}{\Omega^2} + pa^{ij} - \Sigma^{\mathrm{G}ij}, \tag{3.33}$$

the components of the Galilean energy–momentum tensor, following [1]. They are expressed in terms of the fluid velocity, measured in an inertial-like frame, *i.e.* $\mathbf{v}-\mathbf{w}$, and transform under Galilean diffeomorphisms (2.9), (2.10) as a genuine rank-two d-dimensional tensor on $\mathscr{S}$ (one uses (2.11), (2.12), (2.13), (2.15), and (3.26)):

$$\Pi^{\mathrm{G}ij\prime} = J^i_k J^j_l \Pi^{\mathrm{G}kl}. \tag{3.34}$$

Equation $\mathcal{E}=0$ is the Galilean energy conservation equation for a viscous fluid in motion on arbitrary, time-dependent d-dimensional space $\mathscr{S}$, and observed from an arbitrary frame (moving at velocity $-\mathbf{w}(t,\mathbf{x})$ with respect to a local inertial frame). In a short while, we will recast this equation in a suitable form for recognizing the underlying phenomena. Notice that both friction and thermal conduction occur, driven by the viscous stress tensor $\mathbf{\Sigma}^{\mathrm{G}}$ and the heat current $\boldsymbol{Q}^{\mathrm{G}}$. As opposed to the energy-conservation equation at hand, the continuity (mass-conservation) equation depends neither on the motion of the observer ($\mathbf{w}$) nor on the friction properties of the fluid. This is expected because energy is frame-dependent while mass it is not.

One can check that under Galilean diffeomorphisms (2.9), (2.10):

$$\mathcal{C}' = \mathcal{C}, \quad \mathcal{E}' = \mathcal{E}. \tag{3.35}$$

In order to show this, it is convenient to recognize some well-behaved blocks in the expressions at hand, based on the quoted transformation rules. We have gathered this information in App. A.1, Eqs. (A.16)–(A.19). For (3.35), we also need (3.25), (3.26).

Euler equation

Following the same pattern, we can process the large-c behaviour of the relativistic momentum-conservation equations. Along with (3.20) we find:

$$\mathcal{M}_i = \frac{1}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}\varrho\frac{v_i-w_i}{\Omega}\right) + \frac{1}{\Omega}\nabla_j\left(\varrho w^j\left(\frac{v_i-w_i}{\Omega}\right)\right) + \frac{\varrho}{\Omega}\left(\frac{v^j-w^j}{\Omega}\right)\nabla_i w_j + \nabla_j \Pi^{\mathrm{G}\,j}_{\ \ i} \tag{3.36}$$

with $\Pi^{\mathrm{G}\,j}_{\ \ i}$ as in (3.33). The equation $\mathcal{M}_i=0$ is the ultimate generalization of the standard Euler equation, displayed *e.g.* in Ref. [1]. It is remarkably simple. The second and third terms in (3.36) contribute to inertial forces (Coriolis, centrifugal etc.), and are usually absent in Euclidean space with inertial frames. Together with the first term, they provide the components of a one-form on $\mathscr{S}$ transforming as $\frac{\mathbf{v}-\mathbf{w}}{\Omega}$ (see (A.21), (A.22)). This is also how $\mathcal{M}_i$ behave under Galilean diffeomorphisms (2.9), (2.10):

$$\mathcal{M}'_i = J^{-1\,l}_{\ \ i}\mathcal{M}_l. \tag{3.37}$$

The Euler equation (3.36) contains the *acceleration* $\boldsymbol{\gamma}^{\mathrm{G}} = \gamma^{\mathrm{G}}{}_i \mathrm{d}x^i$ of the Galilean fluid. This is defined covariantly as

$$a_i = \gamma^{\mathrm{G}}{}_i + \mathrm{O}\left({}^1\!/c^2\right) \tag{3.38}$$

with a_i the spatial components of the relativistic fluid acceleration as in (3.6). We find:

$$\Omega^2 \gamma^{\mathrm{G}}{}_i = \Omega \frac{\mathrm{d}^{v_i}\!/\Omega}{\mathrm{d}t} - \Omega \partial_t {}^{w_i}\!/\Omega - \frac{1}{2}\partial_i \mathbf{w}^2 - v^j \left(\partial_j w_i - \partial_i w_j\right) \tag{3.39}$$

with ${}^{\mathrm{d}}\!/\mathrm{d}t$ defined in (3.29). In this expression, $\gamma^{\mathrm{G}}{}_i$ appear as the components of the acceleration in the local inertial frame and $\frac{\mathrm{d}^{v_i}/\Omega}{\Omega \mathrm{d}t}$ are the components of the effectively measured acceleration in the coordinate frame at hand. In the right hand side, the second term is the dragging acceleration, the third accounts for the centrifugal acceleration, and the last is Coriolis contribution. We can alternatively write (3.39) as

$$\gamma^{\mathrm{G}}{}_i = \frac{\mathrm{d}^{(v_i - w_i)}\!/\Omega}{\Omega \mathrm{d}t} - \frac{1}{2}\partial_i \frac{\mathbf{w}^2}{\Omega^2} + \frac{v^j}{\Omega} \nabla_i \frac{w_j}{\Omega} = \frac{\mathrm{D}^{(v_i - w_i)}\!/\Omega}{\Omega \mathrm{d}t}, \tag{3.40}$$

where we used the *Galilean covariant time-derivative* (A.25) in the second equality.

By construction, the $\gamma^{\mathrm{G}}{}_i$s transform as components of a genuine d-dimensional form and $\gamma^{\mathrm{G}i} = a^{ij}\gamma^{\mathrm{G}}{}_j$ as a vector, under Galilean diffeomorphisms (2.9), (2.10):

$$\gamma^{\mathrm{G}\prime}_{\ i} = J^{-1l}{}_i \gamma^{\mathrm{G}}{}_l. \tag{3.41}$$

One can also check explicitly the covariance of (3.39) using (A.22). Using $\gamma^{\mathrm{G}}{}_i$ in (3.39) and the expression (3.33) for the Galilean energy–momentum tensor, we can recast $\mathcal{M}_i$ in (3.36) *à la* Euler:

$$\mathcal{M}_i = \varrho \gamma^{\mathrm{G}}{}_i + \partial_i p - \nabla_j \Sigma^{\mathrm{G}\,j}{}_i. \tag{3.42}$$

Energy and entropy

The momentum equation $\mathcal{M}_i = 0$ together with continuity equation $\mathcal{C} = 0$ can also be used in order to provide a sharper expression for $\mathcal{E}$ given in (3.31), and leading to:

$$\frac{1}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}\varrho\left(e + \frac{\mathbf{v}^2 - \mathbf{w}^2}{2\Omega^2}\right)\right) = -\nabla_i \Pi^{\mathrm{G}i} - \frac{1}{2\Omega}\Pi^{\mathrm{G}ij}\partial_t a_{ij} + \varrho\frac{v_j - w_j}{\Omega^2}\partial_t \frac{w^j}{\Omega}. \tag{3.43}$$

In this equation, $\varrho\left(e + \frac{\mathbf{v}^2 - \mathbf{w}^2}{2\Omega^2}\right)$ is the total energy density of the fluid in the natural, non-inertial frame. The energy density has three contributions: $e\varrho$ as internal energy, the kinetic energy ${}^{\varrho \mathbf{v}^2}\!/2\Omega^2$, and the potential energy of inertial forces ${}^{-\varrho \mathbf{w}^2}\!/2\Omega^2$ (see (2.6) for the free particle paradigm). Furthermore

$$\Pi^{\mathrm{G}i} = \varrho\frac{v^i}{\Omega}\left(h + \frac{\mathbf{v}^2 - \mathbf{w}^2}{2\Omega^2}\right) + Q^{\mathrm{G}i} - \frac{v^j}{\Omega}\Sigma^{\mathrm{G}\,i}_{\ \ j} \tag{3.44}$$

appears as the *Galilean energy flux*. It receives contributions from the enthalpy, the kinetic and inertial-potential energies, as well as from dissipative processes: thermal conduction and friction, with the corresponding heat current $\boldsymbol{Q}^{\mathrm{G}}$ and viscous stress current ${}^{-\mathbf{v}\cdot\boldsymbol{\Sigma}^{\mathrm{G}}}/_{\Omega}$. The general energy conservation equation $\mathcal{E} = 0$ has now a simple interpretation: the time variation of energy in a local domain is due to the energy flux through the frontier plus the external work due to the time dependence of a_{ij} and w^i (as for the free particle (2.7)).

Dissipative processes create entropy. One can readily determine the variation of the latter by recasting the energy variation in a manner slightly different than (3.43). For that we compute $\mathcal{E} - \frac{v^i - w^i}{\Omega}\mathcal{M}_i$ with (3.31), (3.40), (3.42). We find, using continuity and (3.30):

$$\mathcal{E} - \frac{v^i - w^i}{\Omega}\mathcal{M}_i = \frac{\varrho}{\Omega}\frac{\mathrm{d}e}{\mathrm{d}t} + p\theta^{\mathrm{G}} + \nabla_i Q^{\mathrm{G}i} - \frac{1}{\Omega}\Sigma^{\mathrm{G}ij}\left(\nabla_i v_j + \frac{1}{2}\partial_t a_{ij}\right). \tag{3.45}$$

In this expression, we can trade the energy per mass e, for the entropy per mass s, obeying

$$\mathrm{d}e = T\mathrm{d}s - p\mathrm{d}v = T\mathrm{d}s + \frac{p}{\varrho^2}\mathrm{d}\varrho, \tag{3.46}$$

where $v = {}^1/_{\varrho}$. Substituting (3.46) in (3.45), and trading ${}^{\mathrm{d}\varrho}/_{\mathrm{d}t}$ for $-\Omega\varrho\theta^{\mathrm{G}}$ (continuity), we obtain finally, owing to $\mathcal{E} = \mathcal{M}_i = 0$:

$$\frac{\varrho T}{\Omega}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{\Omega}\Sigma^{\mathrm{G}ij}\left(\nabla_i v_j + \frac{1}{2}\partial_t a_{ij}\right) - \nabla_i Q^{\mathrm{G}i}. \tag{3.47}$$

The entropy is not conserved as a consequence of friction and heat conduction, which encode dissipative processes. The latter are globally captured in a *generalized dissipation function*

$$\psi = \frac{1}{\Omega}\Sigma^{\mathrm{G}ij}\left(\nabla_i v_j + \frac{1}{2}\partial_t a_{ij}\right) - \nabla_i Q^{\mathrm{G}i}, \tag{3.48}$$

appearing both in energy and entropy equations (3.45), (3.47). Observe that ψ depends explicitly on Christoffel symbols as well as on the time variation of the metric. Hence time dependence and inertial forces contribute the dissipation phenomena.[16]

[16] The effect of inertial forces on dissipation has been recently studied by simulation of flows on curved static films without heat current (*i.e.* $d = 2$, $\Omega = 1$, $\mathbf{w} = 0$, $\partial_t a_{ij} = 0$, $\boldsymbol{Q}^{\mathrm{G}} = 0$) [8]. One might consider performing similar simulations or experiments for probing the more general sources of dissipation present in (3.48).

First-order Galilean hydrodynamics and incompressible fluids

The viscous stress tensor $\mathbf{\Sigma}^{\mathrm{G}}$ and the heat current $\boldsymbol{Q}^{\mathrm{G}}$ are constructed phenomenologically as velocity and temperature derivative expansions. Since these objects transform tensorially under Galilean diffeomorphisms (see (3.25), (3.26)), they must be expressed in terms of tensorial derivative quantities.

At first order, we have θ^{G} defined in (3.30), which is an invariant, and

$$\frac{1}{\Omega}\left(\nabla_{(k}v_{l)}+\frac{1}{2}\partial_t a_{kl}\right), \tag{3.49}$$

which is a rank-two symmetric tensor (see (A.19)). We can therefore set

$$\Sigma^{\mathrm{G}}_{(1)ij}=2\eta^{\mathrm{G}}\xi^{\mathrm{G}}{}_{ij}+\zeta^{\mathrm{G}}a_{ij}\theta^{\mathrm{G}}, \tag{3.50}$$

$$Q^{\mathrm{G}}_{(1)i}=-\kappa^{\mathrm{G}}\partial_i T. \tag{3.51}$$

The transport coefficients are as usual the shear viscosity η^{G}, coupled to the *Galilean shear*,

$$\xi^{\mathrm{G}}{}_{ij}=\frac{1}{\Omega}\left(\nabla_{(i}v_{j)}+\frac{1}{2}\partial_t a_{ij}\right)-\frac{1}{d}a_{ij}\theta^{\mathrm{G}}, \tag{3.52}$$

which receives also contributions from the derivative of the metric; the bulk viscosity ζ^{G}, coupled to the Galilean expansion, and the thermal conductivity κ^{G} coupled to the temperature gradient.

Using the definitions of relativistic expansion and shear (3.6), (3.7), we can find their behaviour at large c in the Zermelo background:

$$\sigma_{ij}=\xi^{\mathrm{G}}{}_{ij}+\mathrm{O}\left(1/c^2\right), \tag{3.53}$$

$$\Theta=\theta^{\mathrm{G}}+\mathrm{O}\left(1/c^2\right). \tag{3.54}$$

For completeness we also display the leading behaviour of the vorticity (3.8), even though it plays no rôle in first-order hydrodynamics:

$$\omega_{ij}=\frac{1}{\Omega}\left(\partial_{[i}(v-w)_{j]}\right)+\mathrm{O}\left(1/c^2\right). \tag{3.55}$$

Since furthermore the transverse projector (3.9) is $h_{ij}=a_{ij}+\mathrm{O}\left(1/c^2\right)$, using (3.4) and (3.5) together with (3.11) and (3.38), we find indeed (3.50) and (3.51) (by definition $Q^{\mathrm{G}}{}_i=q_i$). It is important to stress at this point that transport coefficients are determined as modes of microscopic correlation functions, and are therefore sensitive to the velocity of light. In writing (3.11), we have assumed the following large-c behaviour:

$$\eta=\eta^{\mathrm{G}}+\mathrm{O}\left(1/c^2\right),\quad \zeta=\zeta^{\mathrm{G}}+\mathrm{O}\left(1/c^2\right),\quad \kappa=\kappa^{\mathrm{G}}+\mathrm{O}\left(1/c^2\right). \tag{3.56}$$

The case $d=2$ is peculiar because $\Sigma^{\mathrm{G}}_{(1)ij}$ admits an extra term:

$$\zeta^{\mathrm{G}}_{\mathrm{H}}\eta_{k(i}\xi^{\mathrm{G}}{}_{j)l}a^{kl} = \frac{\zeta^{\mathrm{G}}_{\mathrm{H}}}{2\Omega}\left(\eta_{k(i}\nabla_{j)}v^k + \eta_{k(i}a_{j)l}\left(\nabla^k v^l - \frac{\partial_t\sqrt{a}\,a^{kl}}{\sqrt{a}} - a^{kl}\nabla_m v^m\right)\right) \tag{3.57}$$

with $\eta_{kl} = \sqrt{a}\,\epsilon_{kl}$. This is indeed (up to a global sign) the infinite-c limit of the relativistic Hall-viscosity contribution in three spacetime dimensions given in (3.10), assuming again $\zeta_{\mathrm{H}} = \zeta^{\mathrm{G}}_{\mathrm{H}} + \mathrm{O}\left(1/c^2\right)$.

We can now combine the first-derivative contribution (3.50) of the viscous stress tensor with expression (3.42) for $\mathcal{M}_i$ in order to obtain the momentum conservation equation $\mathcal{M}_i = 0$ of first-order Galilean hydrodynamics. We obtain

$$\varrho\gamma^{\mathrm{G}}{}_i + \partial_i p - \frac{\eta^{\mathrm{G}}}{\Omega}\left(\Delta v_i + r_i^{\ j}v_j + a_{ik}a^{jl}\partial_t\gamma^k_{jl}\right) - \left(\zeta^{\mathrm{G}} + \frac{d-2}{d}\eta^{\mathrm{G}}\right)\partial_i\theta^{\mathrm{G}} = 0, \tag{3.58}$$

where $\Delta = \nabla^i\nabla_i$ is the Laplacian operator in d dimensions and r_{ij} the Ricci tensor of the d-dimensional Levi–Civita connection γ^k_{ij}. Similarly, substituting (3.50), (3.51) and (3.52) in (3.47), we find the entropy equation in first-order hydrodynamics on general backgrounds:

$$\frac{\varrho T}{\Omega}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{2\eta^{\mathrm{G}}}{\Omega^2}\left(\left(\nabla^i v^j\right)\left(\nabla_i v_j\right) + \left(\nabla^i v^j\right)\partial_t a_{ij} - \frac{1}{4}\left(\partial_t a^{ij}\right)\left(\partial_t a_{ij}\right)\right) + \left(\zeta^{\mathrm{G}} - \frac{2\eta^{\mathrm{G}}}{d}\right)\left(\theta^{\mathrm{G}}\right)^2 + \kappa^{\mathrm{G}}\Delta T, \tag{3.59}$$

where we assumed κ^{G} constant (otherwise the last term would read $\nabla^i(\kappa^{\mathrm{G}}\nabla_i T)$).

A special class of Galilean fluids deserves further analysis. These are the *incompressible fluids* for which $\varrho(t,\mathbf{x})$ obeys

$$\frac{\mathrm{d}\varrho(t,\mathbf{x})}{\mathrm{d}t} = 0 \tag{3.60}$$

with $\mathrm{d}/\mathrm{d}t$ the material derivative defined in (3.29). Using the expressions (3.27) and (3.28), we recast the incompressibility requirement as the vanishing of the effective fluid expansion:

$$\theta^{\mathrm{G}} = 0. \tag{3.61}$$

In this case, the bulk viscosity drops from the stress tensor (3.50) and the Galilean shear (3.52) simplifies. The first-order hydrodynamics momentum equation for an incompressible fluid thus reads:

$$\varrho\frac{\mathrm{d}v_i/\Omega}{\Omega\,\mathrm{d}t} = \varrho\frac{\mathrm{d}w_i/\Omega}{\Omega\,\mathrm{d}t} + \frac{\varrho}{2}\partial_i\frac{\mathbf{w}^2}{\Omega^2} - \varrho\frac{v^j}{\Omega}\nabla_i\frac{w_j}{\Omega} - \partial_i p + \frac{\eta^{\mathrm{G}}}{\Omega}\left(\Delta v_i + r_i^{\ j}v_j + a_{ik}a^{jl}\partial_t\gamma^k_{jl}\right). \tag{3.62}$$

We immediately recognize in this expression the generalized *covariant Navier–Stokes equation*, valid for incompressible fluids on any space $\mathscr{S}$, observed from an arbitrary frame. The first three terms in the right-hand side are contributions of frame inertial forces, the fourth is the pressure force, and next come the friction forces at first-order derivative. For Euclidean space with $\Omega = 1$ and $\mathbf{w} = 0$ we recover the textbook form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\mathbf{grad}\, p}{\varrho} + \frac{\eta^{\mathrm{G}}}{\varrho}\Delta\mathbf{v}. \tag{3.63}$$

3.3 Carrollian fluid dynamics from Randers–Papapetrou background

Preliminary remarks

As Carrollian particles, Carrollian fluids have no motion. From a relativistic perspective this is an observer-dependent statement, since boosts can turn on velocity. In the limit of vanishing velocity of light, however, these transformations are no longer permitted. Hence, being at rest becomes a genuinely intrinsic feature.

The fluid velocity must be set to zero faster than c in order to avoid blow-ups in the energy–momentum conservation. The appropriate scaling, ensuring a non-trivial kinematic contribution is

$$v^i = c^2\Omega\beta^i + \mathrm{O}\left(c^4\right), \tag{3.64}$$

where $v^i = {}^{u^i}\!/\!{}_{\gamma}$. This leaves the Carrollian fluid with a kinematic variable $\boldsymbol{\beta} = \beta^i\partial_i$ of inverse-velocity dimension, as in (2.34) for the one-body Carrollian dynamics studied in Sec. 2.2 – reason why we keep the same symbol. In order to reach covariant Carrollian fluid equations by expanding the relativistic fluid equations at small c, we need to define the β^is in such a way that they transform as components of a genuine Carrollian vector under (2.27), (2.36) already at finite c. This is achieved by setting

$$v^i = \frac{c^2\Omega\beta^i}{1 + c^2\beta^j b_j} \Leftrightarrow \beta^i = \frac{v^i}{c^2\Omega\left(1 - \frac{v^j b_j}{\Omega}\right)}, \tag{3.65}$$

from which one checks that[17]

$$\beta^{i\prime} = J^i_j\beta^j. \tag{3.66}$$

The full fluid congruence reads then:

$$\begin{cases} u^0 = \gamma c = \dfrac{c}{\Omega}\dfrac{1 + c^2\boldsymbol{\beta}\cdot\boldsymbol{b}}{\sqrt{1 - c^2\boldsymbol{\beta}^2}} = \dfrac{c}{\Omega} + \mathrm{O}\left(c^3\right), \quad u_0 = -\dfrac{c\Omega}{\sqrt{1 - c^2\boldsymbol{\beta}^2}} = -c\Omega + \mathrm{O}\left(c^3\right), \\ u^i = \gamma v^i = \dfrac{c^2\beta^i}{\sqrt{1 - c^2\boldsymbol{\beta}^2}} = c^2\beta^i + \mathrm{O}\left(c^4\right), \quad u_i = \dfrac{c^2\left(b_i + \beta_i\right)}{\sqrt{1 - c^2\boldsymbol{\beta}^2}} = c^2\left(b_i + \beta_i\right) + \mathrm{O}\left(c^4\right), \end{cases} \tag{3.67}$$

[17] This is easily proven by observing that $\beta_i + b_i = -\frac{\Omega u_i}{c u_0}$. We define as usual $b^i = a^{ij}b_j$, $\beta_i = a_{ij}\beta^j$, $v_i = a_{ij}v^j$, $\boldsymbol{b}^2 = b_i b^i$, $\boldsymbol{\beta}^2 = \beta_i\beta^i$ and $\boldsymbol{b}\cdot\boldsymbol{\beta} = b_i\beta^i$.

where the Lorentz factor has been obtained by imposing the usual normalization $\|\mathrm{u}\|^2 = -c^2$:

$$\gamma = \frac{1 + c^2 \boldsymbol{\beta} \cdot \boldsymbol{b}}{\Omega \sqrt{1 - c^2 \boldsymbol{\beta}^2}} = \frac{1}{\Omega} \left(1 + \frac{c^2}{2} \boldsymbol{\beta} \cdot (\boldsymbol{\beta} + 2\boldsymbol{b}) + \mathrm{O}\left(c^4\right) \right). \tag{3.68}$$

In the relativistic regime, *i.e.* before taking the zero-c limit, in the Randers–Papapetrou background (2.35) the perfect part of the energy–momentum tensor reads then:

$$\begin{cases} T_{\text{perf}}{}^0{}_0 = -\varepsilon - c^2 (\varepsilon + p) \beta^k (b_k + \beta_k) + \mathrm{O}\left(c^4\right), \\ c\Omega T_{\text{perf}}{}^0{}_i = c^2 (\varepsilon + p) (b_i + \beta_i) + \mathrm{O}\left(c^4\right), \\ \dfrac{c}{\Omega} T_{\text{perf}}{}^j{}_0 = -c^2 (\varepsilon + p) \beta^j + \mathrm{O}\left(c^4\right), \\ T_{\text{perf}}{}^j{}_i = p \delta^j_i + c^2 (\varepsilon + p) \beta^j (b_i + \beta_i) + \mathrm{O}\left(c^4\right). \end{cases} \tag{3.69}$$

The non-perfect part is encoded in Eqs. (3.2), (3.12) and (3.14). Notice, on the one hand, that for vanishing β^i, these expressions are exact at finite c: most of the terms of order c^2 vanish as do all non-displayed higher-order contributions in c^2; on the other hand, for vanishing c, one recovers the perfect energy–momentum of a fluid at rest due to the simultaneous vanishing of v^i as a consequence of (3.64).

The eventual absence of motion, macroscopic or microscopic, and the shrinking of the light-cone raise many fundamental questions regarding the origin of pressure, temperature, thermalization, entropy etc. One may wonder in particular what causes viscosity and thermal conduction, what replaces the temperature derivative expansion of q_i, what justifies its behaviour (3.12). Even the propagation of a signal such as sound, if possible, should be reconsidered. It is tempting to claim that all this physics will be mostly of geometric nature rather than many-body statistics, because as we will see the only kinematic Carrollian-fluid variable $\boldsymbol{\beta}$ enters partly the dynamics.

We have no definite answers to all these questions though, and will not discuss these important issues here, which might possibly require to elaborate on space-filling branes as microscopic objects – see Sec. 2.2. Our approach will be kinematical, aiming at writing the fundamental equations, covariant under Carrollian diffeomorphisms (2.27), (2.36), starting from the relativistic equations (3.1). Alternative paths may exist, allowing to built some Carrollian dynamics without using the zero-c limit of a relativistic fluid.[18]

[18] In this spirit, one should quote the attempt made in [32], inspired by the membrane paradigm – admittedly suited for reaching Galilean rather than ultra-relativistic fluid dynamics, as well as Ref. [33], mostly focused on the structure of the energy–momentum tensor of perfect fluids (3.69), which also touches on Carrollian symmetry.

The structure of the equations

The relativistic equations (conservation of the energy–momentum tensor) should now be presented as

$$\nabla_\mu T^\mu{}_0 = 0, \quad \nabla_\mu T^{\mu i} = 0. \tag{3.70}$$

Under Carrollian diffeomorphisms (2.27), (2.36), the divergence of the energy–momentum tensor transforms as:

$$\nabla'_\mu T'^\mu{}_0 = \frac{1}{J}\nabla_\mu T^\mu{}_0, \quad \nabla'_\mu T'^{\mu i} = J^i_l \nabla_\mu T^{\mu l}. \tag{3.71}$$

In analogy with the Galilean case (3.17), the two sets of equations (3.70) have separately a d-dimensional covariant transformation. This is part of the agenda for the Carrollian dynamics.

Equations (3.70) are relativistic. Using the general energy–momentum tensor (3.2) with perfect part (3.69) and (3.12) as stress tensor, we find generally:

$$\frac{c}{\Omega}\nabla_\mu T^\mu{}_0 = \frac{1}{c^2}\mathcal{F} + \mathcal{E} + \mathrm{O}\left(c^2\right), \tag{3.72}$$

$$\nabla_\mu T^{\mu i} = \frac{1}{c^2}\mathcal{H}^i + \mathcal{G}^i + \mathrm{O}\left(c^2\right). \tag{3.73}$$

For zero β^i, these expressions are *exact* with extra terms of order c^2 only, and requiring they vanish leads to the $d+1$ fully relativistic fluid equations. With $\beta^i \neq 0$, (3.72) and (3.73) are genuinely infinite series. Thanks to the validity of (3.66) at finite c, Carrollian diffeomorphisms do not mix the different orders of these series, making each term Carrollian-covariant. Here, we are interested in the zero-c limit, and in this case Eqs. (3.72) and (3.73) split into $2+2d$ distinct equations:

- energy conservation $\mathcal{E} = 0$;
- momentum conservation $\mathcal{G}^i = 0$;
- constraint equations $\mathcal{F} = 0$ and $\mathcal{H}^i = 0$.

All of these are covariant under Carrollian diffeomorphisms (2.27), (2.36).

The Carrollian fluid, obtained as Carrollian limit of a relativistic fluid in the appropriate (Randers–Papapetrou) background, is described in terms of the d β^is, and the two variables p and ε.[19] The latter are related through an equation of state and the energy-conservation equation $\mathcal{E} = 0$. As we will see soon, the other $2d+1$ equations are setting consistency constraints among the $2d$ components of the heat currents ($Q^{\mathrm{C}}{}_i$ and π_i), the $d(d+1)$ components of the viscous stress tensors ($\Sigma^{\mathrm{C}}{}_{ij}$ and Ξ_{ij}), the inverse-velocity components β^i and the geometric

[19] The proper energy density cannot be split in mass density and energy per mass, because the limit at hand is ultra-relativistic. Observe also that $\boldsymbol{b}$ is not a fluid variable but a Carrollian-frame parameter as was $\mathbf{w}$ in the Galilean case. The fluid kinematical variable is $\boldsymbol{\beta}$, playing the rôle $\frac{\mathbf{v}-\mathbf{w}}{\Omega}$ had in the usual non-relativistic case.

environment. Geometry is therefore expected to interfere more actively in the dynamics of Carrollian fluids, than it did for Galilean hydrodynamics. Some of the aforementioned constraints are possibly rooted to more fundamental microscopic/geometric properties, yet to be unravelled. Their usage will be illustrated in Sec. 4.2.

The dissipative tensors in Randers–Papapetrou background

For a relativistic fluid in the Randers–Papapetrou background (2.35), using the velocity field in (3.64) and (3.67) and the components q^i, the transversality conditions (3.3) lead to

$$q^0 = \frac{c}{\Omega}\left(b_i + \beta_i\right) q^i, \quad q_0 = -c\Omega\beta_i q^i, \quad q_i = \left(a_{ij} + c^2 b_i\beta_j\right) q^j. \tag{3.74}$$

Similarly, the components of the viscous stress tensor are obtained from the τ^{ij}s. For example:

$$\begin{gathered}\tau^{00} = \frac{c^2}{\Omega^2}\left(b_k + \beta_k\right)\left(b_l + \beta_l\right)\tau^{kl}, \quad \tau^{0i} = \frac{c}{\Omega}\left(b_i + \beta_i\right)\tau^{ik}, \quad \tau_{00} = c^2\Omega^2\beta_k\beta_l\tau^{kl}, \\ \tau_{0i} = -c\Omega\beta_j\left(a_{ik} + c^2 b_i\beta_k\right)\tau^{jk}, \quad \tau_{ij} = \left(a_{ik} + c^2 b_i\beta_k\right)\left(a_{jl} + c^2 b_j\beta_l\right)\tau^{kl}, \ldots\end{gathered} \tag{3.75}$$

Under Carrollian diffeomorphisms (2.27), (2.36), we obtain the following transformation rules

$$q'^i = q^j J^i_j, \quad \tau'^{ij} = \tau^{kl} J^i_k J^j_l. \tag{3.76}$$

This suggests to use q^i as components for the Carrolian d-dimensional heat current decomposed as $Q^{\mathrm{C}i} + c^2\pi^i$ (see (3.14)), and τ^{ij} for the Carrolian d-dimensional viscous stress tensors $\Sigma^{\mathrm{C}ij}$ and Ξ^{ij} defined in (3.12). We introduce as usual

$$Q^{\mathrm{C}}{}_i = a_{ij}Q^{\mathrm{C}j}, \quad \Sigma^{\mathrm{C}}{}_i{}^j = a_{ik}\Sigma^{\mathrm{C}kj}, \quad \Sigma^{\mathrm{C}}{}_{ij} = a_{jk}\Sigma^{\mathrm{C}}{}_i{}^k, \tag{3.77}$$

$$\pi_i = a_{ij}\pi^j, \quad \Xi_i^{\ j} = a_{ik}\Xi^{kj}, \quad \Xi_{ij} = a_{jk}\Xi_i^{\ k}. \tag{3.78}$$

Using the generic transformations (3.76) under Carrollian diffeomorphisms (2.27), (2.28), we find that the above quantities transform as they should, for being eligible as d-dimensional tensors:

$$Q^{\mathrm{C}\prime}_i = Q^{\mathrm{C}}{}_j J^{-1j}{}_i, \quad Q^{\mathrm{C}\prime i} = J^i_j Q^{\mathrm{C}j}, \tag{3.79}$$

$$\Sigma^{\mathrm{C}\prime}{}_{ij} = J^{-1k}{}_i J^{-1l}{}_j \Sigma^{\mathrm{C}}{}_{kl}, \quad \Sigma^{\mathrm{C}\prime}{}_i{}^j = J^{-1k}{}_i \Sigma^{\mathrm{C}}{}_k{}^l J^j_l, \quad \Sigma^{\mathrm{C}\prime ij} = \Sigma^{\mathrm{C}kl} J^i_k J^j_l, \tag{3.80}$$

and similarly for π_i and Ξ_{jk}.

Scalar equations

The computation of the spacetime divergence in (3.72) is straightforward and leads to the following:

$$\begin{aligned}
\mathcal{E} &= -\left(\frac{1}{\Omega}\partial_t + \frac{d+1}{d}\theta^{\mathrm{C}}\right)\left(\varepsilon + 2\beta_i Q^{\mathrm{C}i} - \beta_i\beta_j\Sigma^{\mathrm{C}ij}\right) + \frac{1}{d}\theta^{\mathrm{C}}\left(\Xi^i{}_i - \beta_i\beta_j\Sigma^{\mathrm{C}ij} + \varepsilon - dp\right) \\
&\quad - \left(\hat{\nabla}_i + 2\varphi_i\right)\left(Q^{\mathrm{C}i} - \beta_j\Sigma^{\mathrm{C}ij}\right) - \left(2Q^{\mathrm{C}i}\beta^j - \Xi^{ij}\right)\xi^{\mathrm{C}}{}_{ij}, && (3.81) \\
\mathcal{F} &= \Sigma^{\mathrm{C}ij}\xi^{\mathrm{C}}_{ij} + \frac{1}{d}\Sigma^{\mathrm{C}i}{}_i\theta^{\mathrm{C}}, && (3.82)
\end{aligned}$$

where we have introduced a new covariant derivative $\hat{\nabla}_i$, as defined in the appendix, Eqs. (A.45)–(A.53). It is based on a new torsionless and metric-compatible connection (see (A.61)–(A.65)) dubbed *Levi–Civita–Carroll*, which plays for Carrollian geometry the rôle of ordinary Levi–Civita connection for ordinary geometry, *i.e.* it allows to built derivatives covariant under Carrollian diffeomorphisms (2.27), (2.28). Some further properties regarding the curvature of this connection are displayed in (A.66)–(A.78). A deeper investigation of this structure is out of place here. In (3.81) and (3.82) we have moreover defined

$$\begin{aligned}
\varphi_i &= \frac{1}{\Omega}\left(\partial_t b_i + \partial_i\Omega\right), && (3.83) \\
\theta^{\mathrm{C}} &= \frac{1}{\Omega}\partial_t \ln\sqrt{a}. && (3.84)
\end{aligned}$$

These expressions describe a form and a scalar (see (A.42) and (A.41) for their transformation rules under Carrollian diffeomorphisms). They play the rôle of *inertial acceleration* and *expansion* for the Carrollian fluid. These are both geometrical and the qualifier "inertial" refers to the frame (*i.e.* b_i and Ω) origin. We shall see in a moment that there is an extra contribution to the Carrollian fluid acceleration due to the kinematical observable β_i, but none for the expansion (see (3.95), (3.96)). As already stated and readily seen by its equations, most of the fluid properties are of geometrical nature. One similarly defines an *inertial vorticity two-form* with components

$$\varpi_{ij} = \partial_{[i}b_{j]} + b_{[i}\varphi_{j]}, \tag{3.85}$$

and the traceless and symmetric *shear tensor*

$$\xi^{\mathrm{C}}{}_{ij} = \frac{1}{\Omega}\left(\frac{1}{2}\partial_t a_{ij} - \frac{1}{d}a_{ij}\partial_t \ln\sqrt{a}\right). \tag{3.86}$$

These quantities will be related in a short while to the ordinary relativistic counterparts (see (3.98) and (3.97)). The former receives a fluid kinematical contribution, as opposed to the latter. Eventually, we can elegantly check that

$$\mathcal{E}' = \mathcal{E}, \quad \mathcal{F}' = \mathcal{F} \tag{3.87}$$

(we use for that Eqs. (2.31), (3.79), (3.80), (A.42), (A.43), (A.50)–(A.59)).

Equation $\mathcal{F} = 0$ sets a geometrical constraint on the Carrollian stress tensor $\mathbf{\Sigma}^{\mathrm{C}}$, whereas $\mathcal{E} = 0$ is the energy conservation. Using (3.81), the latter can be recast as follows:

$$\left(\frac{1}{\Omega}\partial_t + \theta^{\mathrm{C}}\right) e_{\mathrm{e}} = -\left(\hat{\nabla}_i + 2\varphi_i\right) \Pi^{\mathrm{C}i} - \Pi^{\mathrm{C}ij}\left(\xi^{\mathrm{C}}{}_{ij} + \frac{1}{d}\theta^{\mathrm{C}} a_{ij}\right), \tag{3.88}$$

and in this form it bares some resemblance with the Galilean homologous equation (3.43). It exhibits three Carrollian tensors, which capture the Carrollian energy exchanges:

$$e_{\mathrm{e}} = \varepsilon + 2\beta_i Q^{\mathrm{C}i} - \beta_i\beta_j\Sigma^{\mathrm{C}ij}, \quad \Pi^{\mathrm{C}i} = Q^{\mathrm{C}i} - \beta_j\Sigma^{\mathrm{C}ij}, \quad \Pi^{\mathrm{C}ij} = Q^{\mathrm{C}i}\beta^j + \beta^i Q^{\mathrm{C}j} + pa^{ij} - \Xi^{ij}. \tag{3.89}$$

The first is a scalar e_{e}, which can be interpreted as an *effective Carrollian energy density* (observe the absence of kinetic energy, expected from the vanishing velocity). Its time variation, including the dilution/contraction effects due to the expansion, is driven by the gradient of a *Carrollian energy flux*, which is the vector $\Pi^{\mathrm{C}i}$, and by the coupling of the shear to a *Carrollian energy–momentum tensor* $\Pi^{\mathrm{C}ij}$.

Vector equations

The vectorial part of the divergence is obtained from (3.73) and has two pieces. The first reads:

$$\begin{aligned}\mathcal{G}_j \;=\;& \left(\hat{\nabla}_i + \varphi_i\right) \Pi^{\mathrm{C}i}{}_j + \varphi_j e_{\mathrm{e}} + 2\Pi^{\mathrm{C}i}\varpi_{ij} + \left(\frac{1}{\Omega}\partial_t + \theta^{\mathrm{C}}\right)\left(\pi_j + \beta_j\left(e_{\mathrm{e}} - 2\beta_i\Pi^{\mathrm{C}i} - \beta_i\beta_k\Sigma^{\mathrm{C}ik}\right)\right) \\ & + \left(\frac{1}{\Omega}\partial_t + \theta^{\mathrm{C}}\right)\left(\beta^k\left(\Pi^{\mathrm{C}}{}_{kj} - \frac{1}{2}\beta_k\Pi^{\mathrm{C}}{}_j - \frac{1}{2}\beta_k\beta^i\Sigma^{\mathrm{C}}{}_{ij}\right)\right).\end{aligned} \tag{3.90}$$

The second is as follows:

$$\mathcal{H}_j = -\left(\hat{\nabla}_i + \varphi_i\right) \Sigma^{\mathrm{C}i}{}_j + \left(\frac{1}{\Omega}\partial_t + \theta^{\mathrm{C}}\right) \Pi^{\mathrm{C}}{}_j. \tag{3.91}$$

Equation $\mathcal{G}_j = 0$ involves ε, p and their temporal and/or spatial derivatives, $\boldsymbol{\beta}$, the heat current $\boldsymbol{Q}^{\mathrm{C}}$, and $\mathbf{\Xi}$, expressed in terms of the effective energy density e_{e}, the Carrollian energy flux and energy–momentum tensor $\mathbf{\Pi}^{\mathrm{C}}$, as well as $\boldsymbol{\pi}$ and $\mathbf{\Sigma}^{\mathrm{C}}$. It is a momentum conservation. Notice also the coupling of the energy flux to the inertial vorticity. Equation $\mathcal{H}_j = 0$ depends neither on ε nor on p. This is an equation for the Carrollian energy flux $\mathbf{\Pi}^{\mathrm{C}}$ and the viscous stress tensor $\mathbf{\Sigma}^{\mathrm{C}}$, of geometrical nature as it involves the metric $\boldsymbol{a}$, the Carrollian "frame velocity" $\boldsymbol{b}$ and the inertial acceleration $\boldsymbol{\varphi}$.

Under Carrollian diffeomorphisms (2.27), (2.28), using the already quoted equations, (2.31), (3.79), (3.80), and (A.42)–(A.60), we obtain:

$$\mathcal{G}'^i = J^i_j \mathcal{G}^j, \quad \mathcal{H}'^i = J^i_j \mathcal{H}^j. \tag{3.92}$$

One should observe at this point that the energy–momentum tensor and energy flux associated with a Carrollian fluid and defined in (3.89) are merely a repackaging of part of the dynamical data. They do not capture all perfect and friction quantities, as it happens for Galilean fluids, Eqs. (3.33) and (3.44). Equation $\mathcal{F} = 0$, as well as the vector equations need indeed more information than the energy–momentum tensor and energy flux. There is pressure, energy density and "velocity", on the one hand, and on the other hand, we find the two heat currents and the two viscous stress tensors. The zero-c limit produces a decoupling in the equations, sustained by the scaling assumption (3.12). This is the reason why $\mathcal{H}_j = 0$ appears as an equation for the dissipative pieces of data only, while the non-dissipative ones mix with the heat currents inside $\mathcal{G}_j = 0$.

Carrollian perfect fluids

We would like to end this chapter with a remark on the case of perfect fluids, namely fluids with vanishing dissipative tensors. For those, the dynamical variables are ε, p and β_i, with $e_\mathrm{e} = \varepsilon$, $\Pi^\mathrm{C}{}_j = 0$ and $\Pi^\mathrm{C}{}_{ij} = p a_{ij}$. In this case, $\mathcal{F} = \mathcal{H}^i = 0$ identically, and

$$\begin{aligned} \mathcal{E} &= -\frac{1}{\Omega}\partial_t \varepsilon - (\varepsilon + p)\theta^\mathrm{C}, & (3.93)\\ \mathcal{G}_j &= (\varepsilon + p)\left(\varphi_j + \gamma^\mathrm{C}{}_j + \beta_j \theta^\mathrm{C}\right) + \frac{\beta_j}{\Omega}\partial_t(\varepsilon + p) + \hat{\partial}_j p. & (3.94)\end{aligned}$$

On the one hand, non-trivial energy exchanges can only result from time-dependence of the metric and pressure gradients. The latter, on the other hand, are bound to non-trivial $\boldsymbol{\beta}$, $\boldsymbol{\gamma}^\mathrm{C}$, $\boldsymbol{b}$ and Ω. Here $\gamma^\mathrm{C}{}_j$ is the kinematical acceleration defined later in (3.99).

For perfect fluids, only $\mathcal{E}$ and $\mathcal{G}_i$ survive in the relativistic divergence of the energy–momentum tensor, Eqs. (3.72) and (3.73). Furthermore, for zero $\boldsymbol{\beta}$ these are actually the only terms, at finite c. Hence, the relativistic equations are not affected by the vanishing-c limit, and coincide with the Carrollian ones: $\mathcal{E} = 0$ and $\mathcal{G}_i = 0$. As a consequence, the Carrollian nature of a fluid at $\boldsymbol{\beta} = 0$ can only emerge through interactions. This is to be opposed to the Galilean situation, since Galilean perfect fluids are definitely different from relativistic perfect fluids, even at rest.

First-order Carrollian hydrodynamics

In order to acquire a better perspective of Carrollian fluid dynamics, we can study the first-order in derivative expansion of its viscous tensors and heat currents. The first-derivative relativistic kinematical tensors as acceleration and expansion (3.6), shear (3.7), and vorticity (3.8), for a fluid with velocity vanishing as (3.64) when $c \to 0$ in Randers–Papapetrou background (2.35) read (the only independent components are the spatial ones):

$$a_i = \frac{c^2}{\Omega}\left(\partial_t\left(b_i+\beta_i\right)+\partial_i\Omega\right)+\mathrm{O}\left(c^4\right)=c^2\left(\varphi_i+\gamma^{\mathrm{C}}{}_i\right)+\mathrm{O}\left(c^4\right), \tag{3.95}$$

$$\Theta = \frac{1}{\Omega}\partial_t \ln\sqrt{a}+\mathrm{O}\left(c^2\right)=\theta^{\mathrm{C}}+\mathrm{O}\left(c^2\right), \tag{3.96}$$

$$\sigma_{ij} = \frac{1}{\Omega}\left(\frac{1}{2}\partial_t a_{ij}-\frac{1}{d}a_{ij}\partial_t\ln\sqrt{a}\right)+\mathrm{O}\left(c^2\right)=\xi^{\mathrm{C}}{}_{ij}+\mathrm{O}\left(c^2\right), \tag{3.97}$$

$$\omega_{ij} = c^2\left(\partial_{[i}b_{j]}+\frac{1}{\Omega}b_{[i}\partial_{j]}\Omega+\frac{1}{\Omega}b_{[i}\partial_t b_{j]}+w_{ij}\right)+\mathrm{O}\left(c^4\right)=c^2\left(\varpi_{ij}+w_{ij}\right)+\mathrm{O}\left(c^4\right). \tag{3.98}$$

We find the corresponding Carrollian expansion θ^{C} and shear $\xi^{\mathrm{C}}{}_{ij}$, as already anticipated in (3.84) and (3.86). These quantities are purely geometric and originate from the time dependence of the d-dimensional spatial metric. Similarly, the relativistic acceleration and vorticity allow to define the already introduced Carrollian, inertial acceleration φ_i and vorticity ϖ_{ij}, as well as the kinematical acceleration $\gamma^{\mathrm{C}}{}_i$ and kinematical vorticity w_{ij} defined as:

$$\gamma^{\mathrm{C}}{}_i = \frac{1}{\Omega}\partial_t\beta_i, \tag{3.99}$$

$$w_{ij} = \hat{\partial}_{[i}\beta_{j]}+\beta_{[i}\varphi_{j]}+\beta_{[i}\gamma^{\mathrm{C}}{}_{j]}. \tag{3.100}$$

Starting from the first-order relativistic viscous tensor (3.4) and heat current (3.5), in order to comply with the behaviours (3.12) and the definition of the Carrollian heat currents (3.14), we must assume that (up to possible higher orders in c^2)

$$\eta = \tilde{\eta}+\frac{\eta^{\mathrm{C}}}{c^2}, \quad \zeta = \tilde{\zeta}+\frac{\zeta^{\mathrm{C}}}{c^2}, \quad \kappa = c^2\tilde{\kappa}++\kappa^{\mathrm{C}}. \tag{3.101}$$

Hence, putting these equations together, we find

$$\Sigma^{\mathrm{C}}_{(1)ij} = 2\eta^{\mathrm{C}}\xi^{\mathrm{C}}{}_{ij}+\zeta^{\mathrm{C}}\theta^{\mathrm{C}}a_{ij}, \tag{3.102}$$

$$\begin{aligned} Q^{\mathrm{C}}_{(1)i} &= -\frac{\kappa^{\mathrm{C}}}{\Omega}\left(\partial_t(b_iT)+\beta_i\partial_tT+\partial_i(\Omega T)\right) \\ &= -\kappa^{\mathrm{C}}\left(\hat{\partial}_iT+T\left(\varphi_i+\gamma^{\mathrm{C}}{}_i\right)\right), \end{aligned} \tag{3.103}$$

and similarly for $\Xi_{(1)ij}$ and $\pi_{(1)i}$. These quantities will include respectively terms like $2\tilde{\eta}\xi^{\mathrm{C}}{}_{ij} + \tilde{\zeta}\theta^{\mathrm{C}}a_{ij}$ and $-\tilde{\kappa}\left(\hat{\partial}_i T + T\left(\varphi_i + \gamma^{\mathrm{C}}{}_i\right)\right)$, plus extra terms coupled to η^{C}, ζ^{C} and κ^{C}, and originating from higher-order contributions in the c^2-expansion of the relativistic shear, acceleration and expansion. Notice that these are absent for vanishing β^i because in this case (3.95)–(3.98) are exact.

All the above expressions are covariant under Carrollian diffeomorphisms (2.27), (2.28) (see formulas (A.40)–(A.43) in appendix). The friction phenomena are geometric and due to time evolution of the background metric a_{ij}. The heat conduction, depends also on a temperature, which has not been defined in Carrollian thermodynamics due to the absence of kinetic theory.

In the two-dimensional case one should take into account the Hall viscosity (3.10) in the relativistic viscous tensor at first order. Assuming again $\zeta_{\mathrm{H}} = \zeta^{\mathrm{C}}_{\mathrm{H}}/c^2 + \tilde{\zeta}_{\mathrm{H}}$, the extra term to be added to $\Sigma^{\mathrm{C}}_{(1)ij}$ in (3.102) reads:

$$\zeta^{\mathrm{C}}_{\mathrm{H}} \sqrt{a}\,\epsilon_{k(i}\xi^{\mathrm{C}}{}_{j)l}a^{kl}, \tag{3.104}$$

and similarly for $\Xi_{(1)ij}$ with transport coefficients $\tilde{\zeta}_{\mathrm{H}}$ and $\zeta^{\mathrm{C}}_{\mathrm{H}}$ as already explained.

The final first-order Carrollian equations are obtained by substituting $\Sigma^{\mathrm{C}}_{(1)ij}$ and $Q^{\mathrm{C}}_{(1)i}$ given in (3.102) and (3.103), and similarly for $\Xi_{(1)ij}$, and $\pi_{(1)i}$, inside the general expressions for $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}_i$ and $\mathcal{H}_i$, Eqs. (3.81), (3.82), (3.90) and (3.91).

Conformal Carrollian fluids

Carrollian fluids are ultra-relativistic and are thus compatible with conformal symmetry. For conformal relativistic fluids the energy–momentum tensor (3.2) is traceless and this requires

$$\varepsilon = dp, \quad \tau^{\mu}{}_{\mu} = 0. \tag{3.105}$$

In the Carrollian limit, the latter reads:

$$\Xi^{i}{}_{i} = \beta_i\beta_j\Sigma^{\mathrm{C}ij}, \quad \Sigma^{\mathrm{C}i}{}_{i} = 0. \tag{3.106}$$

In particular, we find $e_{\mathrm{e}} = \Pi^{\mathrm{C}i}{}_{i}$.

The dynamics of conformal fluids is covariant under Weyl transformations. Those act on the fluid variables as

$$\varepsilon \to \mathcal{B}^{d+1}\varepsilon, \quad \pi_i \to \mathcal{B}^{d}\pi_i, \quad Q^{\mathrm{C}}{}_i \to \mathcal{B}^{d}Q^{\mathrm{C}}{}_i, \quad \Xi_{ij} \to \mathcal{B}^{d-1}\Xi_{ij}, \quad \Sigma^{\mathrm{C}}{}_{ij} \to \mathcal{B}^{d-1}\Sigma^{\mathrm{C}}{}_{ij}, \tag{3.107}$$

where $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$ is an arbitrary function. The elements of the Carrollian geometry behave as follows:

$$a_{ij} \to \frac{1}{\mathcal{B}^2}a_{ij}, \quad b_i \to \frac{1}{\mathcal{B}}b_i, \quad \Omega \to \frac{1}{\mathcal{B}}\Omega, \tag{3.108}$$

and similarly for the kinematical variable β_i, the inertial and kinematical vorticity (3.85) and the shear (3.86):

$$\beta_i \to \frac{1}{\mathcal{B}}\beta_i, \quad \varpi_{ij} \to \frac{1}{\mathcal{B}}\varpi_{ij}, \quad w_{ij} \to \frac{1}{\mathcal{B}}w_{ij}, \quad \xi^{\mathrm{C}}{}_{ij} \to \frac{1}{\mathcal{B}}\xi^{\mathrm{C}}{}_{ij}. \tag{3.109}$$

The Carrollian inertial and kinematical accelerations (3.83) and (3.99), and the Carrollian expansion (3.84) transform as connections:

$$\varphi_i \to \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \gamma^{\mathrm{C}}{}_i \to \gamma^{\mathrm{C}}{}_i - \frac{\beta_i}{\Omega}\partial_t \ln \mathcal{B}, \quad \theta^{\mathrm{C}} \to \mathcal{B}\theta^{\mathrm{C}} - \frac{d}{\Omega}\partial_t \mathcal{B}. \tag{3.110}$$

The first and the latter enable to define Weyl–Carroll covariant derivatives $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_t$, as discussed in App. A.2, Eqs. (A.82)–(A.93). With these derivatives, Carrollian expressions (3.81), (3.82), (3.90) and (3.91) read for a conformal fluid:

$$\mathcal{E} = -\frac{1}{\Omega}\hat{\mathscr{D}}_t e_{\mathrm{e}} - \hat{\mathscr{D}}_i \Pi^{\mathrm{C}i} - \Pi^{\mathrm{C}ij}\xi^{\mathrm{C}}{}_{ij}, \tag{3.111}$$

$$\mathcal{F} = \Sigma^{\mathrm{C}ij}\xi^{\mathrm{C}}{}_{ij}, \tag{3.112}$$

$$\begin{aligned}\mathcal{G}_j = {} & \hat{\mathscr{D}}_i \Pi^{\mathrm{C}i}{}_j + 2\Pi^{\mathrm{C}i}\varpi_{ij} + \left(\frac{1}{\Omega}\hat{\mathscr{D}}_t \delta^i_j + \xi^{\mathrm{C}i}{}_j\right)\left(\pi_i + \beta_i\left(e_{\mathrm{e}} - 2\beta_k \Pi^{\mathrm{C}k} - \beta_k\beta_l\Sigma^{\mathrm{C}kl}\right)\right) \\ & + \left(\frac{1}{\Omega}\hat{\mathscr{D}}_t \delta^i_j + \xi^{\mathrm{C}i}{}_j\right)\left(\beta^k\left(\Pi^{\mathrm{C}}{}_{ki} - \frac{1}{2}\beta_k\Pi^{\mathrm{C}}{}_i - \frac{1}{2}\beta_k\beta^l\Sigma^{\mathrm{C}}{}_{li}\right)\right),\end{aligned} \tag{3.113}$$

$$\mathcal{H}_j = -\hat{\mathscr{D}}_i\Sigma^{\mathrm{C}i}{}_j + \frac{1}{\Omega}\hat{\mathscr{D}}_t\Pi^{\mathrm{C}}{}_j + \Pi^{\mathrm{C}}{}_i\xi^{\mathrm{C}i}{}_j. \tag{3.114}$$

These equations are Weyl-covariant of weights $d+2$, $d+2$, $d+1$ and $d+1$.

The case of conformal Carrollian perfect fluids is remarkably simple. As quoted earlier $\mathcal{F} = \mathcal{H}^i = 0$, and here

$$\mathcal{E} = -\frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon, \quad \mathcal{G}_j = \frac{1}{d}\hat{\mathscr{D}}_j\varepsilon + \frac{d+1}{d}\left(\frac{1}{\Omega}\hat{\mathscr{D}}_t\delta^i_j + \xi^{\mathrm{C}i}{}_j\right)\varepsilon\beta_i. \tag{3.115}$$

For these fluids the energy density is covariantly constant with respect to the Weyl–Carroll time derivative.

3.4 A self-dual fluid

A duality relationship between the Zermelo and the Randers–Papapetrou background metrics exist and can be stated as follows [15]: the contravariant form of Zermelo matches the covariant expression of Randers–Papapetrou and vice-versa (see Eqs. (A.1) and (A.28)).

This property is actually closely related to the duality among the Galilean and Carrollian contractions of the Poincaré group [12], and has many simple manifestations. For example, the reduction of a spacetime vector representation with respect to Galilean diffeomorphisms

(2.9), (2.10), (2.18) is performed with the components V^0 and V_i. Indeed, these transform as

$$V'^0 = JV^0, \quad V'_i = V_k J^{-1k}{}_i. \tag{3.116}$$

When reducing under Carrollian diffeomorphisms (2.27), (2.28), (2.36), one should instead use V_0 and V^i since

$$V'_0 = \frac{1}{J} V_0, \quad V'^i = J^i_k V^k. \tag{3.117}$$

The remarkable values $w^i = b_i = 0$ and $\Omega = 1$ define a sort of self-dual background. If furthermore we require the fluid to be at rest, no distinction survives between *perfect* Galilean and Carrollian fluids, as one readily checks that their equations are identical. The velocity of light is immaterial in this case. As soon as the system is driven away from perfection, this property does not hold any longer, because interactions are sensitive to c.

4 Examples

We will now illustrate our general formalism with examples for Galilean and Carrollian fluids. The latter is the first instance of a fluid obeying exact Carrollian dynamics. It is important both mathematically, as it makes contact with Calabi flows, and physically, for it is relevant in gravity and holography.

4.1 Galilean fluids

We provide here two applications: the flat space in rotating frame, which is well known and has the virtue of giving confidence to our methods, and the inflating space, combining both time-dependence and non-flatness of the host $\mathscr{S}$.

Euclidean three-dimensional space in rotating frame

We will present the hydrodynamical equations for a non-perfect fluid moving in Euclidean space E_3 with Cartesian coordinates, and observed from a uniformly rotating frame (see (2.8)):

$$a_{ij} = \delta_{ij}, \quad \Omega = 1, \quad \mathbf{w}(\mathbf{x}) = \mathbf{x} \times \boldsymbol{\omega}. \tag{4.1}$$

For this fluid, the continuity equation is simply

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \varrho \, \mathbf{div}\, \mathbf{v} = 0. \tag{4.2}$$

The Euler equation in first-order hydrodynamics, Eq. (3.58) reads:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = (\boldsymbol{\omega} \times \mathbf{x}) \times \boldsymbol{\omega} + 2\mathbf{v} \times \boldsymbol{\omega} - \frac{\mathbf{grad}\, p}{\varrho} + \frac{\eta^{\mathrm{G}}}{\varrho}\Delta\mathbf{v} + \frac{1}{\varrho}\left(\zeta^{\mathrm{G}} + \frac{\eta^{\mathrm{G}}}{3}\right)\mathbf{grad}(\mathbf{div}\,\mathbf{v}), \tag{4.3}$$

and we recognize the various, already spelled contributions to the dynamics. This equation has been obtained and used in many instances, see *e.g.* [7, 34, 35]. We also find the energy conservation equation (3.43):

$$\partial_t\left(\varrho\left(e + \frac{\mathbf{v}^2 - \boldsymbol{\omega}^2\mathbf{x}^2 + (\boldsymbol{\omega}\cdot\mathbf{x})^2}{2}\right)\right) = -\mathbf{div}\boldsymbol{\Pi}^{\mathrm{G}}, \tag{4.4}$$

with

$$\boldsymbol{\Pi}^{\mathrm{G}} = \varrho\mathbf{v}\left(h + \frac{\mathbf{v}^2 - \boldsymbol{\omega}^2\mathbf{x}^2 + (\boldsymbol{\omega}\cdot\mathbf{x})^2}{2}\right) - \kappa^{\mathrm{G}}\mathbf{grad}\, T - \mathbf{v}\cdot\boldsymbol{\Sigma}^{\mathrm{G}}_{(1)} \tag{4.5}$$

and

$$\Sigma^{\mathrm{G}}_{(1)ij} = \eta^{\mathrm{G}}\left(\partial_i v_j + \partial_j v_i\right) + \left(\zeta^{\mathrm{G}} - \frac{2}{3}\eta^{\mathrm{G}}\right)\delta_{ij}\partial_k v^k. \tag{4.6}$$

Alternatively, using (3.32), the energy equation reads:

$$\varrho\frac{\mathrm{d}}{\mathrm{d}t}\left(e + \frac{\mathbf{v}^2 - \boldsymbol{\omega}^2\mathbf{x}^2 + (\boldsymbol{\omega}\cdot\mathbf{x})^2}{2}\right) = -\mathbf{div}p\mathbf{v} + \kappa^{\mathrm{G}}\Delta T + \mathbf{div}\left(\mathbf{v}\cdot\boldsymbol{\Sigma}^{\mathrm{G}}_{(1)}\right). \tag{4.7}$$

The temporal variation of the total energy per mass is given by the divergences of the pressure, the thermal conduction and the viscous stress fluxes.

Inflating space

The dynamics of a non-perfect fluid moving on an inflating space can be studied considering:

$$a_{ij}(t,\mathbf{x}) = \exp\left(\alpha(t)\right)\tilde{a}_{ij}(\mathbf{x}), \quad \Omega = 1, \quad \mathbf{w} = 0. \tag{4.8}$$

The space dimension d is arbitrary here, therefore:

$$\ln\sqrt{a} = d\frac{\alpha}{2} + \ln\sqrt{\tilde{a}}. \tag{4.9}$$

The fluid equations obtained from (3.27), (3.32) and (3.42) become

$$\partial_t\varrho + \frac{\alpha'}{2}d\varrho + \mathbf{div}\varrho\mathbf{v} = 0, \tag{4.10}$$

$$\varrho\frac{\mathrm{d}}{\mathrm{d}t}\left(e + \frac{\mathbf{v}^2}{2}\right) + \frac{\alpha'}{2}\left(\varrho\mathbf{v}^2 + dp - \mathrm{Tr}\boldsymbol{\Sigma}^{\mathrm{G}}\right) + \mathbf{div}\left(p\mathbf{v} + \mathbf{Q}^{\mathrm{G}} - \mathbf{v}\cdot\boldsymbol{\Sigma}^{\mathrm{G}}\right) = 0, \tag{4.11}$$

$$\varrho\frac{\mathrm{d}v^i}{\mathrm{d}t} + \alpha'\varrho v^i + \nabla^i p - \nabla_j\Sigma^{\mathrm{G}ij} = 0. \tag{4.12}$$

where $\alpha' = {}^{\mathrm{d}\alpha}/_{\mathrm{d}t}$ and $\mathrm{Tr}\boldsymbol{\Sigma}^{\mathrm{G}} = a^{ij}\Sigma^{\mathrm{G}}{}_{ij}$.

The continuity equation (4.10) has an extra term proportional to ϱ. This reflects the change of density due to α'. For a static fluid one finds the familiar result $\varrho = \varrho_0 \mathrm{e}^{-d\alpha/2}$: for a space expanding in time, the density is getting diluted. In Euler's equation (4.12), a similar term creates a force proportional to the velocity field. For positive α', time dependence acts effectively like a friction. A similar conclusion is drawn from the energy conservation equation (4.11).

4.2 Two-dimensional Carrollian fluids and the Robinson–Trautman dynamics

Consider now a two-dimensional surface $\mathscr{S}$, endowed with a complex chart $(\zeta, \bar{\zeta})$ and a time-dependent metric of the form

$$\mathrm{d}\ell^2 = \frac{2}{P(t,\zeta,\bar{\zeta})^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}. \tag{4.13}$$

In this case the Carrollian shear $\boldsymbol{\xi}^{\mathrm{C}}$ (3.86) vanishes. We assume that the Carrollian frame has $\boldsymbol{b} = 0$ and $\Omega = 1$, and that the Carrollian kinematical variable $\boldsymbol{\beta}$ also vanishes. Hence, the Carrollian inertial acceleration $\boldsymbol{\varphi}$ (3.83) and inertial vorticity $\boldsymbol{\varpi}$ (3.85) vanish together with the kinematical acceleration $\boldsymbol{\gamma}^{\mathrm{C}}$ (3.99) and kinematical vorticity $\boldsymbol{w}$ (3.100). We further assume that $\boldsymbol{\pi}$ and $\boldsymbol{\Xi}$ vanish, so that the friction and heat-transport phenomena are captured exclusively by $\boldsymbol{Q}^{\mathrm{C}}$ and $\boldsymbol{\Sigma}^{\mathrm{C}}$. Hence $e_{\mathrm{e}} = \varepsilon$, $\Pi^{\mathrm{C}}{}_j = Q^{\mathrm{C}}{}_j$ and $\Pi^{\mathrm{C}}{}_{ij} = p a_{ij}$.

We will here study a conformal Carrollian fluid. In this case (see (3.106)), the Gibbs–Duhem equation reads

$$\varepsilon(t,\zeta,\bar{\zeta}) = 2p(t,\zeta,\bar{\zeta}), \tag{4.14}$$

and the viscous tensor is traceless:

$$\Sigma^{\mathrm{C}\zeta\bar{\zeta}} = 0. \tag{4.15}$$

The generic set of equations of motion for the Carrollian fluid at hand is (see (3.111), (3.113), (3.114))

$$\mathcal{E} = 3\varepsilon\partial_t \ln P - \partial_t\varepsilon - \mathbf{div}\,\boldsymbol{Q}^{\mathrm{C}} = 0, \tag{4.16}$$

$$\boldsymbol{\mathcal{G}} = \mathbf{grad}\, p = 0, \tag{4.17}$$

$$\boldsymbol{\mathcal{H}} = \partial_t\boldsymbol{Q}^{\mathrm{C}} - 2\boldsymbol{Q}^{\mathrm{C}}\partial_t \ln P - \mathbf{div}\boldsymbol{\Sigma}^{\mathrm{C}} = 0, \tag{4.18}$$

together with Eq. (3.112), $\mathcal{F} = 0$, identically satisfied due to the absence of shear. Equations (4.16), (4.17) and (4.18) are covariant under Weyl transformations mapping $P(t,\zeta,\bar{\zeta})$ onto $\mathcal{B}(t,\zeta,\bar{\zeta})P(t,\zeta,\bar{\zeta})$ with $\mathcal{B}(t,\zeta,\bar{\zeta})$ an arbitrary function.

The momentum equation (4.17) states that the pressure p is space-independent, which is not a surprise for a fluid at $\boldsymbol{\beta} = 0$ in a Carrollian frame with vanishing $\boldsymbol{b}$ and constant Ω. The same holds for the energy, due to the equation of state.

In order to proceed we must introduce some further assumptions regarding the heat current and the viscous stress tensor. These quantities are rooted to the unknown microscopic properties of the Carrollian fluids. As already mentioned earlier in Sec. 3.3, due to the absence of motion even at a microscopic level, it is tempting to assign a geometric rather than a statistical or kinetic origin to Carrollian thermodynamics. We may therefore define the *Carrollian temperature* as

$$\kappa^{\mathrm{C}} T(t,\zeta,\bar{\zeta}) = \left\langle \kappa^{\mathrm{C}} T \right\rangle (t) + \kappa' K(t,\zeta,\bar{\zeta}) - \kappa' \langle K \rangle (t), \tag{4.19}$$

where K the Gaussian curvature of (4.13):

$$K = \Delta \ln P \tag{4.20}$$

with $\Delta = 2P^2 \partial_{\bar{\zeta}} \partial_{\zeta}$ the ordinary two-dimensional Laplacian operator. The thermal conductivity κ^{C} is not constant in general because the identification with the curvature scalar endows the product $\kappa^{\mathrm{C}} T$ with a conformal weight 2, whereas the temperature T has weight 1. We also introduced a constant κ' for matching the dimensions. In expression (4.19), $\left\langle \kappa^{\mathrm{C}} T \right\rangle (t)$ is an *a priori* arbitrary time-dependent reference temperature (times thermal conductivity), and the brackets are meant to average over $\mathscr{S}$:[20]

$$\langle f \rangle (t) = \frac{1}{A} \int_{\mathscr{S}} \frac{\mathrm{d}^2\zeta}{P^2} f(t,\zeta,\bar{\zeta}), \quad A = \int_{\mathscr{S}} \frac{\mathrm{d}^2\zeta}{P^2}. \tag{4.21}$$

Equipped with a temperature, we define next the heat current as its gradient

$$\boldsymbol{Q}^{\mathrm{C}} = -\,\mathbf{grad}\, \kappa^{\mathrm{C}} T = -\kappa' \,\mathbf{grad}\, K, \tag{4.22}$$

following first-order Carrollian hydrodynamics, Eq. (3.103). Here, we assume this expression be exact, *i.e.* without higher-derivative contributions. With these definitions, the heat equation (4.16) for the Carrollian fluid at hand reads:

$$3\varepsilon \partial_t \ln P - \partial_t \varepsilon + \kappa' \Delta K = 0, \tag{4.23}$$

where we have used the equation of state (4.14). This is a dynamical equation for $P(t,\zeta,\bar{\zeta})$, given $\varepsilon(t)$. Carrollian dynamics, within the framework set by our definitions of temperature and heat current, is therefore purely geometrical and describes the evolution of the hosting space $\mathscr{S}$ rather than the fluid itself. This is not a surprise because the fluid does not move. Going in the Carrollian limit from a relativistic set-up, amounts to trading the dynamics of the fluid for that of the supporting geometry.

[20] Here $\mathrm{d}^2\zeta = -i\,\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$. If $\mathscr{S}$ is non-compact a limiting procedure is required for defining the integrals.

We must finally impose Eq. (4.18). As we mentioned in the general discussion of Sec. 3.3, this is not an evolution equation, but instead a constraint among the heat current, the viscous stress tensor and the ambient geometry. Thus, we can integrate it using (4.22). We find

$$\mathbf{\Sigma}^{\mathrm{C}} = -\frac{2\kappa'}{P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)\mathrm{d}\zeta^2 + \partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right)\mathrm{d}\bar\zeta^2\right), \tag{4.24}$$

up to a divergence-free, trace-free symmetric tensor. The viscous stress tensor for the Carrollian fluid at hand is therefore geometric, as is the heat current, and both appear as third-order derivatives of the metric. Actually, the effective expansion generally defined for Carrollian fluids as in (3.96), reads here:

$$\theta^{\mathrm{C}} = -2\partial_t \ln P. \tag{4.25}$$

It enables to view $\mathbf{\Sigma}^{\mathrm{C}}$ as a velocity third derivative through the writing

$$\Sigma^{\mathrm{C}}{}_{ij} = \kappa'\left(\nabla_i\nabla_j\theta^{\mathrm{C}} - \frac{1}{2}a_{ij}\nabla^k\nabla_k\theta^{\mathrm{C}}\right). \tag{4.26}$$

Notice that in the two-dimensional background under consideration (4.13), the viscous tensor $\mathbf{\Sigma}^{\mathrm{C}}$ could not have received an η^{C}-induced first-order derivative correction as in (3.102) because the Carrollian shear ξ^{C}_{ij} given in (3.97) vanishes here identically. However, since the Carrollian expansion θ^{C} is non-zero, the absence of first-order derivative correction (3.102) implies that for the fluid at hand $\zeta^{\mathrm{C}} = 0$.

Equation (4.23), which is at the heart of two-dimensional conformal Carrollian fluid dynamics, is actually known as Robinson–Trautman. It emerges when solving four-dimensional Einstein equations, assuming the existence of a null, geodesic and shearless congruence [36]. In vacuum or in the presence of a cosmological constant, Goldberg–Sachs theorems state that the corresponding spacetime is algebraically special and the whole dynamics boils down to the Robinson–Trautman equation with $\varepsilon(t) = 4\kappa' M(t)$ and $\kappa' = 1/16\pi G$ (using (4.20)):

$$\Delta\Delta \ln P + 12M\partial_t \ln P - 4\partial_t M = 0. \tag{4.27}$$

In that framework, the time dependence of the mass function $M(t)$ can be reabsorbed by an appropriate coordinate transformation (see *e.g.* [37]) and Robinson–Trautman equation becomes then

$$2\partial_{\bar\zeta}\partial_\zeta P^2 \partial_{\bar\zeta}\partial_\zeta \ln P = 3M\partial_t\left(\frac{1}{P^2}\right) \tag{4.28}$$

with M constant related to the Bondi mass. This is a parabolic equation describing a Calabi flow on a two-surface [38].

The reason why Robinson–Trautman appears both as a heat equation in conformal Carrollian fluids and as a remnant of four-dimensional Einstein equations is the holographic relationship between gravity and fluid dynamics. The two-dimensional conformal Carrollian fluid studied here originates from flat Robinson–Trautman spacetime holography [16]. Similarly Robinson–Trautman equation is the heat equation for $2+1$-dimensional relativistic boundary fluids emerging holographically from four-dimensional anti-de Sitter Robinson–Trautman spacetimes [28].

5 Conclusions

We can summarize our method and results as follows.

A general relativistic spacetime metric is covariant under diffeomorphisms. When put in Zermelo form, the data $\Omega(t)$, $w^i(t, \mathbf{x})$ and $a_{ij}(t, \mathbf{x})$ transform under Galilean diffeomorphisms $t' = t'(t)$ and $\mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$ as they should to comply with the infinite-c non-relativistic expectations. This observation is made by analyzing the relativistic particle motion and its classical limit. It provides the appropriate framework for studying the general non-relativistic Galilean fluid dynamics as an infinite-c limit of the relativistic one. In this manner, we have obtained the general equations *i.e.* continuity, energy-conservation and Euler, valid on any spatial background, potentially time-dependent, and observed from an arbitrary frame. These equations transform covariantly under Galilean diffeomorphisms.

Alternatively, one can study relativistic instantonic space-filling branes and the small-c behaviour of their dynamics. The latter is invariant under Carrollian diffeomorphisms $t' = t'(t, \mathbf{x})$ and $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$, and Randers–Papapetrou form is the best designed spacetime metric because the data $\Omega(t, \mathbf{x})$, $b_i(t, \mathbf{x})$ and $a_{ij}(t, \mathbf{x})$ transform as expected from the non-relativistic limit (which is actually ultra-relativistic). In Randers–Papapetrou backgrounds one can study relativistic fluids and their Carrollian limit at vanishing velocity of light. This limit exhibits a new connection, which naturally fits into the emerging Carrollian geometry. One obtains in this way the general equations for the Carrollian fluids, manifestly covariant under Carrollian diffeomorphisms.

Several comments are in order here.

The Carrollian set we have reached is made of two scalar and two vector equations. The first scalar is an energy conservation, whereas the first vector is a momentum conservation. As there is no motion (due to $c = 0$), there is no velocity field. Nonetheless there is a kinematical fluid variable (an "inverse velocity") accompanied by the pressure and energy density, related through an equation of state. We also find two heat currents and two viscous stress tensors. The Carrollian-fluid data cannot be naturally encapsulated all together in an energy–momentum tensor or an energy flux, as it happens in the Galilean case. Half of the equations concern exclusively the heat currents and the viscous stress tensors, relating them intimately to the ambient geometry and the Carrollian frame. We should stress here that we have made a specific assumption on the small-c behaviour of the relativistic viscous stress tensor and heat current, or equivalently of the transport coefficients. The number and the structure of the equations finally obtained reflects this unavoidable ansatz, inspired from the holographic Carrollian fluids met in flat-space gravity/fluid correspondence [16].[21] Going further in understanding this ansatz, and the physics behind the equations of motion, would require a microscopic analysis of Carrollian fluids.

Despite the absence of velocity field in Carrollian hydrodynamics, the concept of derivative expansion still holds. At each order one can define covariant tensors build on time and space derivatives of a_{ij}, b_i and β_i, as we met at first order with the shear and the expansion. The heat current and the viscous stress tensor can be expanded in these tensors, introducing phenomenological transport coefficients of increasing order.

Regarding Carrollian hydrodynamics, one could exploit a radically different perspective. Instead of defining a Carrollian fluid as the zero-c limit of a relativistic fluid in some Randers–Papapetrou background, one could simply try to build a fluid-like – *i.e.* continuous – generalization of an instantonic d-brane, directly within a Carrollian structure. This would promote the "inverse velocity" $\partial_i t$ of the elementary d-brane described by $t = t(\mathbf{x})$ into an "inverse velocity field" reminiscent of $\beta_i + b_i$ and transforming as in (2.32) under a Carrollian diffeomorphism. This could be the starting point for designing the dynamics of this new continuous Carrollian medium. Irrespective of the viewpoint chosen for describing Carrollian continuous media, zero-c limit of ordinary relativistic fluids or d-brane continuums, a great deal of fundamental thermodynamics, kinetic theory, derivative expansions, equilibrium and transport dynamics remains to be unravelled.

In conclusion of our general work, we have presented some examples. Those on Galilean hydrodynamics illustrate the power of the formalism for handling general, time-dependent and curved host spaces, potentially observed from non-inertial frames. The example of two-dimensional Carrollian fluid is interesting because it introduces the concept of geometric temperature and treats dissipative phenomena exactly *i.e.* by solving explicitly all the equations but one, finally brought in the canonical form of a Calabi flow on the two-dimensional surface. The Carrollian fluid dynamics translates into a dynamics for the geometry. This example has important implications in asymptotically flat holography [16] of Robinson–Trautman spacetimes.

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[21] Concrete examples of exact Carrollian fluids are described in this reference.

A Christoffel symbols, transformations and connections

We provide here a toolbox for working out the Galilean and Carrollian limits in the Zermelo and Randers–Papapetrou backgrounds, and checking the covariance properties of the set of equations reached by this method. These properties are bound to the emergence of novel Galilean and Carrollian connections, and covariant derivatives, which are discussed together with the associated curvature tensors. In the Carrollian case, an extra conformal connection is also presented, relevant when studying conformal Carrollian fluids.

A.1 Zermelo metric

Christoffel symbols

The Zermelo metric (2.17) has components (in the coframe $\left\{\mathrm{d}x^0 = c\mathrm{d}t, \mathrm{d}x^i\right\}$):

$$g^{Z}_{\mu\nu} \to \begin{pmatrix} -\Omega^2 + \frac{\mathbf{w}^2}{c^2} & -\frac{w_k}{c} \\ -\frac{w_i}{c} & a_{ik} \end{pmatrix}, \quad g^{Z\mu\nu} \to \frac{1}{\Omega^2}\begin{pmatrix} -1 & -\frac{w^j}{c} \\ -\frac{w^i}{c} & \Omega^2 a^{ij} - \frac{w^i w^j}{c^2} \end{pmatrix}, \tag{A.1}$$

where $w_k = a_{kj} w^j$. Its determinant reads:

$$\sqrt{-g} = \Omega \sqrt{a}, \tag{A.2}$$

where a is the determinant of a_{ij}. We remind that Ω depends on time only, whereas a_{ij} and w_i also depend on space.

The Christoffel symbols are easily computed. We are interested in their large-c behaviour for which one obtains the following:

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{c}\partial_t \ln\Omega + + \frac{w^i}{2c^3\Omega^2}\left(\partial_i \mathbf{w}^2 + w^j \partial_t a_{ij}\right) + \mathrm{O}\left(1/c^5\right), && \text{(A.3)}\\
\Gamma^0_{0i} &= -\frac{1}{2c^2\Omega^2}\left(w_j \partial_i w^j + w^j \partial_j w_i + w^j \partial_t a_{ij}\right) + \mathrm{O}\left(1/c^4\right), && \text{(A.4)}\\
\Gamma^0_{ij} &= \frac{1}{c\Omega^2}\left(\frac{1}{2}\left(\partial_i w_j + \partial_j w_i + \partial_t a_{ij}\right) - w_k \gamma^k_{ij}\right), && \text{(A.5)}\\
\Gamma^i_{00} &= \frac{1}{c^2}\left(w^i \partial_t \ln\Omega - a^{ik}\left(\partial_t w_k + \partial_k \frac{\mathbf{w}^2}{2}\right)\right) + \mathrm{O}\left(1/c^4\right), && \text{(A.6)}\\
\Gamma^i_{j0} &= \frac{a^{ik}}{2c}\left(\partial_k w_j - \partial_j w_k + \partial_t a_{jk}\right) + \mathrm{O}\left(1/c^3\right), && \text{(A.7)}\\
\Gamma^i_{jk} &= \gamma^i_{jk} + \mathrm{O}\left(1/c^2\right), && \text{(A.8)}
\end{aligned}$$

where

$$\gamma^i_{jk} = \frac{a^{il}}{2}\left(\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}\right) \tag{A.9}$$

are the Christoffel symbols for the d-dimensional metric a_{ij}. Note also

$$\Gamma^{\mu}_{\mu 0} = \frac{1}{c}\partial_t \ln\left(\sqrt{a}\,\Omega\right), \quad \Gamma^{\mu}_{\mu i} = \partial_i \ln\sqrt{a}. \tag{A.10}$$

With these data it is possible to compute the divergence of the fluid energy–momentum tensor (3.19) and (3.20).

Covariance

In order to check the covariance (3.35) and (3.37),

$$\mathcal{C}' = \mathcal{C}, \quad \mathcal{E}' = \mathcal{E} \quad \mathcal{M}'_i = J^{-1}{}^{l}_{i}\mathcal{M}_l,$$

for the Galilean fluid dynamics under Galilean diffeomorphisms (2.9)

$$t' = t'(t) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(t, \mathbf{x}),$$

with Jacobian functions (2.10)

$$J(t) = \frac{\partial t'}{\partial t}, \quad j^i(t, \mathbf{x}) = \frac{\partial x^{i\prime}}{\partial t}, \quad J^i_j(t, \mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j},$$

we can use several simple covariant blocks. We first remind (2.11), (2.12), (2.13), (2.15):

$$a'_{ij} = a_{kl} J^{-1}{}^{k}_{i} J^{-1}{}^{l}_{j}, \quad v'^k = \frac{1}{J}\left(J^k_i v^i + j^k\right), \quad w'^k = \frac{1}{J}\left(J^k_i w^i + j^k\right), \quad \Omega' = \frac{\Omega}{J},$$

implying in particular

$$v'_k = \frac{J^{-1}{}^{i}_{k}}{J}\left(v_i + a_{ij} J^{-1}{}^{j}_{l} j^l\right), \quad w'_k = \frac{J^{-1}{}^{i}_{k}}{J}\left(w_i + a_{ij} J^{-1}{}^{j}_{l} j^l\right) \tag{A.11}$$

with

$$\partial'_t = \frac{1}{J}\left(\partial_t - j^k J^{-1}{}^{i}_{k}\partial_i\right), \tag{A.12}$$

$$\partial'_j = J^{-1}{}^{i}_{j}\partial_i. \tag{A.13}$$

Consider now A^k and B^k, the components of fields transforming like v^k or w^k (gauge-like transformation) and V^k a field transforming like $\frac{v^k - w^k}{\Omega}$ *i.e.* like a genuine vector:

$$A'^k = \frac{1}{J}\left(J^k_i A^i + j^k\right), \quad B'^k = \frac{1}{J}\left(J^k_i B^i + j^k\right), \quad V'^k = J^k_i V^i. \tag{A.14}$$

Consider also a scalar and a rank-two tensor

$$\Phi' = \Phi, \quad S'_{ij} = S_{kl} J^{-1k}{}_{i} J^{-1l}{}_{j}. \tag{A.15}$$

The basic transformation rules are as follows:

$$\begin{aligned}
\frac{A'^k - B'^k}{\Omega'} &= J^k_i \frac{A^i - B^i}{\Omega}, && \text{(A.16)}\\
\frac{1}{\sqrt{a'}}\partial'_t\left(\sqrt{a'}\Phi'\right) + \nabla'_i\left(\Phi' A'^i\right) &= \frac{1}{J}\left(\frac{1}{\sqrt{a}}\partial_t\left(\sqrt{a}\Phi\right) + \nabla_i\left(\Phi A^i\right)\right), && \text{(A.17)}\\
\nabla'_i V'^i &= \nabla_i V^i, && \text{(A.18)}\\
\nabla'_{(i}A'_{j)} + \frac{1}{2}\partial'_t a'_{ij} &= \frac{1}{J}\left(\nabla_{(k}A_{l)} + \frac{1}{2}\partial_t a_{kl}\right) J^{-1k}{}_{i} J^{-1l}{}_{j}, && \text{(A.19)}\\
\nabla'^{(i}A'^{j)} - \frac{1}{2}\partial'_t a'^{ij} &= \frac{1}{J}\left(\nabla^{(k}A^{l)} - \frac{1}{2}\partial_t a^{kl}\right) J^i_k J^j_l, && \text{(A.20)}\\
\nabla'_i S'^{ij} &= J^j_l \nabla_i S^{il}, && \text{(A.21)}\\
\frac{1}{\Omega'}\left(\partial'_t V'_i + A'^j \nabla'_j V'_i + V'_j \nabla'_i B'^j\right) &= \frac{J^{-1k}{}_{i}}{\Omega}\left(\partial_t V_k + A^j \nabla_j V_k + V_j \nabla_k B^j\right), && \text{(A.22)}\\
\Delta' A'_i + r'^{m}_{i} A'_m + a'_{ik} a'^{mn} \partial'_t \gamma'^k_{mn} &= \frac{J^{-1j}{}_{i}}{J}\left(\Delta A_j + r_j{}^m A_m + a_{jk} a^{mn} \partial_t \gamma^k_{mn}\right). && \text{(A.23)}
\end{aligned}$$

In the above expressions, ∇_i, Δ and r_{ij} are associated with the d-dimensional Levi–Civita connection γ^i_{jk} displayed in (A.9).

As a final comment regarding Galilean covariance properties, we would like to stress that the action of ∂_t spoils the transformation rules displayed in (A.14) and (A.15). This is both due to the transformation property of the partial time derivative (A.12), and to the time dependence of the Jacobian matrix J^i_j. A Galilean covariant time-derivative can be introduced, acting as follows on a vector:[22]

$$\frac{1}{\Omega}\frac{\mathrm{D}V^i}{\mathrm{d}t} = \frac{1}{\Omega}\left[\left(\partial_t + v^j \nabla_j\right) V^i - V^j \nabla_j w^i\right] = \frac{1}{\Omega}\frac{\mathrm{d}V^i}{\mathrm{d}t} - \frac{1}{\Omega} V^j \nabla_j w^i, \tag{A.24}$$

and resulting in a genuine vector under Galilean diffeomorphisms. Here, the frame velocity w^k plays the rôle of a connection, and the Galilean covariant time-derivative generalizes the material derivative $\mathrm{d}/\mathrm{d}t$ introduced in (3.29). The latter is covariant only when acting on scalar functions f, hence we set $\frac{\mathrm{D}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}$. Expression (A.24) is easily extended for tensors of arbitrary rank using the Leibniz rule, as *e.g.* for one-forms:

$$\frac{1}{\Omega}\frac{\mathrm{D}V_i}{\mathrm{d}t} = \frac{1}{\Omega}\frac{\mathrm{d}V_i}{\mathrm{d}t} + \frac{1}{\Omega} V_j \nabla_i w^j. \tag{A.25}$$

[22] For a detailed and general presentation of Galilean affine connections see [23,24].

Notice that the Galilean covariant time-derivative at hand is not "metric compatible":

$$\frac{1}{\Omega}\frac{\mathrm{D}a_{ij}}{\mathrm{d}t} = \frac{1}{\Omega}\left(\partial_t a_{ij} + 2\nabla_{(i}w_{j)}\right). \tag{A.26}$$

This result is actually expected because a covariant time-derivative of the metric should be interpreted as an extrinsic curvature. Indeed, expression (A.26) divided by $2c$ is exactly identified with the spatial components K_{ij} of constant-t hypersurfaces extrinsic curvature in the Zermelo background (2.17), (A.1).

The commutator of covariant time and space derivatives reveals a new piece of curvature, which appears in Galilean geometries, on top of the standard Riemann tensor associated with the spatial covariant derivative ∇_i. It is encapsulated in a one-form $\mathrm{d}\theta^{\mathrm{G}}$, as one observes from:

$$\left[\frac{1}{\Omega}\frac{\mathrm{D}}{\mathrm{d}t}, \nabla_i\right] V^i = V^i\partial_i\theta^{\mathrm{G}} + \nabla_j\left(V^i\nabla_i\left(\frac{w^j - v^j}{\Omega}\right)\right), \tag{A.27}$$

where θ^{G} is a scalar function introduced in (3.30) as the Galilean effective expansion:

$$\theta^{\mathrm{G}} = \frac{1}{\Omega}\left(\partial_t \ln\sqrt{a} + \nabla_i v^i\right).$$

This extra piece of curvature should not come as a surprise. It is a Galilean remnant of some ordinary components of Riemannian curvature in the original Zermelo spacetime.

A.2 Randers–Papapetrou metric

Christoffel symbols

The Randers–Papapetrou metric (2.35) has components (in the coframe $\{\mathrm{d}x^0 = c\mathrm{d}t, \mathrm{d}x^i\}$):

$$g^{\mathrm{RP}}_{\mu\nu} \to \begin{pmatrix} -\Omega^2 & c\Omega b_j \\ c\Omega b_i & a_{ij} - c^2 b_i b_j \end{pmatrix}, \quad g^{\mathrm{RP}\mu\nu} \to \frac{1}{\Omega^2}\begin{pmatrix} -1 + c^2\boldsymbol{b}^2 & c\Omega b^k \\ c\Omega b^i & \Omega^2 a^{ik} \end{pmatrix}, \tag{A.28}$$

where $b^k = a^{kj}b_j$. The metric determinant is again given in (A.2):

$$\sqrt{-g} = \Omega\sqrt{a}. \tag{A.29}$$

Here, Ω, a_{ij} and b_i depend on time t and space $\mathbf{x}$.

The Christoffel symbols are computed exactly in the present case:

$$
\begin{aligned}
\Gamma^0_{00} &= \frac{1}{c}\partial_t \ln\Omega + c\left(b^i\partial_i\Omega + \frac{1}{2}\left(\partial_t \boldsymbol{b}^2 - b_i b_j \partial_t a^{ij}\right)\right), &\text{(A.30)}\\
\Gamma^0_{0i} &= \left(1-\frac{1}{2}c^2\boldsymbol{b}^2\right)\partial_i\ln\Omega + \frac{1}{2}c^2 b^j\left(\partial_i b_j - \partial_j b_i - b_i\partial_j\ln\Omega\right)\\
&\quad + \frac{1}{2\Omega}b^j\partial_t\left(a_{ij} - c^2 b_i b_j\right), &\text{(A.31)}\\
\Gamma^0_{ij} &= -\frac{c}{2\Omega}\left(\partial_i b_j + \partial_j b_i + c^2 b^k\left(b_i\left(\partial_j b_k - \partial_k b_j\right) + b_j\left(\partial_i b_k - \partial_k b_i\right)\right)\right)\\
&\quad + \frac{c b_k}{\Omega}\gamma^k_{ij} + \frac{1-c^2\boldsymbol{b}^2}{2\Omega^2}\left(\frac{1}{c}\partial_t a_{ij} - c b_j\left(\partial_t b_i + \partial_i\Omega\right) - c b_i\left(\partial_t b_j + \partial_j\Omega\right)\right), &\text{(A.32)}\\
\Gamma^i_{00} &= \Omega a^{ij}\left(\partial_t b_j + \partial_j\Omega\right), &\text{(A.33)}\\
\Gamma^i_{j0} &= \frac{1}{2c}a^{ik}\left(\partial_t\left(a_{kj} - c^2 b_k b_j\right) + c^2\Omega\left(\partial_j b_k - \partial_k b_j\right) - c^2\left(b_k\partial_j\Omega + b_j\partial_k\Omega\right)\right), &\text{(A.34)}\\
\Gamma^i_{jk} &= \frac{c^2}{2}\left(\frac{b^i}{\Omega}\left(b_j\left(\partial_t b_k + \partial_k\Omega\right) + b_k\left(\partial_t b_j + \partial_j\Omega\right)\right) - a^{il}\left(b_j\left(\partial_k b_l - \partial_l b_k\right) + b_k\left(\partial_j b_l - \partial_l b_j\right)\right)\right)\\
&\quad + \gamma^i_{jk} - \frac{b^i}{2\Omega}\partial_t a_{jk}, &\text{(A.35)}
\end{aligned}
$$

where γ^k_{ij} are the d-dimensional Christoffel symbols:

$$\gamma^i_{jk} = \frac{a^{il}}{2}\left(\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}\right). \tag{A.36}$$

Note also

$$\Gamma^\mu_{\mu 0} = \frac{1}{c}\partial_t \ln\left(\sqrt{a}\,\Omega\right), \quad \Gamma^\mu_{\mu i} = \partial_i \ln\left(\sqrt{a}\,\Omega\right). \tag{A.37}$$

With these data it is possible to compute the divergence of the fluid energy–momentum tensor (3.72) and (3.73).

Covariance and the Levi–Civita–Carroll connection

In order to check the covariance (3.87) and (3.92),

$$\mathcal{E}' = \mathcal{E}, \quad \mathcal{F}' = \mathcal{F}, \quad \mathcal{G}'^i = J^i_j \mathcal{G}^j, \quad \mathcal{H}'^i = J^i_j \mathcal{H}^j$$

for the Carrollian fluid dynamics under Carrollian diffeomorphisms (2.27)

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}),$$

with Jacobian functions (2.28)

$$J(t,\mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t,\mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J^i_j(\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j},$$

we can use several simple covariant blocks. We first remind (2.15), (2.31), (2.33):

$$a'_{ij} = a_{kl} J^{-1}{}^k_i J^{-1}{}^l_j, \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1}{}^i_k, \quad \Omega' = \frac{\Omega}{J},$$

and

$$\begin{aligned} \partial'_t &= \frac{1}{J}\partial_t, & \text{(A.38)} \\ \partial'_j &= J^{-1}{}^i_j \left(\partial_i - \frac{j_i}{J}\partial_t\right). & \text{(A.39)} \end{aligned}$$

From the above transformation rules we obtains:

$$\begin{aligned} \frac{1}{\Omega'}\partial'_t a'_{ij} &= \frac{1}{\Omega}\partial_t a_{kl} J^{-1}{}^k_i J^{-1}{}^l_j, & \text{(A.40)} \\ \frac{1}{\Omega'}\partial'_t \ln\sqrt{a'} &= \frac{1}{\Omega}\partial_t \ln\sqrt{a}, & \text{(A.41)} \\ \partial'_t b'_i + \partial'_i \Omega' &= \frac{1}{J} J^{-1}{}^j_i \left(\partial_t b_j + \partial_j \Omega\right), & \text{(A.42)} \\ \hat{\partial}'_i &= J^{-1}{}^j_i \hat{\partial}_j, & \text{(A.43)} \end{aligned}$$

where we have defined

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega}\partial_t. \tag{A.44}$$

In view of the basic rules (A.38), (A.39) and (A.40)–(A.43), it is tempting to introduce a new connection for Carrollian geometry that we will call *Levi–Civita–Carroll*, whose coefficients will be generalizations of the Christoffel symbols (A.36):

$$\begin{aligned} \hat{\gamma}^i_{jk} &= \frac{a^{il}}{2}\left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}\right) \\ &= \frac{a^{il}}{2}\left(\left(\partial_j + \tfrac{b_j}{\Omega}\partial_t\right) a_{lk} + \left(\partial_k + \tfrac{b_k}{\Omega}\partial_t\right) a_{lj} - \left(\partial_l + \tfrac{b_l}{\Omega}\partial_t\right) a_{jk}\right) \\ &= \gamma^i_{jk} + c^i_{jk} \end{aligned} \tag{A.45}$$

with γ^i_{jk} and $\hat{\partial}_i$ defined in (A.36) and (A.44). We will refer to those as *Christoffel–Carroll* symbols. They transform under Carrollian diffeomorphisms as ordinary Christoffel symbols under ordinary diffeomorphisms:

$$\hat{\gamma}'^k_{ij} = J^k_n J^{-1}{}^l_i J^{-1}{}^m_j \hat{\gamma}^n_{lm} - J^{-1}{}^l_i J^{-1}{}^n_j \partial_l J^k_n. \tag{A.46}$$

The emergence of this new set of connection coefficients should not be a surprise. Indeed one readily shows that

$$h_i{}^\mu \Gamma^k_{\mu\nu} h^\nu{}_j = \hat{\gamma}^k_{ij}, \tag{A.47}$$

where $\Gamma^k_{\mu\nu}$ are the $d+1$-dimensional Randers–Papapetrou Christoffel symbols (A.30)–(A.35), and $h_\nu{}^\mu$ the projector orthogonal to $\mathrm{u} = {}^{\partial_t}/\Omega$ (as in (3.9), (3.67)).

The Levi–Civita–Carroll covariant derivative acts symbolically as

$$\hat{\boldsymbol{\nabla}} = \hat{\boldsymbol{\partial}} + \hat{\boldsymbol{\gamma}} = \boldsymbol{\partial} + \frac{\boldsymbol{b}}{\Omega}\partial_t + \boldsymbol{\gamma} + \boldsymbol{c} = \boldsymbol{\nabla} + \frac{\boldsymbol{b}}{\Omega}\partial_t + \boldsymbol{c}. \tag{A.48}$$

For example, consider Φ, V^k and S_{kl}, the components of a scalar, a vector, and rank-two symmetric tensor:

$$\Phi' = \Phi, \quad V'^i = J^i_j V^j, \quad S'_{ij} = S_{kl} J^{-1k}_{\ \ i} J^{-1l}_{\ \ j}, \tag{A.49}$$

the action of this new covariant derivative is

$$\hat{\partial}_i \Phi = \partial_i \Phi + \frac{b_i}{\Omega}\partial_t \Phi, \tag{A.50}$$

$$\begin{aligned} \hat{\nabla}_i V^j &= \partial_i V^j + \frac{b_i}{\Omega}\partial_t V^j + \hat{\gamma}^j_{il} V^l \\ &= \nabla_i V^j + \frac{b_i}{\Omega}\partial_t V^j + c^j_{il} V^l, \end{aligned} \tag{A.51}$$

$$\hat{\nabla}_i V^i = \frac{1}{\sqrt{a}}\hat{\partial}_i\left(\sqrt{a}V^i\right) \tag{A.52}$$

$$\begin{aligned} \hat{\nabla}_i S_{jk} &= \partial_i S_{jk} + \frac{b_i}{\Omega}\partial_t S_{jk} - \hat{\gamma}^l_{ij} S_{lk} - \hat{\gamma}^l_{ik} S_{jl} \\ &= \nabla_i S_{jk} + \frac{b_i}{\Omega}\partial_t S_{jk} - c^l_{ij} S_{lk} - c^l_{ik} S_{jl}. \end{aligned} \tag{A.53}$$

All these transform as genuine tensors, namely:

$$\hat{\partial}'_i \Phi' = J^{-1j}_{\ \ i}\hat{\partial}_j \Phi, \tag{A.54}$$

$$\hat{\nabla}'_i V'^j = J^{-1k}_{\ \ i} J^j_l \hat{\nabla}_k V^l, \tag{A.55}$$

$$\hat{\nabla}'_i V'^i = \hat{\nabla}_i V^i, \tag{A.56}$$

$$\hat{\nabla}'_i S'_{jk} = J^{-1m}_{\ \ i} J^{-1n}_{\ \ j} J^{-1l}_{\ \ k} \hat{\nabla}_m S_{nl}. \tag{A.57}$$

Further elementary transformation rules are as follows:

$$\frac{1}{\Omega'}\partial'_t \Phi' = \frac{1}{\Omega}\partial_t \Phi, \quad \frac{1}{\Omega'}\partial'_t V'^i = J^i_j \frac{1}{\Omega}\partial_t V^j, \quad \frac{1}{\Omega'}\partial'_t S'^{ij} = J^i_k J^j_l \frac{1}{\Omega}\partial_t S^{kl}, \tag{A.58}$$

as well as

$$\nabla'_i V'^i + \frac{b'_i}{\Omega'\sqrt{a'}}\partial'_t\left(\sqrt{a'}V'^i\right) = \hat{\nabla}'_i V'^i = \hat{\nabla}_i V^i = \nabla_i V^i + \frac{b_i}{\Omega\sqrt{a}}\partial_t\left(\sqrt{a}V^i\right), \tag{A.59}$$

and

$$\nabla'_k S'^{ki} + \frac{b'_k}{\Omega'\sqrt{a'}}\left(\partial'_t\left(\sqrt{a'}S'^{ki}\right) - \sqrt{a'}S'^{k}{}_j\partial'_t a'^{ij}\right) - \frac{b'^i}{2\Omega'}S'^{kl}\partial'_t a'_{kl} = \hat{\nabla}'_k S'^{ki} =$$
$$= J^i_j \hat{\nabla}_k S^{kj} = J^i_j\left(\nabla_k S^{kj} + \frac{b_k}{\Omega\sqrt{a}}\left(\partial_t\left(S^{kj}\sqrt{a}\right) - \sqrt{a}S^k{}_l\partial_t a^{jl}\right) - \frac{b^j}{2\Omega}S^{kl}\partial_t a_{kl}\right). \tag{A.60}$$

Curvature, effective torsion and further properties of the Levi–Civita–Carroll connection

The Levi–Civita–Carroll connection is metric,

$$\hat{\nabla}_i a_{jk} = 0. \tag{A.61}$$

Furthermore, the usual torsion tensor vanishes:[23]

$$\hat{t}^k{}_{ij} = 2\hat{\gamma}^k_{[ij]} = 0. \tag{A.62}$$

However, the new ordinary (as opposed to covariant) derivatives $\hat{\partial}_i$ defined in (A.44) do not commute. Indeed, acting on any arbitrary function they lead to

$$\left[\hat{\partial}_i, \hat{\partial}_j\right]\Phi = \frac{2}{\Omega}\varpi_{ij}\partial_t\Phi, \tag{A.63}$$

where ϖ_{ij} are the components of the Carrollian vorticity defined in (3.85) (explicitly in (3.98)) using the Carrollian acceleration φ_i (3.83):

$$\varpi_{ij} = \partial_{[i}b_{j]} + b_{[i}\varphi_{j]}, \quad \varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i\Omega\right). \tag{A.64}$$

Therefore, the Levi–Civita–Carroll connection has an *effective torsion* as one can see from

$$\left[\hat{\nabla}_i, \hat{\nabla}_j\right]\Phi = \varpi_{ij}\frac{2}{\Omega}\partial_t\Phi, \tag{A.65}$$

where Φ is a scalar.

Similarly, one can compute the commutator of the Levi–Civita–Carroll covariant derivatives acting on a vector field:

$$\begin{aligned}\left[\hat{\nabla}_k, \hat{\nabla}_l\right]V^i &= \left(\hat{\partial}_k\hat{\gamma}^i_{lj} - \hat{\partial}_l\hat{\gamma}^i_{kj} + \hat{\gamma}^i_{km}\hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm}\hat{\gamma}^m_{kj}\right)V^j + \left[\hat{\partial}_k, \hat{\partial}_l\right]V^i \\ &= \hat{r}^i{}_{jkl}V^j + \varpi_{kl}\frac{2}{\Omega}\partial_t V^i.\end{aligned} \tag{A.66}$$

In this expression we have defined $\hat{r}^i{}_{jkl}$, which are by construction components of a genuine tensor under Carrollian diffeomorphisms in d dimensions. This should be called the *Riemann–Carroll* tensor. It is made of several pieces, among which $\partial_k\gamma^i_{lj} - \partial_l\gamma^i_{kj} + \gamma^i_{km}\gamma^m_{lj} -$

[23] Discussions on Carrollian affine connections can be found *e.g.* in [24, 39, 40]. In particular, Ref. [24] provides a general classification of connections with or without torsion.

$\gamma^i_{lm}\gamma^m_{kj}$, which is *not* covariant under Carrollian diffeomorphisms – it is under ordinary d-dimensional diffeomorphisms though. The Ricci–Carroll tensor and the Carroll scalar curvature are thus

$$\hat{r}_{ij} = \hat{r}^k{}_{ikj}, \quad \hat{r} = a^{ij}\hat{r}_{ij}. \tag{A.67}$$

Notice that the Ricci–Carroll tensor is *not* symmetric in general: $\hat{r}_{ij} \neq \hat{r}_{ji}$.

We would like to close this part with two remarks regarding Carrollian geometry and in particular Carrollian time. As readily seen in (A.58), acting on any object tensorial under Carrollian diffeomorphisms, the time derivative $\hat{\partial}_t$ provides another tensor. For this reason, it was not necessary to define any "temporal covariant derivative". Our first remark is that the ordinary time derivative has an unsatisfactory feature: its action on the metric does not vanish. One is tempted therefore to set a new time derivative $\hat{\partial}_t$ such that

$$\hat{\partial}_t a_{jk} = 0, \tag{A.68}$$

while keeping the transformation rule under Carrollian diffeomorphisms:

$$\hat{\partial}'_t = \frac{1}{J}\hat{\partial}_t. \tag{A.69}$$

This is achieved by introducing a "temporal Carrollian connection"

$$\hat{\gamma}^i{}_j = \frac{1}{2\Omega}a^{ik}\partial_t a_{kj}. \tag{A.70}$$

Calling this a connection is actually inappropriate because it transforms as a genuine tensor under Carrollian diffeomorphisms:

$$\hat{\gamma}'^k{}_j = J^k_n J^{-1m}{}_j \hat{\gamma}^n{}_m. \tag{A.71}$$

In fact, the trace of this object is the Carrollian expansion introduced in (3.84):

$$\theta^{\mathrm{C}} = \frac{1}{\Omega}\partial_t \ln\sqrt{a} = \hat{\gamma}^i{}_i, \tag{A.72}$$

whereas its traceless part is the Carrollian shear defined in (3.86):

$$\xi^{\mathrm{C}i}{}_j = \hat{\gamma}^i{}_j - \frac{1}{d}\delta^i_j\hat{\gamma}^i{}_i = \hat{\gamma}^i{}_j - \frac{1}{d}\delta^i_j\theta^{\mathrm{C}}. \tag{A.73}$$

The temporal connection $\hat{\gamma}^i{}_j$ appears also as the zero-c remnant of the mixed projected relativistic Randers–Papapetrou Christoffel symbols, as in (A.47):

$$\frac{c}{\Omega}U_0{}^\mu\Gamma^k_{\mu\nu}h^\nu{}_j = \hat{\gamma}^k{}_j. \tag{A.74}$$

The action of $\hat{\partial}_t$ on scalars is simply ∂_t:

$$\hat{\partial}_t \Phi = \partial_t \Phi, \tag{A.75}$$

whereas on vectors or forms it is defined as

$$\frac{1}{\Omega}\hat{\partial}_t V^i = \frac{1}{\Omega}\partial_t V^i + \hat{\gamma}^i{}_j V^j, \quad \frac{1}{\Omega}\hat{\partial}_t V_i = \frac{1}{\Omega}\partial_t V_i - \hat{\gamma}^j{}_i V_j. \tag{A.76}$$

Leibniz rule generalizes the latter to any tensor and allows to demonstrate the property (A.68). Indices can now be raised and lowered with the metric passing through $\hat{\partial}_t$.

The above Riemann–Carroll curvature tensor of a Carrollian geometry appears actually as the zero-c limit of the spatial components of the ordinary Riemann curvature in the Randers–Papapetrou background.[24] In the same spirit, one may also wonder what the Carrollian limit is for the temporal components of the relativistic Randers–Papapetrou curvature, and this is our second and last remark. In order to answer this question, we must compute the commutator of time and space covariant derivatives acting on scalar and vector fields, as in (A.65) and (A.66). We find:

$$\left[\frac{1}{\Omega}\hat{\partial}_t, \hat{\partial}_i\right]\Phi = \left(\varphi_i \frac{1}{\Omega}\partial_t - \hat{\gamma}^j{}_i \hat{\partial}_j\right)\Phi, \tag{A.77}$$

and

$$\left[\frac{1}{\Omega}\hat{\partial}_t, \hat{\nabla}_i\right]V^i = \left(\hat{\partial}_i\theta^{\mathrm{C}} - \hat{\nabla}_j\hat{\gamma}^j{}_i\right)V^i + \left(\theta^{\mathrm{C}}\delta^j_i - \hat{\gamma}^j{}_i\right)\varphi_j V^i + \left(\varphi_i \frac{1}{\Omega}\hat{\partial}_t - \hat{\gamma}^j{}_i\hat{\nabla}_j\right)V^i \tag{A.78}$$

with φ_i and θ^{C} the Carrollian acceleration and expansion (A.64), (A.72). We can define from this expression the components of a time-curvature Carrollian form:

$$\hat{r}_i = \frac{1}{d}\left(\hat{\nabla}_j\hat{\gamma}^j{}_i - \hat{\partial}_i\theta^{\mathrm{C}}\right) = \frac{1}{d}\left(\hat{\nabla}_j\hat{\xi}^{\mathrm{C}j}{}_i + \frac{1-d}{d}\hat{\partial}_i\theta^{\mathrm{C}}\right). \tag{A.79}$$

Using ϖ_{kl}, $\hat{r}_i$ and time derivative in the framework at hand, many new curvature-like (*i.e.* two-derivative) tensorial objects can be defined. We will not elaborate any longer on these issues, which would naturally fit in a more thorough analysis of Carrollian geometry.

[24] This statement is accurate but comes without a proof. Evaluating the zero-c (or infinite-c, as we would do in the Galilean counterpart) limit is a subtle task because in this kind of limits several components of the curvature usually diverge (see *e.g.* [16], where the rôle of curvature is prominent). From the perspective of the final geometry this does not produce any harm because the involved components decouple.

The Weyl–Carroll connection

The Levi–Civita–Carroll covariant derivatives $\hat{\nabla}$ and $\hat{\partial}_t$ defined in (A.48), (A.75) and (A.76) for Carrollian geometry are not covariant with respect to Weyl transformations (3.108),

$$a_{ij} \to \frac{1}{\mathcal{B}^2} a_{ij}, \quad b_i \to \frac{1}{\mathcal{B}} b_i, \quad \Omega \to \frac{1}{\mathcal{B}} \Omega. \tag{A.80}$$

We can define *Weyl–Carroll* covariant spatial and time derivatives using the Carrollian acceleration φ_i defined in (A.64) and the Carrollian expansion (A.72), which transform as connections (see (3.109)):

$$\varphi_i \to \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \theta^{\mathrm{C}} \to \mathcal{B}\theta^{\mathrm{C}} - \frac{d}{\Omega} \partial_t \mathcal{B}. \tag{A.81}$$

For a weight-w scalar function Φ, *i.e.* a function scaling with $\mathcal{B}^w$ under (A.80), we introduce

$$\hat{\mathscr{D}}_j \Phi = \hat{\partial}_j \Phi + w \varphi_j \Phi, \tag{A.82}$$

such that under a Weyl transformation

$$\hat{\mathscr{D}}_j \Phi \to \mathcal{B}^w \hat{\mathscr{D}}_j \Phi. \tag{A.83}$$

Similarly, for a vector with weight-w components V^l:

$$\hat{\mathscr{D}}_j V^l = \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta^l_j V^i \varphi_i. \tag{A.84}$$

The action on any other tensor is obtained using the Leibniz rule, as in example for rank-two tensors:

$$\hat{\mathscr{D}}_j t_{kl} = \hat{\nabla}_j t_{kl} + (w+2)\varphi_j t_{kl} + \varphi_k t_{jl} + \varphi_l t_{kj} - a_{jl} t_{ki} \varphi^i - a_{jk} t_{il} \varphi^i. \tag{A.85}$$

The Weyl–Carroll spatial derivative does not modify the weight of the tensor it acts on. Furthermore, it is metric as (a_{kl} has weight -2):

$$\hat{\mathscr{D}}_j a_{kl} = 0. \tag{A.86}$$

It has an effective torsion because

$$\left[\hat{\mathscr{D}}_i, \hat{\mathscr{D}}_j\right] \Phi = \frac{2}{\Omega} \varpi_{ij} \hat{\mathscr{D}}_t \Phi + w \Omega_{ij} \Phi, \tag{A.87}$$

although this expression does not contain terms of the type $\hat{\mathscr{D}}_k \Phi$. We have introduced here

$$\Omega_{ij} = \varphi_{ij} - \frac{2}{d} \varpi_{ij} \theta^{\mathrm{C}}, \tag{A.88}$$

where ϖ_{ij} are the components of the Carrollian vorticity defined in (A.64), and

$$\varphi_{ij} = \hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i. \tag{A.89}$$

Both Ω_{ij} and ϖ_{ij} are components of genuine Carrollian two-forms, and Weyl-covariant of weight 0 and -1. However, φ_{ij} are not Weyl-covariant, although they are also by construction components of a good Carrollian two-form.

In Eq. (A.87), we have used a Weyl–Carroll derivative with respect to time $\hat{\mathscr{D}}_t$. Its action on a weight-w function Φ is defined as:

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi = \frac{1}{\Omega}\hat{\partial}_t\Phi + \frac{w}{d}\theta^{\mathrm{C}}\Phi = \frac{1}{\Omega}\partial_t\Phi + \frac{w}{d}\theta^{\mathrm{C}}\Phi, \tag{A.90}$$

which is a scalar of weight $w+1$ under (A.80):

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi \to \mathcal{B}^{w+1}\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi. \tag{A.91}$$

Accordingly, on a weight-w vector the action of the Weyl–Carroll time derivative is

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t V^l = \frac{1}{\Omega}\hat{\partial}_t V^l + \frac{w-1}{d}\theta^{\mathrm{C}}V^l = \frac{1}{\Omega}\partial_t V^l + \frac{w}{d}\theta^{\mathrm{C}}V^l + \xi^{\mathrm{C}l}{}_i V^i. \tag{A.92}$$

These are the components of a genuine Carrollian vector of weight $w+1$ (the tensor $\xi^{\mathrm{C}l}{}_i$ is Weyl-covariant of weight 1). We have used (A.75), (A.76) and (A.73) for the second equalities in (A.90) and (A.92). The same pattern applies for any tensor by Leibniz rule, and in particular:

$$\hat{\mathscr{D}}_t a_{kl} = 0. \tag{A.93}$$

We will close the present appendix with the Weyl–Carroll curvature tensors, obtained by studying the commutation of Weyl–Carroll covariant derivatives acting on vectors. We find

$$\left[\hat{\mathscr{D}}_k, \hat{\mathscr{D}}_l\right] V^i = \left(\hat{\mathscr{R}}^i{}_{jkl} - 2\xi^{\mathrm{C}i}{}_j\varpi_{kl}\right) V^j + \varpi_{kl}\frac{2}{\Omega}\hat{\mathscr{D}}_t V^i + w\Omega_{kl}V^i, \tag{A.94}$$

where

$$\begin{aligned}\hat{\mathscr{R}}^i{}_{jkl} \quad = \quad & \hat{r}^i{}_{jkl} - \delta^i_j\varphi_{kl} - a_{jk}\hat{\nabla}_l\varphi^i + a_{jl}\hat{\nabla}_k\varphi^i + \delta^i_k\hat{\nabla}_l\varphi_j - \delta^i_l\hat{\nabla}_k\varphi_j \\ & + \varphi^i\left(\varphi_k a_{jl} - \varphi_l a_{jk}\right) - \left(\delta^i_k a_{jl} - \delta^i_l a_{jk}\right)\varphi_m\varphi^m + \left(\delta^i_k\varphi_l - \delta^i_l\varphi_k\right)\varphi_j\end{aligned} \tag{A.95}$$

are the components of the Riemann–Weyl–Carroll weight-0 tensor, from which we define

$$\hat{\mathscr{R}}_{ij} = \hat{\mathscr{R}}^k{}_{ikj}, \quad \hat{\mathscr{R}} = a^{ij}\hat{\mathscr{R}}_{ij}. \tag{A.96}$$

Notice that the Ricci–Weyl–Carroll tensor is *not* symmetric in general: $\hat{\mathscr{R}}_{ij} \neq \hat{\mathscr{R}}_{ji}$.

Eventually, we quote

$$\left[\frac{1}{\Omega}\hat{\mathscr{D}}_t, \hat{\mathscr{D}}_i\right]\Phi = w\hat{\mathscr{R}}_i\Phi - \xi^{\mathrm{C}j}{}_i\hat{\mathscr{D}}_j\Phi \tag{A.97}$$

and

$$\left[\frac{1}{\Omega}\hat{\mathscr{D}}_t, \hat{\mathscr{D}}_i\right]V^i = (w-d)\hat{\mathscr{R}}_i V^i - V^i\hat{\mathscr{D}}_j\xi^{\mathrm{C}j}{}_i - \xi^{\mathrm{C}j}{}_i\hat{\mathscr{D}}_j V^i, \tag{A.98}$$

with

$$\hat{\mathscr{R}}_i = \hat{r}_i + \frac{1}{\Omega}\hat{\partial}_t\varphi_i - \frac{1}{d}\hat{\nabla}_j\hat{\gamma}^j{}_i + \xi^{\mathrm{C}j}{}_i\varphi_j = \frac{1}{\Omega}\partial_t\varphi_i - \frac{1}{d}\left(\hat{\partial}_i + \varphi_i\right)\theta^{\mathrm{C}} \tag{A.99}$$

the components of a Weyl-covariant weight-1 Carrollian curvature one-form, where $\hat{r}_i$ is given in (A.79).

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Flat holography and Carrollian fluids

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Abstract

We show that a holographic description of four-dimensional asymptotically locally flat spacetimes is reached smoothly from the zero-cosmological-constant limit of anti-de Sitter holography. To this end, we use the derivative expansion of fluid/gravity correspondence. From the boundary perspective, the vanishing of the bulk cosmological constant appears as the zero velocity of light limit. This sets how Carrollian geometry emerges in flat holography. The new boundary data are a two-dimensional spatial surface, identified with the null infinity of the bulk Ricci-flat spacetime, accompanied with a Carrollian time and equipped with a Carrollian structure, plus the dynamical observables of a conformal Carrollian fluid. These are the energy, the viscous stress tensors and the heat currents, whereas the Carrollian geometry is gathered by a two-dimensional spatial metric, a frame connection and a scale factor. The reconstruction of Ricci-flat spacetimes from Carrollian boundary data is conducted with a flat derivative expansion, resummed in a closed form in Eddington–Finkelstein gauge under further integrability conditions inherited from the ancestor anti-de Sitter set-up. These conditions are hinged on a duality relationship among fluid friction tensors and Cotton-like geometric data. We illustrate these results in the case of conformal Carrollian perfect fluids and Robinson–Trautman viscous hydrodynamics. The former are dual to the asymptotically flat Kerr–Taub–NUT family, while the latter leads to the homonymous class of algebraically special Ricci-flat spacetimes.

Contents

1 Introduction

Ever since its conception, there have been many attempts to extend the original holographic anti-de Sitter correspondence along various directions, including asymptotically flat or de Sitter bulk spacetimes. Since the genuine microscopic correspondence based on type IIB string and maximally supersymmetric Yang–Mills theory is deeply rooted in the anti-de Sitter background, phenomenological extensions such as fluid/gravity correspondence have been considered as more promising for reaching a flat spacetime generalization.

The mathematical foundations of holography are based on the existence of the Fefferman–Graham expansion for asymptotically anti-de Sitter Einstein spaces [1, 2]. Indeed, on the one hand, putting an asymptotically anti-de Sitter Einstein metric in the Fefferman–Graham gauge allows to extract the two independent boundary data *i.e.* the boundary metric and the conserved boundary conformal energy–momentum tensor. On the other hand, given a pair of suitable boundary data the Fefferman–Graham expansion makes it possible to reconstruct, order by order, an Einstein space.

More recently, fluid/gravity correspondence has provided an alternative to Fefferman–Graham, known as derivative expansion [3–6]. It is inspired from the fluid derivative expansion (see *e.g.* [7, 8]), and is implemented in Eddington–Finkelstein coordinates. The metric of an Einstein spacetime is expanded in a light-like direction and the information on the boundary fluid is made available in a slightly different manner, involving explicitly a velocity field whose derivatives set the order of the expansion. Conversely, the boundary fluid data, including the fluid's congruence, allow to reconstruct an exact bulk Einstein spacetime.

Although less robust mathematically, the derivative expansion has several advantages over Fefferman–Graham. Firstly, under some particular conditions it can be resummed leading to algebraically special Einstein spacetimes in a closed form [9–14]. Such a resummation is very unlikely, if at all possible, in the context of Fefferman–Graham. Secondly, boundary geometrical terms appear packaged at specific orders in the derivative expansion, which is performed in Eddington–Finkelstein gauge. These terms feature precisely whether the bulk is asymptotically globally or locally anti-de Sitter. Thirdly, and contrary to Fefferman–Graham again, the derivative expansion admits a consistent limit of vanishing scalar curvature. Hence it appears to be applicable to Ricci-flat spacetimes and emerges as a valuable tool for setting up flat holography. Such a smooth behaviour is not generic, as in most coordinate systems switching off the scalar curvature for an Einstein space leads to plain Minkowski spacetime.[1]

The observations above suggest that it is relevant to wonder whether a Ricci-flat spacetime admits a dual fluid description. This can be recast into two sharp questions:

1. Which surface $\mathscr{S}$ would replace the AdS conformal boundary $\mathscr{I}$, and what is the geometry that this new boundary should be equipped with?

2. Which are the degrees of freedom hosted by $\mathscr{S}$ and succeeding the relativistic-fluid energy–momentum tensor, and what is the dynamics these degrees of freedom obey?

Many proposals have been made for answering these questions. Most of them were inspired by the seminal work [17, 18], where Navier–Stokes equations were shown to capture the dynamics of black-hole horizon perturbations. This result is taken as the crucial evidence regarding the deep relation between gravity, without cosmological constant, and fluid dynamics.

A more recent approach has associated Ricci-flat spacetimes in $d+1$ dimensions with d-dimensional fluids [19–24]. This is based on the observation that the Brown–York energy–momentum tensor on a Rindler hypersurface of a flat metric has the form of a perfect fluid [25]. In this particular framework, one can consider a non-relativistic limit, thus showing

[1] This phenomenon is well known in supergravity, when studying the gravity decoupling limit of scalar manifolds. For this limit to be non-trivial, one has to chose an appropriate gauge (see [15, 16] for a recent discussion and references).

that the Navier–Stokes equations coincide with Einstein's equations on the Rindler hypersurface. Paradoxically, it has simultaneously been argued that all information can be stored in a relativistic d-dimensional fluid.

Outside the realm of fluid interpretation, and on the more mathematical side of the problem, some solid works regarding flat holography are [26–28] (see also [29]). The dual theories reside at null infinity emphasizing the importance of the null-like formalisms of [30–32]. In this line of thought, results where also reached focusing on the expected symmetries, in particular for the specific case of three-dimensional bulk versus two-dimensional boundary [33–39].[2] These achievements *are not* unconditionally transferable to four or higher dimensions, and can possibly infer inaccurate expectations due to features holding exclusively in three dimensions.

The above wanderings between relativistic and non-relativistic fluid dynamics in relation with Ricci-flat spacetimes are partly due to the incomplete understanding on the rôle played by the null infinity. On the one hand, it has been recognized that the Ricci-flat limit is related to some contraction of the Poincaré algebra [33–37, 40, 41]. On the other hand, this observation was tempered by a potential confusion among the Carrollian algebra and its dual contraction, the conformal Galilean algebra, as they both lead to the decoupling of time. This phenomenon was exacerbated by the equivalence of these two algebras in two dimensions, and has somehow obscured the expectations on the nature and the dynamics of the relevant boundary degrees of freedom. Hence, although the idea of localizing the latter on the spatial surface at null infinity was suggested (as *e.g.* in [42–45]), their description has often been accustomed to the relativistic-fluid or the conformal-field-theory approaches, based on the revered energy–momentum tensor and its conservation law.[3]

From this short discussion, it is clear that the attempts implemented so far follow different directions without clear overlap and common views. Although implicitly addressed in the literature, the above two questions have not been convincingly answered, and the treatment of boundary theories in the zero cosmological constant limit remains nowadays tangled.

In this work we make a precise statement, which clarifies unquestionably the situation. Our starting point is a four-dimensional bulk Einstein spacetime with $\Lambda = -3k^2$, dual to a boundary relativistic fluid. In this set-up, we consider the $k \to 0$ limit, which has the following features:

- The derivative expansion is generically well behaved. We will call its limit the *flat derivative expansion*. Under specified conditions it can be resummed in a closed form.

- Inside the boundary metric, and in the complete boundary fluid dynamics, k plays the

[2] Reference [37] is the first where a consistent and non-trivial $k \to 0$ limit was taken, mapping the entire family of three-dimensional Einstein spacetimes (locally AdS) to the family of Ricci–flat solutions (locally flat).

[3] This is manifest in the very recent work of Ref. [46].

rôle of *velocity of light*. Its vanishing is thus a *Carrollian limit*.

- The boundary is the two-dimensional *spatial* surface $\mathscr{S}$ emerging as the future null infinity of the limiting Ricci-flat bulk spacetime. It replaces the AdS conformal boundary and is endowed with a *Carrollian geometry i.e.* is covariant under *Carrollian diffeomorphisms*.

- The degrees of freedom hosted by this surface are captured by a *conformal Carrollian fluid* : energy density and pressure related by a conformal equation of state, heat currents and traceless viscous stress tensors. These macroscopic degrees of freedom obey *conformal Carrollian fluid dynamics*.

Any two-dimensional conformal Carrollian fluid hosted by an arbitrary spatial surface $\mathscr{S}$, and obeying conformal Carrollian fluid dynamics on this surface, is therefore mapped onto a Ricci-flat four-dimensional spacetime using the flat derivative expansion. The latter is invariant under boundary Weyl transformations. Under a set of resummability conditions involving the Carrollian fluid and its host $\mathscr{S}$, this derivative expansion allows to reconstruct exactly algebraically special Ricci-flat spacetimes. The results summarized above answer in the most accurate manner the two questions listed earlier.

Carrollian symmetry has sporadically attracted attention following the pioneering work or Ref. [47], where the Carroll group emerged as a new contraction of the Poincaré group: the ultra-relativistic contraction, dual to the usual non-relativistic one leading to the Galilean group. Its conformal extensions were explored latterly [48–51], showing in particular its relationship to the BMS group, which encodes the asymptotic symmetries of asymptotically flat spacetimes along a null direction [53–56].[4]

It is therefore quite natural to investigate on possible relationships between Carrollian asymptotic structure and flat holography and, by the logic of fluid/gravity correspondence, to foresee the emergence of Carrollian hydrodynamics rather than any other, relativistic or Galilean fluid. Nonetheless searches so far have been oriented towards the near-horizon membrane paradigm, trying to comply with the inevitable BMS symmetries as in [59, 60]. The power of the derivative expansion and its flexibility to handle the zero-k limit has been somehow dismissed. This expansion stands precisely at the heart of our method. Its actual implementation requires a comprehensive approach to Carrollian hydrodynamics, as it emanates from the ultra-relativistic limit of relativistic fluid dynamics, made recently available in [52].

The aim of the present work is to provide a detailed analysis of the various statements presented above, and exhibit a precise expression for the Ricci-flat line element as reconstructed from the boundary Carrollian geometry and Carrollian fluid dynamics. As already

[4] Carroll symmetry has also been explored in connection to the tensionless-string limit, see *e.g.* [57, 58].

stated, the tool for understanding and implementing operationally these ideas is the derivative expansion and, under conditions, its resummed version. For this reason, Sec. 2 is devoted to its thorough description in the framework of ordinary anti-de Sitter fluid/gravity holography. This chapter includes the conditions, stated in a novel fashion with respect to [12, 13], for the expansion to be resummed in a closed form, representing generally an Einstein spacetime of algebraically special Petrov type.

In Sec. 3 we discuss how the Carrollian geometry emerges at null infinity and describe in detail conformal Carrollian hydrodynamics following [52]. The formulation of the Ricci-flat derivative expansion is undertaken in Sec. 4. Here we discuss the important issue of resumming in a closed form the generic expansion. This requires the investigation of another uncharted territory: the higher-derivative curvature-like Carrollian tensors. The Carrollian geometry on the spatial boundary $\mathscr{S}$ is naturally equipped with a (conformal) Carrollian connection, which comes with various curvature tensors presented in Sec. 3. The relevant object for discussing the resummability in the anti-de Sitter case is the Cotton tensor, as reviewed in Sec. 2. It turns out that this tensor has well-defined Carrollian descendants, which we determine and exploit. With those, the resummability conditions are well-posed and set the framework for obtaining exact Ricci-flat spacetimes in a closed form from conformal-Carrollian-fluid data.

In order to illustrate our results, we provide examples starting from Sec. 3 and pursuing systematically in Sec. 5. Generic Carrollian perfect fluids are meticulously studied and shown to be dual to the general Ricci-flat Kerr–Taub–NUT family. The non perfect Carrollian fluid called Robinson–Trautman fluid is discussed both as the limiting Robinson–Trautman relativistic fluid (Sec. 3), and alternatively from Carrollian first principles (Sec. 5, following [52]). It is shown to be dual to the Ricci-flat Robinson–Trautman spacetime, of which the line element is obtained thanks to our flat resummation procedure.

One of the resummability requirements is the absence of shear for the Carrollian fluid. This is a geometric quantity, which, if absent, makes possible for using holomorphic coordinates. In App. A, we gather the relevant formulas in this class of coordinates.

2 Fluid/gravity in asymptotically locally AdS spacetimes

We present here an executive summary of the holographic reconstruction of four-dimensional asymptotically locally anti-de Sitter spacetimes from three-dimensional relativistic boundary fluid dynamics. The tool we use is the fluid-velocity derivative expansion. We show that exact Einstein spacetimes written in a closed form can arise by resumming this expansion. It appears that the key conditions allowing for such an explicit resummation are the absence of shear in the fluid flow, as well as the relationship among the non-perfect components of the fluid energy–momentum tensor (*i.e.* the heat current and the viscous stress tensor) and the boundary Cotton tensor.

2.1 The derivative expansion

The spirit

Due to the Fefferman–Graham ambient metric construction [61], asymptotically locally anti-de Sitter four-dimensional spacetimes are determined by a set of independent boundary data, namely a three-dimensional metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ and a rank-2 tensor $\mathrm{T} = T_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, symmetric ($T_{\mu\nu} = T_{\nu\mu}$), traceless ($T^\mu{}_\mu = 0$) and conserved:

$$\nabla^\mu T_{\mu\nu} = 0. \tag{2.1}$$

Perhaps the most well known subclass of asymptotically locally AdS spacetimes are those whose boundary metrics are conformally flat (see *e.g.* [62, 63]). These are asymptotically *globally* anti-de Sitter. The asymptotic symmetries of such spacetimes comprise the finite dimensional conformal group, *i.e.* $SO(3,2)$ in four dimensions [64], and AdS/CFT is at work giving rise to a boundary conformal field theory. Then, the rank-2 tensor $T_{\mu\nu}$ is interpreted as the expectation value over a boundary quantum state of the conformal-field-theory energy–momentum tensor. Whenever hydrodynamic regime is applicable, this approach gives rise to the so-called fluid/gravity correspondence and all its important spinoffs (see [4] for a review).

For a long time, all the work on fluid/gravity correspondence was confined to asymptotically globally AdS spacetimes, hence to holographic boundary fluids that flow on conformally flat backgrounds. In a series of works [9–14] we have extended the fluid/gravity correspondence into the realm of asymptotically locally AdS_4 spacetimes. In the following, we present and summarize our salient findings.

The energy–momentum tensor

Given the energy–momentum tensor of the boundary fluid and assuming that it represents a state in a hydrodynamic regime, one should be able to pick a boundary congruence u, playing the rôle of fluid velocity. Normalizing the latter as[5] $\|\mathrm{u}\|^2 = -k^2$ we can in general decompose the energy–momentum tensor as

$$T_{\mu\nu} = (\varepsilon + p)\frac{u_\mu u_\nu}{k^2} + p g_{\mu\nu} + \tau_{\mu\nu} + \frac{u_\mu q_\nu}{k^2} + \frac{u_\nu q_\mu}{k^2}. \tag{2.2}$$

[5] This unconventional normalization ensures that the derivative expansion is well-behaved in the $k \to 0$ limit. In the language of fluids, it naturally incorporates the scaling introduced in [37] – see footnote 2.

We assume local thermodynamic equilibrium with p the local pressure and ε the local energy density:

$$\varepsilon = \frac{1}{k^2} T_{\mu\nu} u^\mu u^\nu. \tag{2.3}$$

A local-equilibrium thermodynamic equation of state $p = p(T)$ is also needed for completing the system, and we omit the chemical potential as no independent conserved current, *i.e.* no gauge field in the bulk, is considered here.

The symmetric viscous stress tensor $\tau_{\mu\nu}$ and the heat current q_μ are purely transverse:

$$u^\mu \tau_{\mu\nu} = 0, \quad u^\mu q_\mu = 0, \quad q_\nu = -\varepsilon u_\nu - u^\mu T_{\mu\nu}. \tag{2.4}$$

For a conformal fluid in 3 dimensions

$$\varepsilon = 2p, \quad \tau^\mu{}_\mu = 0. \tag{2.5}$$

The quantities at hand are usually expressed as expansions in temperature and velocity derivatives, the coefficients of which characterize the transport phenomena occurring in the fluid. In first-order hydrodynamics

$$\tau_{(1)\mu\nu} = -2\eta\sigma_{\mu\nu} - \zeta h_{\mu\nu}\Theta, \tag{2.6}$$

$$q_{(1)\mu} = -\kappa h_\mu{}^\nu \left(\partial_\nu T + \frac{T}{k^2} a_\nu\right), \tag{2.7}$$

where $h_{\mu\nu}$ is the projector onto the space transverse to the velocity field:

$$h_{\mu\nu} = \frac{u_\mu u_\nu}{k^2} + g_{\mu\nu}, \tag{2.8}$$

and[6]

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \tag{2.9}$$

$$\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)} + \tfrac{1}{k^2} u_{(\mu} a_{\nu)} - \tfrac{1}{2}\Theta h_{\mu\nu}, \tag{2.10}$$

$$\omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]} + \tfrac{1}{k^2} u_{[\mu} a_{\nu]}, \tag{2.11}$$

are the acceleration (transverse), the expansion, the shear and the vorticity (both rank-two tensors are transverse and traceless). As usual, η, ζ are the shear and bulk viscosities, and κ is the thermal conductivity.

It is customary to introduce the vorticity two-form

$$\omega = \frac{1}{2}\omega_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}\left(\mathrm{du} + \frac{1}{k^2}\mathrm{u} \wedge \mathrm{a}\right), \tag{2.12}$$

[6]Our conventions for (anti-) symmetrization are: $A_{(\mu\nu)} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)$ and $A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right)$.

as well as the Hodge–Poincaré dual of this form, which is proportional to u (we are in $2+1$ dimensions):

$$k\gamma \mathrm{u} = \star\omega \quad \Leftrightarrow \quad k\gamma u_\mu = \frac{1}{2}\eta_{\mu\nu\sigma}\omega^{\nu\sigma}, \tag{2.13}$$

where $\eta_{\mu\nu\sigma} = \sqrt{-g}\epsilon_{\mu\nu\sigma}$. In this expression γ is a scalar, that can also be expressed as

$$\gamma^2 = \frac{1}{2k^4}\omega_{\mu\nu}\omega^{\mu\nu}. \tag{2.14}$$

In three spacetime dimensions and in the presence of a vector field, one naturally defines a fully antisymmetric two-index tensor as

$$\eta_{\mu\nu} = -\frac{u^\rho}{k}\eta_{\rho\mu\nu}, \tag{2.15}$$

obeying

$$\eta_{\mu\sigma}\eta_\nu{}^\sigma = h_{\mu\nu}. \tag{2.16}$$

With this tensor the vorticity reads:

$$\omega_{\mu\nu} = k^2\gamma\eta_{\mu\nu}. \tag{2.17}$$

Weyl covariance, Weyl connection and the Cotton tensor

In the case when the boundary metric $g_{\mu\nu}$ is conformally flat, it was shown that using the above set of boundary data it is possible to reconstruct the four-dimensional bulk Einstein spacetime order by order in derivatives of the velocity field [3–6]. The guideline for the spacetime reconstruction based on the derivative expansion is *Weyl covariance*: the bulk geometry should be insensitive to a conformal rescaling of the boundary metric (weight -2)

$$\mathrm{d}s^2 \to \frac{\mathrm{d}s^2}{\mathcal{B}^2}, \tag{2.18}$$

which should correspond to a bulk diffeomorphism and be reabsorbed into a redefinition of the radial coordinate: $r \to \mathcal{B}r$. At the same time, u_μ is traded for $u_\mu/\mathcal{B}$ (velocity one-form), $\omega_{\mu\nu}$ for $\omega_{\mu\nu}/\mathcal{B}$ (vorticity two-form) and $T_{\mu\nu}$ for $\mathcal{B}T_{\mu\nu}$. As a consequence, the pressure and energy density have weight 3, the heat-current q_μ weight 2, and the viscous stress tensor $\tau_{\mu\nu}$ weight 1.

Covariantization with respect to rescaling requires to introduce a Weyl connection one-form:[7]

$$\mathrm{A} = \frac{1}{k^2}\left(\mathrm{a} - \frac{\Theta}{2}\mathrm{u}\right), \tag{2.19}$$

which transforms as $\mathrm{A} \to \mathrm{A} - \mathrm{d}\ln\mathcal{B}$. Ordinary covariant derivatives ∇ are thus traded

[7] The explicit form of A is obtained by demanding $\mathscr{D}_\mu u^\mu = 0$ and $u^\lambda \mathscr{D}_\lambda u_\mu = 0$.

for Weyl covariant ones $\mathscr{D} = \nabla + w\,\mathrm{A}$, w being the conformal weight of the tensor under consideration. We provide for concreteness the Weyl covariant derivative of a weight-w form v_μ:

$$\mathscr{D}_\nu v_\mu = \nabla_\nu v_\mu + (w+1)A_\nu v_\mu + A_\mu v_\nu - g_{\mu\nu}A^\rho v_\rho. \tag{2.20}$$

The Weyl covariant derivative is metric with effective torsion:

$$\mathscr{D}_\rho g_{\mu\nu} = 0, \tag{2.21}$$

$$\left(\mathscr{D}_\mu \mathscr{D}_\nu - \mathscr{D}_\nu \mathscr{D}_\mu\right) f = w f F_{\mu\nu}, \tag{2.22}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.23}$$

is Weyl-invariant.

Commuting the Weyl-covariant derivatives acting on vectors, as usual one defines the Weyl covariant Riemann tensor

$$\left(\mathscr{D}_\mu \mathscr{D}_\nu - \mathscr{D}_\nu \mathscr{D}_\mu\right) V^\rho = \mathscr{R}^\rho{}_{\sigma\mu\nu} V^\sigma + w V^\rho F_{\mu\nu} \tag{2.24}$$

(V^ρ are weight-w) and the usual subsequent quantities. In three spacetime dimensions, the covariant Ricci (weight 0) and the scalar (weight 2) curvatures read:

$$\mathscr{R}_{\mu\nu} = R_{\mu\nu} + \nabla_\nu A_\mu + A_\mu A_\nu + g_{\mu\nu}\left(\nabla_\lambda A^\lambda - A_\lambda A^\lambda\right) - F_{\mu\nu}, \tag{2.25}$$

$$\mathscr{R} = R + 4\nabla_\mu A^\mu - 2A_\mu A^\mu. \tag{2.26}$$

The Weyl-invariant Schouten tensor[8] is

$$\mathscr{S}_{\mu\nu} = \mathscr{R}_{\mu\nu} - \frac{1}{4}\mathscr{R} g_{\mu\nu} = S_{\mu\nu} + \nabla_\nu A_\mu + A_\mu A_\nu - \frac{1}{2}A_\lambda A^\lambda g_{\mu\nu} - F_{\mu\nu}. \tag{2.27}$$

Other Weyl-covariant velocity-related quantities are

$$\begin{aligned} \mathscr{D}_\mu u_\nu &= \nabla_\mu u_\nu + \frac{1}{k^2}u_\mu a_\nu - \frac{\Theta}{2}h_{\mu\nu} \\ &= \sigma_{\mu\nu} + \omega_{\mu\nu}, \end{aligned} \tag{2.28}$$

$$\mathscr{D}_\nu \omega^\nu{}_\mu = \nabla_\nu \omega^\nu{}_\mu, \tag{2.29}$$

$$\mathscr{D}_\nu \eta^\nu{}_\mu = 2\gamma u_\mu, \tag{2.30}$$

$$u^\lambda \mathscr{R}_{\lambda\mu} = \mathscr{D}_\lambda\left(\sigma^\lambda{}_\mu - \omega^\lambda{}_\mu\right) - u^\lambda F_{\lambda\mu}, \tag{2.31}$$

of weights -1, 1, 0 and 1 (the scalar vorticity γ has weight 1).

The remarkable addition to the fluid/gravity dictionary came with the realization that

[8] The ordinary Schouten tensor in three spacetime dimensions is given by $R_{\mu\nu} - \frac{1}{4}R g_{\mu\nu}$.

the derivative expansion can be used to reconstruct Einstein metrics which are asymptotically locally AdS. For the latter, the boundary metric has a non zero Cotton tensor [9–13]. The Cotton tensor is generically a three-index tensor with mixed symmetries. In three dimensions, which is the case for our boundary geometry, the Cotton tensor can be dualized into a two-index, symmetric and traceless tensor. It is defined as

$$C_{\mu\nu} = \eta_{\mu}{}^{\rho\sigma}\mathscr{D}_{\rho}\left(\mathscr{S}_{\nu\sigma} + F_{\nu\sigma}\right) = \eta_{\mu}{}^{\rho\sigma}\nabla_{\rho}\left(R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}\right). \tag{2.32}$$

The Cotton tensor is Weyl-covariant of weight 1 (*i.e.* transforms as $C_{\mu\nu} \to \mathcal{B}C_{\mu\nu}$), and is *identically* conserved:

$$\mathscr{D}_{\rho}C^{\rho}{}_{\nu} = \nabla_{\rho}C^{\rho}{}_{\nu} = 0, \tag{2.33}$$

sharing thereby all properties of the energy–momentum tensor. Following (2.2) we can decompose the Cotton tensor into longitudinal, transverse and mixed components with respect to the fluid velocity u:[9]

$$C_{\mu\nu} = \frac{3c}{2}\frac{u_{\mu}u_{\nu}}{k} + \frac{ck}{2}g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{u_{\mu}c_{\nu}}{k} + \frac{u_{\nu}c_{\mu}}{k}. \tag{2.34}$$

Such a decomposition naturally defines the weight-3 *Cotton scalar density*

$$c = \frac{1}{k^3}C_{\mu\nu}u^{\mu}u^{\nu}, \tag{2.35}$$

as the longitudinal component. The symmetric and traceless *Cotton stress tensor* $c_{\mu\nu}$ and the *Cotton current* c_{μ} (weights 1 and 2, respectively) are purely transverse:

$$c_{\mu}{}^{\mu} = 0, \quad u^{\mu}c_{\mu\nu} = 0, \quad u^{\mu}c_{\mu} = 0, \tag{2.36}$$

and obey

$$c_{\mu\nu} = -kh^{\rho}{}_{\mu}h^{\sigma}{}_{\nu}C_{\rho\sigma} + \frac{ck^2}{2}h_{\mu\nu}, \quad c_{\nu} = -cu_{\nu} - \frac{u^{\mu}C_{\mu\nu}}{k}. \tag{2.37}$$

One can use the definition (2.32) to further express the Cotton density, current and stress tensor as ordinary or Weyl derivatives of the curvature. We find

$$\begin{aligned} c &= \frac{1}{k^2}u^{\nu}\eta^{\sigma\rho}\mathscr{D}_{\rho}\left(\mathscr{S}_{\nu\sigma} + F_{\nu\sigma}\right), & (2.38)\\ c_{\nu} &= \eta^{\rho\sigma}\mathscr{D}_{\rho}\left(\mathscr{S}_{\nu\sigma} + F_{\nu\sigma}\right) - cu_{\nu}, & (2.39)\\ c_{\mu\nu} &= -h^{\lambda}{}_{\mu}\left(k\eta_{\nu}{}^{\rho\sigma} - u_{\nu}\eta^{\rho\sigma}\right)\mathscr{D}_{\rho}\left(\mathscr{S}_{\lambda\sigma} + F_{\lambda\sigma}\right) + \frac{ck^2}{2}h_{\mu\nu}. & (2.40) \end{aligned}$$

[9]Notice that the energy–momentum tensor has an extra factor of k with respect to the Cotton tensor, see (2.60), due to their different dimensions.

The bulk Einstein derivative expansion

Given the ingredients above, the leading terms in a $1/r$ expansion for a four-dimensional Einstein metric are of the form:[10]

$$
\begin{aligned}
\mathrm{d}s^2_{\text{bulk}} &= 2\frac{\mathrm{u}}{k^2}(\mathrm{d}r + r\mathrm{A}) + r^2\mathrm{d}s^2 + \frac{\mathrm{S}}{k^4} \\
&\quad + \frac{\mathrm{u}^2}{k^4 r^2}\left(1 - \frac{1}{2k^4 r^2}\omega_{\alpha\beta}\omega^{\alpha\beta}\right)\left(\frac{8\pi G T_{\lambda\mu}u^\lambda u^\mu}{k^2} r + \frac{C_{\lambda\mu}u^\lambda \eta^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^4}\right) \\
&\quad + \text{terms with } \sigma, \sigma^2, \nabla\sigma, \ldots + \mathrm{O}\left(\mathscr{D}^4\mathrm{u}\right).
\end{aligned}
\tag{2.41}
$$

In this expression

- S is a Weyl-invariant tensor:

$$
\mathrm{S} = S_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -2\mathrm{u}\mathscr{D}_\nu \omega^\nu{}_\mu \mathrm{d}x^\mu - \omega_\mu{}^\lambda \omega_{\lambda\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu - \mathrm{u}^2\frac{\mathscr{R}}{2};
\tag{2.42}
$$

- the boundary metric is parametrized *à la* Randers–Papapetrou:

$$
\mathrm{d}s^2 = -k^2\left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j;
\tag{2.43}
$$

- the boundary conformal fluid velocity field and the corresponding one form are

$$
\mathrm{u} = \frac{1}{\Omega}\partial_t \quad \Leftrightarrow \quad \mathrm{u} = -k^2\left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right),
\tag{2.44}
$$

  i.e. the fluid is at rest in the frame associated with the coordinates in (2.43) – this is not a limitation, as one can always choose a local frame where the fluid is at rest, in which the metric reads (2.43) (with Ω, b_i and a_{ij} functions of all coordinates);

- $\omega_{\mu\nu}$ is the vorticity of u as given in (2.11), which reads:

$$
\omega = \frac{1}{2}\omega_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{k^2}{2}\left(\partial_i b_j + \frac{1}{\Omega}b_i\partial_j\Omega + \frac{1}{\Omega}b_i\partial_t b_j\right)\mathrm{d}x^i \wedge \mathrm{d}x^j;
\tag{2.45}
$$

- $\gamma^2 = \frac{1}{2}a^{ik}a^{jl}\left(\partial_{[i}b_{j]} + \frac{1}{\Omega}b_{[i}\partial_{j]}\Omega + \frac{1}{\Omega}b_{[i}\partial_t b_{j]}\right)\left(\partial_{[k}b_{l]} + \frac{1}{\Omega}b_{[k}\partial_{l]}\Omega + \frac{1}{\Omega}b_{[k}\partial_t b_{l]}\right);$

[10] We have traded here the usual advanced-time coordinate used in the quoted literature on fluid/gravity correspondence for the retarded time, spelled t (see (2.44)).

- the expansion and acceleration are

$$\Theta = \frac{1}{\Omega}\partial_t \ln\sqrt{a}, \tag{2.46}$$

$$\mathrm{a} = k^2\left(\partial_i \ln\Omega + \frac{1}{\Omega}\partial_t b_i\right)\mathrm{d}x^i, \tag{2.47}$$

leading to the Weyl connection

$$\mathrm{A} = \frac{1}{\Omega}\left(\partial_i\Omega + \partial_t b_i - \frac{1}{2}b_i\partial_t \ln\sqrt{a}\right)\mathrm{d}x^i + \frac{1}{2}\partial_t \ln\sqrt{a}\,\mathrm{d}t, \tag{2.48}$$

with a the determinant of a_{ij};

- $\frac{1}{k^2}T_{\mu\nu}u^\mu u^\nu$ is the energy density ε of the fluid (see (2.3)), and in the Randers–Papapetrou frame associated with (2.43), (2.44), q_0, τ_{00} , $\tau_{0i} = \tau_{i0}$ entering in (2.2) all vanish due to (2.4);

- $\frac{1}{2k^4}C_{\lambda\mu}u^\lambda\eta^{\mu\nu\sigma}\omega_{\nu\sigma} = c\gamma$, where we have used (2.13) and (2.35), and similarly $c_0 = c_{00} = c_{0i} = c_{i0} = 0$ as a consequence of (2.36) with (2.43), (2.44);

- σ, σ^2, $\nabla\sigma$ stand for the shear of u and combinations of it, as computed from (2.10):

$$\sigma = \frac{1}{2\Omega}\left(\partial_t a_{ij} - a_{ij}\partial_t \ln\sqrt{a}\right)\mathrm{d}x^i\mathrm{d}x^j. \tag{2.49}$$

We have not exhibited explicitly shear-related terms because we will ultimately assume the absence of shear for our congruence. This raises the important issue of choosing the fluid velocity field, not necessary in the Fefferman–Graham expansion, but fundamental here. In relativistic fluids, the absence of sharp distinction between heat and matter fluxes leaves a freedom in setting the velocity field. This choice of *hydrodynamic frame* is not completely arbitrary though, and one should stress some reservations, which are often dismissed, in particular in the already quoted fluid/gravity literature.

As was originally exposed in [65] and extensively discussed *e.g.* in [7], the fluid-velocity ambiguity is well posed in the presence of a conserved current J, naturally decomposed into a longitudinal perfect piece and a transverse part:

$$J^\mu = \varrho u^\mu + j^\mu. \tag{2.50}$$

The velocity freedom originates from the redundancy in the heat current q and the non-perfect piece of the matter current j. One may therefore set $\mathrm{j} = 0$ and reach the Eckart frame. Alternatively $\mathrm{q} = 0$ defines the Landau–Lifshitz frame. In the absence of matter current, nothing guarantees that one can still move to the Landau–Lifshitz frame, and setting $\mathrm{q} = 0$ appears as a constraint on the fluid, rather than a choice of frame for describing arbitrary fluids. This important issue was recently discussed in the framework of holography [66], from which it is clear that setting $\mathrm{q} = 0$ in the absence of a conserved current would simply inhibit certain classes of Einstein spaces to emerge holographically from boundary data, and possibly blur the physical phenomena occurring in the fluids under consideration. Consequently, we will not make any such assumption, keeping the heat current as part of the physical data.

We would like to close this section with an important comment on asymptotics. The reconstructed bulk spacetime can be asymptotically locally or globally anti-de Sitter. This property is read off directly inside terms appearing at designated orders in the radial expansion, and built over specific boundary tensors. For $d+1$-dimensional boundaries, the boundary energy–momentum contribution first appears at order $1/r^{d-1}$, whereas the boundary Cotton tensor[11] emerges at order $1/r^2$. This behaviour is rooted in the Eddington–Finkelstein gauge used in (2.41), but appears also in the slightly different Bondi gauge. It is however absent in the Fefferman–Graham coordinates, where the Cotton cannot be possibly isolated in the expansion.

2.2 The resummation of AdS spacetimes

Resummation and exact Einstein spacetimes in closed form

In order to further probe the derivative expansion (2.41), we will impose the fluid velocity congruence be shearless. This choice has the virtue of reducing considerably the number of terms compatible with conformal invariance in (2.41), and potentially making this expansion resummable, thus leading to an Einstein metric written in a closed form. Nevertheless, this shearless condition reduces the class of Einstein spacetimes that can be reconstructed holographically to the algebraically special ones [10–14]. Going beyond this class is an open problem that we will not address here.

Following [6, 10–14], it is tempting to try a resummation of (2.41) using the following substitution:

$$1 - \frac{\gamma^2}{r^2} \to \frac{r^2}{\rho^2} \tag{2.51}$$

with

$$\rho^2 = r^2 + \gamma^2. \tag{2.52}$$

The resummed expansion would then read

$$\boxed{\mathrm{d}s^2_{\text{res. Einstein}} = 2\frac{\mathrm{u}}{k^2}(\mathrm{d}r + r\mathrm{A}) + r^2\mathrm{d}s^2 + \frac{\mathrm{S}}{k^4} + \frac{\mathrm{u}^2}{k^4\rho^2}\left(8\pi G\varepsilon r + c\gamma\right),} \tag{2.53}$$

which is indeed written in a closed form. Under the conditions listed below, the metric (2.53)

[11] Actually, the object appearing in generic dimension is the Weyl divergence of the boundary Weyl tensor, which contains also the Cotton tensor (see [67] for a preliminary discussion on this point).

defines the line element of an *exact* Einstein space with $\Lambda = -3k^2$.

- *The congruence* u *is shearless.* This requires (see (2.49))

$$\partial_t a_{ij} = a_{ij}\partial_t \ln \sqrt{a}. \tag{2.54}$$

Actually (2.54) is equivalent to ask that the two-dimensional spatial section $\mathscr{S}$ defined at every time t and equipped with the metric $\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is conformally flat. This may come as a surprise because every two-dimensional metric is conformally flat. However, a_{ij} generally depends on space $\mathbf{x}$ *and* time t, and the transformation required to bring it in a form proportional to the flat-space metric might depend on time. This would spoil the three-dimensional structure (2.43) and alter the *a priori* given u. Hence, $\mathrm{d}\ell^2$ is conformally flat within the three-dimensional spacetime (2.43) under the condition that the transformation used to reach the explicit conformally flat form be of the type $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$. This exists if and only if (2.54) is satisfied.[12] Under this condition, one can always choose $\zeta = \zeta(\mathbf{x})$, $\bar{\zeta} = \bar{\zeta}(\mathbf{x})$ such that

$$\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta} \tag{2.55}$$

with $P = P(t,\zeta,\bar{\zeta})$ a real function. Even though this does not hold for arbitrary $\mathrm{u} = \partial_t/\Omega$, one can show that there exists always a congruence for which it does [68], and this will be chosen for the rest of the paper.

- *The heat current of the boundary fluid introduced in* (2.2) *and* (2.4) *is identified with the transverse-dual of the Cotton current defined in* (2.34) *and* (2.37). The Cotton current being transverse to u, it defines a field on the conformally flat two-surface $\mathscr{S}$, the existence of which is guaranteed by the absence of shear. This surface is endowed with a natural hodge duality mapping a vector onto another, which can in turn be lifted back to the three-dimensional spacetime as a new transverse vector. This whole process is taken care of by the action of $\eta^\nu{}_\mu$ defined in (2.15):

$$\boxed{q_\mu = \frac{1}{8\pi G}\eta^\nu{}_\mu c_\nu = \frac{1}{8\pi G}\eta^\nu{}_\mu \eta^{\rho\sigma}\mathscr{D}_\rho\left(\mathscr{S}_{\nu\sigma} + F_{\nu\sigma}\right),} \tag{2.56}$$

where we used (2.39) in the last expression. Using holomorphic and antiholomorphic coordinates $\zeta,\bar{\zeta}$ as in (2.55)[13] leads to $\eta^\zeta{}_\zeta = \mathrm{i}$ and $\eta^{\bar{\zeta}}{}_{\bar{\zeta}} = -\mathrm{i}$, and thus

$$\mathrm{q} = \frac{\mathrm{i}}{8\pi G}\left(c_\zeta \mathrm{d}\zeta - c_{\bar{\zeta}}\mathrm{d}\bar{\zeta}\right). \tag{2.57}$$

[12] A peculiar subclass where this works is when ∂_t is a Killing field.

[13] Orientation is chosen such that in the coordinate frame $\eta_{0\zeta\bar{\zeta}} = \sqrt{-g}\,\epsilon_{0\zeta\bar{\zeta}} = \frac{\mathrm{i}\Omega}{P^2}$, where $x^0 = kt$.

- *The viscous stress tensor of the boundary conformal fluid introduced in* (2.2) *is identified with the transverse-dual of the Cotton stress tensor defined in* (2.34) *and* (2.37). Following the same pattern as for the heat current, we obtain:

$$\boxed{\begin{array}{rcl}\tau_{\mu\nu} & = & -\frac{1}{8\pi G k^2}\eta^{\rho}{}_{\mu}c_{\rho\nu} \\ & = & \frac{1}{8\pi G k^2}\left(-\frac{1}{2}u^{\lambda}\eta_{\mu\nu}\eta^{\rho\sigma}+\eta^{\lambda}_{\mu}\left(k\eta_{\nu}{}^{\rho\sigma}-u_{\nu}\eta^{\rho\sigma}\right)\right)\mathscr{D}_{\rho}\left(\mathscr{S}_{\lambda\sigma}+F_{\lambda\sigma}\right),\end{array}} \tag{2.58}$$

  where we also used (2.40) in the last equality. The viscous stress tensor $\tau_{\mu\nu}$ is transverse symmetric and traceless because these are the properties of the Cotton stress tensor $c_{\mu\nu}$. Similarly, we find in complex coordinates:

$$\tau=-\frac{\mathrm{i}}{8\pi G k^2}\left(c_{\zeta\zeta}\mathrm{d}\zeta^2-c_{\bar{\zeta}\bar{\zeta}}\mathrm{d}\bar{\zeta}^2\right). \tag{2.59}$$

- *The energy–momentum tensor defined in* (2.2) *with* $p=\varepsilon/2$, *heat current as in* (2.56) *and viscous stress tensor as in* (2.58) *must be conserved, i.e. obey Eq.* (2.1). These are differential constraints that from a bulk perspective can be thought of as a generalization of the Gauss law.

Identifying parts of the energy–momentum tensor with the Cotton tensor may be viewed as setting integrability conditions, similar to the electric–magnetic duality conditions in electromagnetism, or in Euclidean gravitational dynamics. As opposed to the latter, it is here implemented in a rather unconventional manner, on the conformal boundary.

It is important to emphasize that the conservation equations (2.1) concern *all* boundary data. On the fluid side the only remaining unknown piece is the energy density $\varepsilon(x)$, whereas for the boundary metric $\Omega(x)$, $b_i(x)$ and $a_{ij}(x)$ are available and must obey (2.1), together with $\varepsilon(x)$. Given these ingredients, (2.1) turns out to be precisely the set of equations obtained by demanding bulk Einstein equations be satisfied with the metric (2.53). This observation is at the heart of our analysis.

The bulk algebraic structure and the physics of the boundary fluid

The pillars of our approach are (i) the requirement of a shearless fluid congruence and (ii) the identification of the non-perfect energy–momentum tensor pieces with the corresponding Cotton components by transverse dualization.

What does motivate these choices? The answer to this question is rooted to the Weyl tensor and to the remarkable integrability properties its structure can provide to the system.

Let us firstly recall that from the bulk perspective, u is a manifestly null congruence associated with the vector ∂_r. One can show (see [13]) that this bulk congruence is also *geodesic* and *shear-free*. Therefore, accordingly to the generalizations of the Goldberg–Sachs theorem, if the bulk metric (2.41) is an Einstein space, then it is algebraically special, *i.e.* of

Petrov type II, III, D, N or O. Owing to the close relationship between the algebraic structure and the integrability properties of Einstein equations, it is clear why the absence of shear in the fluid congruence plays such an instrumental rôle in making the tentatively resummed expression (2.53) an exact Einstein space.

The structure of the bulk Weyl tensor makes it possible to go deeper in foreseeing how the boundary data should be tuned in order for the resummation to be successful. Indeed the Weyl tensor can be expanded for large-r, and the dominant term ($1/r^3$) exhibits the following combination of the boundary energy–momentum and Cotton tensors [69–73]:

$$T^{\pm}_{\mu\nu} = T_{\mu\nu} \pm \frac{\mathrm{i}}{8\pi G k} C_{\mu\nu}, \tag{2.60}$$

satisfying a conservation equation, analogue to (2.1)

$$\nabla^{\mu} T^{\pm}_{\mu\nu} = 0. \tag{2.61}$$

For algebraically special spaces, these complex-conjugate tensors simplify considerably (see detailed discussions in [10–14]), and this suggests the transverse duality enforced between the Cotton and the energy–momentum non-perfect components. Using (2.57) and (2.59), we find indeed for the tensor T^{+} in complex coordinates:

$$\mathrm{T}^{+} = \left(\varepsilon + \frac{\mathrm{i}c}{8\pi G}\right)\left(\frac{\mathrm{u}^2}{k^2} + \frac{1}{2}\mathrm{d}\ell^2\right) + \frac{\mathrm{i}}{4\pi G k^2}\left(2c_{\zeta}\mathrm{d}\zeta\mathrm{u} - c_{\zeta\zeta}\mathrm{d}\zeta^2\right), \tag{2.62}$$

and similarly for T^{-} obtained by complex conjugation with

$$\varepsilon_{\pm} = \varepsilon \pm \frac{\mathrm{i}c}{8\pi G}. \tag{2.63}$$

The bulk Weyl tensor and consequently the Petrov class of the bulk Einstein space are encoded in the three complex functions of the boundary coordinates: ε_{+}, c_{ζ} and $c_{\zeta\zeta}$.

The proposed resummation procedure, based on boundary relativistic fluid dynamics of non-perfect fluids with heat current and stress tensor designed from the boundary Cotton tensor, allows to reconstruct all algebraically special four-dimensional Einstein spaces. The simplest correspond to a Cotton tensor of the perfect form [10]. The complete class of Plebański–Demiański family [74] requires non-trivial b_i with two commuting Killing fields [13], while vanishing b_i without isometry leads to the Robinson–Trautman Einstein spaces [12]. For the latter, the heat current and the stress tensor obtained from the Cotton by the transverse duality read:

$$q = -\frac{1}{16\pi G}\left(\partial_\zeta K \mathrm{d}\zeta + \partial_{\bar\zeta} K \mathrm{d}\bar\zeta\right), \tag{2.64}$$

$$\tau = \frac{1}{8\pi G k^2 P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)\mathrm{d}\zeta^2 + \partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right)\mathrm{d}\bar\zeta^2\right), \tag{2.65}$$

where $K = 2P^2\partial_{\bar\zeta}\partial_\zeta \ln P$ is the Gaussian curvature of (2.55). With these data the conservation of the energy–momentum tensor (2.1) enforces the absence of spatial dependence in $\varepsilon = 2p$, and leads to a single independent equation, the heat equation:

$$12M\partial_t \ln P + \Delta K = 4\partial_t M. \tag{2.66}$$

This is the Robinson–Trautman equation, here expressed in terms of $M(t) = 4\pi G\varepsilon(t)$.

The boundary fluids emerging in the systems considered here have a specific physical behaviour. This behaviour is inherited from the boundary geometry, since their excursion away from perfection is encoded in the Cotton tensor via the transverse duality. In the hydrodynamic frame at hand, this implies in particular that the derivative expansion of the energy–momentum tensor terminates at third order. Discussing this side of the holography is not part of our agenda. We shall only stress that such an analysis does not require to change hydrodynamic frame. Following [66], it is possible to show that the frame at hand is the Eckart frame. Trying to discard the heat current in order to reach a Landau–Lifshitz-like frame (as in [75–78] for Robinson–Trautman) is questionable, as already mentioned earlier, because of the absence of conserved current, and distorts the physical phenomena occurring in the holographic conformal fluid.

3 The Ricci-flat limit I: Carrollian geometry and Carrollian fluids

The Ricci-flat limit is achieved at vanishing k. Although no conformal boundary exists in this case, a two-dimensional spatial conformal structure emerges at null infinity. Since the Einstein bulk spacetime derivative expansion is performed along null tubes, it provides the appropriate arena for studying both the nature of the two-dimensional "boundary" and the dynamics of the degrees of freedom it hosts as "holographic duals" to the bulk Ricci-flat spacetime.

3.1 The Carrollian boundary geometry

The emergence of a boundary

For vanishing k, time decouples in the boundary geometry (2.43). There exist two decoupling limits, associated with two distinct contractions of the Poincaré group: the Galilean, reached at infinite velocity of light and referred to as "non-relativistic", and the Carrollian, emerging at zero velocity of light [47] – often called "ultra-relativistic". In (2.43), k plays effectively the rôle of velocity of light and $k \to 0$ is indeed a *Carrollian limit*.

This very elementary observation sets precisely and unambiguously the fate of asymptotically flat holography: *the reconstruction of four-dimensional Ricci-flat spacetimes is based on Carrollian boundary geometry*.

The appearance of Carrollian symmetry, or better, conformal Carrollian symmetry at null infinity of asymptotically flat spacetimes is not new [48–51]. It has attracted attention in the framework of flat holography, mostly from the algebraic side [79, 80], or in relation with its dual geometry emerging in the Galilean limit, known as Newton–Cartan (see [81]). The novelties we bring in the present work are twofold. On the one hand, the Carrollian geometry emerging at null infinity is generally non-flat, *i.e.* it is not isometric under the Carroll group, but under a more general group associated with a time-dependent positive-definite spatial metric and a Carrollian time arrow, this general Carrollian geometry being covariant under a subgroup of the diffeomorphisms dubbed Carrollian diffeomorphisms. On the other hand, the Carrollian surface is the natural host for a Carrollian fluid, zero-k limit of the relativistic boundary fluid dual to the original Einstein space of which we consider the flat limit. This Carrollian fluid must be considered as the holographic dual of a Ricci-flat spacetime, and its dynamics (studied in Sec. 3.2) as the dual of gravitational bulk dynamics at zero cosmological constant. From the hydrodynamical viewpoint, this gives a radically new perspective on the subject of flat holography.

The Carrollian geometry: connection and curvature

The Carrollian geometry consists of a spatial surface $\mathscr{S}$ endowed with a positive-definite metric

$$\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{3.1}$$

and a Carrollian time $t \in \mathbb{R}$.[14] The metric on $\mathscr{S}$ is generically time-dependent: $a_{ij} = a_{ij}(t, \mathbf{x})$. Much like a Galilean space is observed from a spatial frame moving with respect to a local inertial frame with velocity $\mathbf{w}$, a Carrollian frame is described by a form $\boldsymbol{b} = b_i(t, \mathbf{x})\,\mathrm{d}x^i$. The latter is *not* a velocity because in Carrollian spacetimes motion is forbidden. It is rather an inverse velocity, describing a "temporal frame" and plays a dual rôle. A scalar $\Omega(t, \mathbf{x})$ is also introduced (as in the Galilean case, see [52] – this reference will be useful along the present section), as it may naturally arise from the $k \to 0$ limit.

[14] We are genuinely describing a spacetime $\mathbb{R} \times \mathscr{S}$ endowed with a Carrollian structure, and this is actually how the boundary geometry should be spelled. In order to make the distinction with the relativistic pseudo-Riemannian three-dimensional spacetime boundary $\mathscr{I}$ of AdS bulks, we quote only the spatial surface $\mathscr{S}$ when referring to the Carrollian boundary geometry of a Ricci-flat bulk spacetime. For a complete description of such geometries we recommend [82].

We define the Carrollian diffeomorphisms as

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \tag{3.2}$$

with Jacobian functions

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J^i_j(\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j}. \tag{3.3}$$

Those are the diffeomorphisms adapted to the Carrollian geometry since under such transformations, $\mathrm{d}\ell^2$ remains a positive-definite metric (it does not produce terms involving $\mathrm{d}t'$). Indeed,

$$a'_{ij} = a_{kl} J^{-1k}_{i} J^{-1l}_{j}, \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1i}_{k}, \quad \Omega' = \frac{\Omega}{J}, \tag{3.4}$$

whereas the time and space derivatives become

$$\partial'_t = \frac{1}{J}\partial_t, \quad \partial'_j = J^{-1i}_{j}\left(\partial_i - \frac{j_i}{J}\partial_t\right). \tag{3.5}$$

We will show in a short while that the Carrollian fluid equations are precisely covariant under this particular set of diffeomorphisms.

Expression (3.5) shows that the ordinary exterior derivative of a scalar function does not transform as a form. To overcome this issue, it is desirable to introduce a Carrollian derivative as

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega}\partial_t, \tag{3.6}$$

transforming as

$$\hat{\partial}'_i = J^{-1j}_{i}\hat{\partial}_j. \tag{3.7}$$

Acting on scalars this provides a form, whereas for any other tensor it must be covariantized by introducing a new connection for Carrollian geometry, called *Levi–Civita–Carroll* connection, whose coefficients are the *Christoffel–Carroll* symbols,[15]

$$\hat{\gamma}^i_{jk} = \frac{a^{il}}{2}\left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}\right) = \gamma^i_{jk} + c^i_{jk}. \tag{3.8}$$

The Levi–Civita–Carroll covariant derivative acts symbolically as $\hat{\boldsymbol{\nabla}} = \hat{\boldsymbol{\partial}} + \hat{\boldsymbol{\gamma}}$. It is metric and torsionless: $\hat{\nabla}_i a_{jk} = 0$, $\hat{t}^k_{\ ij} = 2\hat{\gamma}^k_{[ij]} = 0$. There is however an effective torsion, since the derivatives $\hat{\nabla}_i$ do not commute, even when acting of scalar functions Φ – where they are identical to $\hat{\partial}_i$:

$$[\hat{\nabla}_i, \hat{\nabla}_j]\Phi = \frac{2}{\Omega}\varpi_{ij}\partial_t\Phi. \tag{3.9}$$

[15] We remind that the ordinary Christoffel symbols are $\gamma^i_{jk} = \frac{a^{il}}{2}\left(\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}\right)$.

Here ϖ_{ij} is a two-form identified as the Carrollian vorticity defined using the Carrollian acceleration one-form φ_i:

$$\varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right) = \partial_t \frac{b_i}{\Omega} + \hat{\partial}_i \ln \Omega, \tag{3.10}$$

$$\varpi_{ij} = \partial_{[i} b_{j]} + b_{[i}\varphi_{j]} = \frac{\Omega}{2}\left(\hat{\partial}_i \frac{b_j}{\Omega} - \hat{\partial}_j \frac{b_i}{\Omega}\right). \tag{3.11}$$

Since the original relativistic fluid is at rest, the kinematical "inverse-velocity" variable potentially present in the Carrollian limit vanishes.[16] Hence the various kinematical quantities such as the vorticity and the acceleration are purely geometric and originate from the temporal Carrollian frame used to describe the surface $\mathscr{S}$. As we will see later, they turn out to be $k \to 0$ counterparts of their relativistic homologues defined in (2.9), (2.10), (2.11) (see also (3.14) for the expansion and shear).

The time derivative transforms as in (3.5), and acting on any tensor under Carrollian diffeomorphisms, it provides another tensor. This ordinary time derivative has nonetheless an unsatisfactory feature: its action on the metric does not vanish. One is tempted therefore to set a new time derivative $\hat{\partial}_t$ such that $\hat{\partial}_t a_{jk} = 0$, while keeping the transformation rule under Carrollian diffeomorphisms: $\hat{\partial}'_t = \frac{1}{J}\hat{\partial}_t$. This is achieved by introducing a "temporal Carrollian connection"

$$\hat{\gamma}^i{}_j = \frac{1}{2\Omega} a^{ik}\partial_t a_{kj}, \tag{3.12}$$

which allows us to define the time covariant derivative on a vector field:

$$\frac{1}{\Omega}\hat{\partial}_t V^i = \frac{1}{\Omega}\partial_t V^i + \hat{\gamma}^i{}_j V^j, \tag{3.13}$$

while on a scalar the action is as the ordinary time derivative: $\hat{\partial}_t \Phi = \partial_t \Phi$. Leibniz rule allows extending the action of this derivative to any tensor.

Calling $\hat{\gamma}^i{}_j$ a connection is actually misleading because it transforms as a genuine tensor under Carrollian diffeomorphisms: $\hat{\gamma}'^k{}_j = J^k_n J^{-1m}_{j} \hat{\gamma}^n{}_m$. Its trace and traceless parts have a well-defined kinematical interpretation, as the expansion and shear, completing the acceleration and vorticity introduced earlier in (3.10), (3.11):

$$\theta = \hat{\gamma}^i{}_i = \frac{1}{\Omega}\partial_t \ln\sqrt{a}, \quad \xi^i{}_j = \hat{\gamma}^i{}_j - \frac{1}{2}\delta^i_j \theta = \frac{1}{2\Omega} a^{ik}\left(\partial_t a_{kj} - a_{kj}\partial_t \ln\sqrt{a}\right). \tag{3.14}$$

We can define the curvature associated with a connection, by computing the commutator

[16] A Carrollian fluid is always at rest, but could generally be obtained from a relativistic fluid moving at $v^i = k^2\beta^i + \mathrm{O}\left(k^4\right)$. In this case, the "inverse velocity" β^i would contribute to the kinematics and the dynamics of the fluid (see [52]). Here, $v^i = 0$ before the limit $k \to 0$ is taken, so $\beta^i = 0$.

of covariant derivatives acting on a vector field. We find

$$\left[\hat{\nabla}_k, \hat{\nabla}_l\right] V^i = \hat{r}^i{}_{jkl} V^j + \varpi_{kl} \frac{2}{\Omega} \partial_t V^i, \tag{3.15}$$

where

$$\hat{r}^i{}_{jkl} = \hat{\partial}_k \hat{\gamma}^i_{lj} - \hat{\partial}_l \hat{\gamma}^i_{kj} + \hat{\gamma}^i_{km} \hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm} \hat{\gamma}^m_{kj} \tag{3.16}$$

is a genuine tensor under Carrollian diffeomorphisms, the Riemann–Carroll tensor.

As usual, the Ricci–Carroll tensor is

$$\hat{r}_{ij} = \hat{r}^k{}_{ikj}. \tag{3.17}$$

It is *not* symmetric in general ($\hat{r}_{ij} \neq \hat{r}_{ji}$) and carries four independent components:

$$\hat{r}_{ij} = \hat{s}_{ij} + \hat{K} a_{ij} + \hat{A} \eta_{ij}. \tag{3.18}$$

In this expression $\hat{s}_{ij}$ is symmetric and traceless, whereas[17]

$$\hat{K} = \frac{1}{2} a^{ij} \hat{r}_{ij} = \frac{1}{2} \hat{r}, \quad \hat{A} = \frac{1}{2} \eta^{ij} \hat{r}_{ij} = *\varpi\theta \tag{3.19}$$

are the scalar-electric and scalar-magnetic Gauss–Carroll curvatures, with

$$* \varpi = \frac{1}{2} \eta^{ij} \varpi_{ij}. \tag{3.20}$$

Since time and space are intimately related in Carrollian geometry, curvature extends also in time. This can be seen by computing the covariant time and space derivatives commutator:

$$\left[\frac{1}{\Omega} \hat{\partial}_t, \hat{\nabla}_i\right] V^i = -2\hat{r}_i V^i + \left(\theta \delta^j_i - \hat{\gamma}^j{}_i\right) \varphi_j V^i + \left(\varphi_i \frac{1}{\Omega} \hat{\partial}_t - \hat{\gamma}^j{}_i \hat{\nabla}_j\right) V^i. \tag{3.21}$$

A Carroll curvature one-form emerges thus as

$$\hat{r}_i = \frac{1}{2} \left(\hat{\nabla}_j \xi^j{}_i - \frac{1}{2} \hat{\partial}_i \theta\right). \tag{3.22}$$

The Ricci–Carroll curvature tensor $\hat{r}_{ij}$ and the Carroll curvature one-form $\hat{r}_i$ are actually the Carrollian vanishing-k contraction of the ordinary Ricci tensor $R_{\mu\nu}$ associated with the original three-dimensional pseudo-Riemannian AdS boundary $\mathscr{I}$, of Randers–Papapetrou type (2.43). The identification of the various pieces is however a subtle task because in this

[17] We use $\eta_{ij} = \sqrt{a}\epsilon_{ij}$, which matches, in the zero-k limit, with the spatial components of the $\eta_{\mu\nu}$ introduced in (2.15). To avoid confusion we also quote that $\eta^{il}\eta_{jl} = \delta^i_j$ and $\eta^{ij}\eta_{ij} = 2$.

kind of limit, where the size of one dimension shrinks, the curvature usually develops divergences. From the perspective of the final Carrollian geometry this does not produce any harm because the involved components decouple.

The metric (3.1) of the Carrollian geometry on $\mathscr{S}$ may or may not be recast in conformally flat form (2.55) using Carrollian diffeomorphisms (3.2), (3.3). A necessary and sufficient condition is the vanishing of the Carrollian shear ξ_{ij}, displayed in (3.14). Assuming this holds, one proves that the traceless and symmetric piece of the Ricci-Carroll tensor is zero,

$$\hat{s}_{ij} = 0. \tag{3.23}$$

We gather in App. A various expressions when holomorphic coordinates are used and the metric is given in conformally flat form. The absence of shear will be imposed again in Sec. 4, where it plays a crucial rôle in the resummation of the derivative expansion.

The conformal Carrollian geometry

In the present set-up, the spatial surface $\mathscr{S}$ appears as the null infinity of the resulting Ricci-flat geometry *i.e.* as $\mathscr{I}^+$. This is not surprising. The bulk congruence tangent to ∂_r is light-like. Hence the holographic limit $r \to \infty$ is lightlike, already at finite k, which is a well known feature of the derivative expansion, expressed by construction in Eddington–Finkelstein-like coordinates [3,4,6]. What is specific about $k = 0$ is the decoupling of time.

The geometry of $\mathscr{I}^+$ is equipped with a conformal class of metrics rather than with a metric. From a representative of this class, we must be able to explore others by Weyl transformations, and this amounts to study conformal Carrollian geometry as opposed to plain Carrollian geometry (see [48]).

The action of Weyl transformations on the elements of the Carrollian geometry on a surface $\mathscr{S}$ is inherited from (2.18):

$$a_{ij} \to \frac{a_{ij}}{\mathcal{B}^2}, \quad b_i \to \frac{b_i}{\mathcal{B}}, \quad \Omega \to \frac{\Omega}{\mathcal{B}}, \tag{3.24}$$

where $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$ is an arbitrary function. The Carrollian vorticity (3.11) and shear (3.14) transform covariantly under (3.24): $\varpi_{ij} \to \frac{1}{\mathcal{B}}\varpi_{ij}$, $\xi_{ij} \to \frac{1}{\mathcal{B}}\xi_{ij}$. However, the Levi–Civita–Carroll covariant derivatives $\hat{\boldsymbol{\nabla}}$ and $\hat{\partial}_t$ defined previously for Carrollian geometry are not covariant under (3.24). Following [52], they must be replaced with Weyl–Carroll covariant spatial and time derivatives built on the Carrollian acceleration φ_i (3.10) and the Carrollian expansion (3.14), which transform as connections:

$$\varphi_i \to \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \theta \to \mathcal{B}\theta - \frac{2}{\Omega}\partial_t \mathcal{B}. \tag{3.25}$$

In particular, these can be combined in[18]

$$\alpha_i = \varphi_i - \frac{\theta}{2} b_i, \tag{3.26}$$

transforming under Weyl rescaling as:

$$\alpha_i \to \alpha_i - \partial_i \ln \mathcal{B}. \tag{3.27}$$

The Weyl–Carroll covariant derivatives $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_t$ are defined according to the pattern (2.19), (2.20). They obey

$$\hat{\mathscr{D}}_j a_{kl} = 0, \quad \hat{\mathscr{D}}_t a_{kl} = 0. \tag{3.28}$$

For a weight-w scalar function Φ, or a weight-w vector V^i, *i.e.* scaling with $\mathcal{B}^w$ under (3.24), we introduce

$$\hat{\mathscr{D}}_j \Phi = \hat{\partial}_j \Phi + w \varphi_j \Phi, \quad \hat{\mathscr{D}}_j V^l = \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta^l_j V^i \varphi_i, \tag{3.29}$$

which leave the weight unaltered. Similarly, we define

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t \Phi = \frac{1}{\Omega}\hat{\partial}_t \Phi + \frac{w}{2}\theta\Phi = \frac{1}{\Omega}\partial_t \Phi + \frac{w}{2}\theta\Phi, \tag{3.30}$$

and

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t V^l = \frac{1}{\Omega}\hat{\partial}_t V^l + \frac{w-1}{2}\theta V^l = \frac{1}{\Omega}\partial_t V^l + \frac{w}{2}\theta V^l + \xi^l{}_i V^i, \tag{3.31}$$

where $\frac{1}{\Omega}\hat{\mathscr{D}}_t$ increases the weight by one unit. The action of $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_t$ on any other tensor is obtained using the Leibniz rule.

The Weyl–Carroll connection is torsion-free because

$$\left[\hat{\mathscr{D}}_i, \hat{\mathscr{D}}_j\right] \Phi = \frac{2}{\Omega}\varpi_{ij}\hat{\mathscr{D}}_t \Phi + w\left(\varphi_{ij} - \varpi_{ij}\theta\right)\Phi \tag{3.32}$$

does not contain terms of the type $\hat{\mathscr{D}}_k \Phi$. Here $\varphi_{ij} = \hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i$ is a Carrollian two-form, not conformal though. Connection (3.32) is accompanied with its own curvature tensors, which emerge in the commutation of Weyl–Carroll covariant derivatives acting *e.g.* on vectors:

$$\left[\hat{\mathscr{D}}_k, \hat{\mathscr{D}}_l\right] V^i = \left(\hat{\mathscr{R}}^i{}_{jkl} - 2\xi^i{}_j \varpi_{kl}\right) V^j + \varpi_{kl}\frac{2}{\Omega}\hat{\mathscr{D}}_t V^i + w\left(\varphi_{kl} - \varpi_{kl}\theta\right) V^i. \tag{3.33}$$

The combination $\varphi_{kl} - \varpi_{kl}\theta$ forms a weight-0 conformal two-form, whose dual $*\varphi - *\varpi\theta$ is

[18] Contrary to φ_i, α_i is not a Carrollian one-form, *i.e.* it does not transform covariantly under Carrollian diffeomorphisms (3.2).

conformal of weight 2 ($*\varpi$ is defined in (3.20) and similarly $*\varphi = \frac{1}{2}\eta^{ij}\varphi_{ij}$). Moreover

$$\begin{aligned}\hat{\mathscr{R}}^{i}{}_{jkl} \quad &= \quad \hat{r}^{i}{}_{jkl} - \delta^{i}_{j}\varphi_{kl} - a_{jk}\hat{\nabla}_{l}\varphi^{i} + a_{jl}\hat{\nabla}_{k}\varphi^{i} + \delta^{i}_{k}\hat{\nabla}_{l}\varphi_{j} - \delta^{i}_{l}\hat{\nabla}_{k}\varphi_{j} \\ &\quad + \varphi^{i}\left(\varphi_{k}a_{jl} - \varphi_{l}a_{jk}\right) - \left(\delta^{i}_{k}a_{jl} - \delta^{i}_{l}a_{jk}\right)\varphi_{m}\varphi^{m} + \left(\delta^{i}_{k}\varphi_{l} - \delta^{i}_{l}\varphi_{k}\right)\varphi_{j}\end{aligned} \tag{3.34}$$

is the Riemann–Weyl–Carroll weight-0 tensor, from which we define

$$\hat{\mathscr{R}}_{ij} = \hat{\mathscr{R}}^{k}{}_{ikj} = \hat{r}_{ij} + a_{ij}\hat{\nabla}_{k}\varphi^{k} - \varphi_{ij}. \tag{3.35}$$

We also quote

$$\left[\frac{1}{\Omega}\hat{\mathscr{D}}_{t},\hat{\mathscr{D}}_{i}\right]\Phi = w\hat{\mathscr{R}}_{i}\Phi - \xi^{j}{}_{i}\hat{\mathscr{D}}_{j}\Phi \tag{3.36}$$

and

$$\left[\frac{1}{\Omega}\hat{\mathscr{D}}_{t},\hat{\mathscr{D}}_{i}\right]V^{i} = (w-2)\hat{\mathscr{R}}_{i}V^{i} - V^{i}\hat{\mathscr{D}}_{j}\xi^{j}{}_{i} - \xi^{j}{}_{i}\hat{\mathscr{D}}_{j}V^{i}, \tag{3.37}$$

with

$$\hat{\mathscr{R}}_{i} = \hat{r}_{i} + \frac{1}{\Omega}\hat{\partial}_{t}\varphi_{i} - \frac{1}{2}\hat{\nabla}_{j}\hat{\gamma}^{j}{}_{i} + \xi^{j}{}_{i}\varphi_{j} = \frac{1}{\Omega}\partial_{t}\varphi_{i} - \frac{1}{2}\left(\hat{\partial}_{i} + \varphi_{i}\right)\theta. \tag{3.38}$$

This is a Weyl-covariant weight-1 curvature one-form, where $\hat{r}_i$ is given in (3.22).

The Ricci–Weyl–Carroll tensor (3.35) is *not* symmetric in general: $\hat{\mathscr{R}}_{ij} \neq \hat{\mathscr{R}}_{ji}$. Using (3.17) we can recast it as

$$\hat{\mathscr{R}}_{ij} = \hat{s}_{ij} + \hat{\mathscr{K}}a_{ij} + \hat{\mathscr{A}}\eta_{ij}, \tag{3.39}$$

where we have introduced the Weyl-covariant scalar-electric and scalar-magnetic Gauss–Carroll curvatures

$$\hat{\mathscr{K}} = \frac{1}{2}a^{ij}\hat{\mathscr{R}}_{ij} = \hat{K} + \hat{\nabla}_{k}\varphi^{k}, \quad \hat{\mathscr{A}} = \frac{1}{2}\eta^{ij}\hat{\mathscr{R}}_{ij} = \hat{A} - *\varphi \tag{3.40}$$

both of weight 2.

Before closing the present section, it is desirable to make a clarification: Weyl transformations (3.24) should not be confused with the action of the conformal Carroll group, which is a subset of Carrollian diffeomorphisms defined as[19]

$$\mathbf{CCarr}_{2}\left(\mathbb{R}\times\mathscr{S},\mathrm{d}\ell^{2},\mathrm{u}\right) = \left\{\phi \in \mathrm{Diff}(\mathbb{R}\times\mathscr{S}), \quad \mathrm{d}\ell^{2} \xrightarrow{\phi} \mathrm{e}^{-2\Phi}\mathrm{d}\ell^{2} \quad \mathrm{u} \xrightarrow{\phi} e^{\Phi}\mathrm{u}\right\}, \tag{3.41}$$

where $\Phi \in \mathcal{C}^{\infty}(\mathbb{R}\times\mathscr{S})$, $\mathrm{d}\ell^2$ is the spatial metric on $\mathscr{S}$ as in (3.1), and $\mathrm{u} = \frac{1}{\Omega}\partial_t$ the Carrollian time arrow. This group is actually the zero-k contraction of $\mathbf{CIsom}\left(\mathscr{I},\mathrm{d}s^2\right)$, the group of conformal isometries of the original finite-k relativistic metric $\mathrm{d}s^2$ on the boundary $\mathscr{I}$ of the

[19]The subscript 2 stands for level-2 conformal Carroll group. For a detailed discussion, see [49] .

corresponding AdS bulk:

$$\mathbf{CIsom}\left(\mathscr{I},\mathrm{d}s^2\right)=\left\{\phi\in\mathrm{Diff}(\mathscr{I}),\quad \mathrm{d}s^2\xrightarrow{\phi}\mathrm{e}^{-2\Phi}\mathrm{d}s^2\right\}\tag{3.42}$$

with $\Phi\in\mathcal{C}^\infty(\mathscr{I})$. Indeed, consider the Lie algebra of conformal symmetries of $\mathrm{d}s^2$, denoted $\mathfrak{cisom}\left(\mathscr{I},\mathrm{d}s^2\right)$ and spanned by vector fields $\mathrm{X}=X^0\partial_0+X^i\partial_i$ such that

$$\mathscr{L}_{\mathrm{X}}\mathrm{d}s^2=-2\lambda\mathrm{d}s^2\tag{3.43}$$

for some function λ on $\mathscr{I}$. In order to perform the zero-k contraction we write the generators as $\mathrm{X}=kX^t\partial_0+X^i\partial_i$ (here $x^0=kt$, thus $X^0=kX^t$) and the metric $\mathrm{d}s^2$ in the Randers–Papapetrou form (2.43). At zero k Eq. (3.43) splits into:[20]

$$\mathscr{L}_{\mathrm{X}}\mathrm{u}=\lambda\mathrm{u},\quad \mathscr{L}_{\mathrm{X}}\mathrm{d}\ell^2=-2\lambda\mathrm{d}\ell^2.\tag{3.44}$$

These are the equations the field X must satisfy for belonging to $\mathfrak{ccarr}_2\left(\mathbb{R}\times\mathscr{S},\mathrm{d}\ell^2,\mathrm{u}\right)$, the Lie algebra of the corresponding conformal Carroll group. This confirms that

$$\mathbf{CIsom}\left(\mathscr{I},\mathrm{d}s^2\right)\underset{k\to0}{\longrightarrow}\mathbf{CCarr}_2\left(\mathbb{R}\times\mathscr{S},\mathrm{d}\ell^2,\mathrm{u}\right).\tag{3.45}$$

At last, if $\mathscr{S}$ is chosen to be the two-sphere and $\mathrm{d}\ell^2$ the round metric, it can be shown (see [49]) that the corresponding conformal Carroll group is precisely the BMS(4) group, which describes the asymptotic symmetries of an asymptotically flat $3+1$-dimensional metric.

3.2 Carrollian conformal fluid dynamics

Physical data and hydrodynamic equations

More on the physics underlying the Carrollian limit can be found in [52], with emphasis on hydrodynamics. This is precisely what we need here, since the original asymptotically AdS bulk Einstein spacetime is the holographic dual of a relativistic fluid hosted by its $2+1$-dimensional boundary. This relativistic fluid satisfying Eq. (2.1), will obey Carrollian dynamics at vanishing k. Even though the fluid has no velocity, it has non-trivial hydrodynamics based on the following data:

- the energy density $\varepsilon(t,\mathbf{x})$ and the pressure $p(t,\mathbf{x})$, related here through a conformal equation of state $\varepsilon=2p$;

[20] In coordinates, defining $\chi=\Omega X^t-b_jX^j$, these equations are written as:

$$\frac{1}{\Omega}\partial_t\chi+\varphi_jX^j=-\lambda,\quad \frac{1}{\Omega}\partial_tX^i=0,\quad \hat{\nabla}^{(i}X^{j)}+\chi\left(\xi^{ij}+\frac{1}{2}a^{ij}\theta\right)=-\lambda a^{ij},$$

which are manifestly covariant under Carrollian diffeomorphisms.

- the heat currents $\boldsymbol{Q} = Q_i(t,\mathbf{x})\mathrm{d}x^i$ and $\boldsymbol{\pi} = \pi_i(t,\mathbf{x})\mathrm{d}x^i$;
- the viscous stress tensors $\boldsymbol{\Sigma} = \Sigma_{ij}(t,\mathbf{x})\,\mathrm{d}x^i\mathrm{d}x^j$ and $\boldsymbol{\Xi} = \Xi_{ij}(t,\mathbf{x})\mathrm{d}x^i\mathrm{d}x^j$.

The latter quantities are inherited from the relativistic ones (see (2.2)) as the following limits:

$$Q_i = \lim_{k\to 0} q_i, \quad \pi_i = \lim_{k\to 0} \tfrac{1}{k^2}\left(q_i - Q_i\right), \tag{3.46}$$

$$\Sigma_{ij} = -\lim_{k\to 0} k^2 \tau_{ij}, \quad \Xi_{ij} = -\lim_{k\to 0}\left(\tau_{ij} + \tfrac{1}{k^2}\Sigma_{ij}\right). \tag{3.47}$$

Compared with the corresponding ones in the Galilean fluids, they are doubled because two orders seem to be required for describing the Carrollian dynamics. They obey

$$\Sigma_{ij} = \Sigma_{ji}, \quad \Sigma^i{}_i = 0, \quad \Xi_{ij} = \Xi_{ji}, \quad \Xi^i{}_i = 0. \tag{3.48}$$

The Carrollian energy and pressure are just the zero-k limits of the corresponding relativistic quantities. In order to avoid symbols inflation, we have kept the same notation, ε and p.

All these objects are Weyl-covariant with conformal weights 3 for the pressure and energy density, 2 for the heat currents, and 1 for the viscous stress tensors (when all indices are lowered). They are well-defined in all examples we know from holography. Ultimately they should be justified within a microscopic quantum/statistical approach, missing at present since the microscopic nature of a Carrollian fluid has not been investigated so far, except for [52], where some elementary issues were addressed.

Following this reference, the equations for a Carrollian fluid are as follows:

- a set of two scalar equations, both weight-4 Weyl-covariant:

$$-\frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon - \hat{\mathscr{D}}_i Q^i + \Xi^{ij}\xi_{ij} = 0, \tag{3.49}$$

$$\Sigma^{ij}\xi_{ij} = 0; \tag{3.50}$$

- two vector equations, Weyl-covariant of weight 3:

$$\hat{\mathscr{D}}_j p + 2Q^i\varpi_{ij} + \frac{1}{\Omega}\hat{\mathscr{D}}_t\pi_j - \hat{\mathscr{D}}_i\Xi^i{}_j + \pi_i\xi^i{}_j = 0, \tag{3.51}$$

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t Q_j - \hat{\mathscr{D}}_i\Sigma^i{}_j + Q_i\xi^i{}_j = 0. \tag{3.52}$$

Equation (3.49) is the energy conservation, whereas (3.50) sets a geometrical constraint on the Carrollian viscous stress tensor Σ_{ij}. Equations (3.51) and (3.52) are dynamical equations involving the pressure $p = \varepsilon/2$, the heat currents Q_i and π_i, and the viscous stress tensors Σ_{ij} and Ξ_{ij}. They are reminiscent of a momentum conservation, although somewhat degenerate due to the absence of fluid velocity.

An example of Carrollian fluid

The simplest non-trivial example of a Carrollian fluid is obtained as the Carrollian limit of the relativistic Robinson–Trautman fluid, studied at the end of Sec. 2.2 (see also [66] and [52] for the relativistic and Carrollian approaches, respectively).

The geometric Carrollian data are in this case

$$\mathrm{d}\ell^2 = \frac{2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}, \tag{3.53}$$

$b_i = 0$ and $\Omega = 1$. Hence the Carrollian shear vanishes ($\xi_{ij} = 0$), whereas the expansion reads:

$$\theta = -2\partial_t \ln P. \tag{3.54}$$

Similarly $\varpi_{ij} = 0$, $\varphi_i = 0$, $\varphi_{ij} = 0$, and using results from App. A, we find

$$\hat{\mathscr{K}} = 2P^2\partial_{\bar{\zeta}}\partial_{\zeta} \ln P, \quad \hat{\mathscr{A}} = 0 \tag{3.55}$$

(in fact $\hat{\mathscr{K}} = \hat{K} = K$), while

$$\hat{\mathscr{R}}_{\bar{\zeta}} = \partial_{\bar{\zeta}}\partial_t \ln P, \quad \hat{\mathscr{R}}_{\bar{\zeta}} = \partial_{\bar{\zeta}}\partial_t \ln P. \tag{3.56}$$

From the relativistic heat current q and viscous stress tensor τ displayed in (2.64) and (2.65), we obtain the Carrollian descendants:[21]

$$\boldsymbol{Q} = -\tfrac{1}{16\pi G}\left(\partial_\zeta K \mathrm{d}\zeta + \partial_{\bar{\zeta}} K \mathrm{d}\bar{\zeta}\right), \quad \boldsymbol{\pi} = 0, \tag{3.57}$$

$$\boldsymbol{\Sigma} = -\tfrac{1}{8\pi G P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)\mathrm{d}\zeta^2 + \partial_{\bar{\zeta}}\left(P^2\partial_t\partial_{\bar{\zeta}} \ln P\right)\mathrm{d}\bar{\zeta}^2\right), \quad \boldsymbol{\Xi} = 0. \tag{3.58}$$

Due to the absence of shear, the hydrodynamic equation (3.50) is identically satisfied, whereas (3.49), (3.51), (3.52) are recast as:

$$3\varepsilon\partial_t \ln P - \partial_t\varepsilon - \nabla_i Q^i = 0, \tag{3.59}$$

$$\partial_i p = 0, \tag{3.60}$$

$$\partial_t Q_i - 2Q_i\partial_t \ln P - \nabla_j \Sigma^j{}_i = 0. \tag{3.61}$$

In agreement with the relativistic Robinson–Trautman fluid, the pressure p (and so the energy density, since the fluid is conformal) must be space-independent. Furthermore, as expected from the relativistic case, Eq. (3.61) is satisfied with Q_i and Σ_{ij} given in (3.57) and (3.58). Hence we are left with a single non-trivial equation, Eq. (3.59), the heat equation of

[21] Notice a useful identity: $\partial_t\left(\frac{\partial_\zeta^2 P}{P}\right) = \frac{1}{P^2}\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)$.

the Carrollian fluid:

$$3\varepsilon\partial_t \ln P - \partial_t\varepsilon + \frac{1}{16\pi G}\Delta K = 0 \tag{3.62}$$

with $\Delta = \nabla_j\nabla^j$ the Laplacian operator on $\mathscr{S}$.

Equation (3.62) is exactly Robinson–Trautman's, Eq. (2.66). We note that the relativistic and the Carrolian dynamics lead to the same equations – and hence to the same solutions $\varepsilon = \varepsilon(t)$. This is specific to the case under consideration, and it is actually expected since the bulk Einstein equations for a geometry with a shearless and vorticity-free null congruence lead to the Robinson–Trautman equation, irrespective of the presence of a cosmological constant, $\Lambda = -3k^2$: asymptotically locally AdS or locally flat spacetimes lead to the same dynamics. This is not the case in general though, because there is no reason for the relativistic dynamics to be identical to the Carrollian (see [52] for a detailed account of this statement). For example, when switching on more data, as in the case of the Plebański–Demiański family, where all b_i, φ_i, ϖ_{ij}, as well as π_i and Ξ_{ij}, are on, the Carrollian equations are different from the relativistic ones.

4 The Ricci-flat limit II: derivative expansion and resummation

We can summarize our observations as follows. Any four-dimensional Ricci-flat spacetime is associated with a two-dimensional spatial surface, emerging at null infinity and equipped with a conformal Carrollian geometry. This geometry is the host of a Carrollian fluid, obeying Carrollian hydrodynamics. Thanks to the relativistic-fluid/AdS-gravity duality, one can also safely claim that, conversely, any Carrollian fluid evolving on a spatial surface with Carrollian geometry is associated with a Ricci-flat geometry. This conclusion is reached by considering the simultaneous zero-k limit of both sides of the quoted duality. In order to make this statement operative, this limit must be performed inside the derivative expansion. When the latter is resummable in the sense discussed in Sec. 2.2, the zero-k limit will also affect the resummability conditions, and translate them in terms of Carrollian fluid dynamics.

4.1 Back to the derivative expansion

Our starting point is the derivative expansion of an asymptotically locally AdS spacetime, Eq. (2.41). The fundamental question is whether the latter admits a smooth zero-k limit.

We have implicitly assumed that the Randers–Papapetrou data of the three-dimensional pseudo-Riemannian conformal boundary $\mathscr{I}$ associated with the original Einstein spacetime, a_{ij}, b_i and Ω, remain unaltered at vanishing k, providing therefore directly the Carrollian data for the new spatial two-dimensional boundary $\mathscr{S}$ emerging at $\mathscr{I}^+$.[22] Following again the

[22] Indeed our ultimate goal is to set up a derivative expansion (in a closed resummed form under appropriate

detailed analysis performed in [52], we can match the various three-dimensional Riemannian quantities with the corresponding two-dimensional Carrollian ones:

$$\mathrm{u} = -k^2 \left(\Omega \mathrm{d}t - \boldsymbol{b}\right) \tag{4.1}$$

and

$$\begin{aligned} \omega &= \tfrac{k^2}{2}\varpi_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j, \\ \gamma &= *\varpi, \\ \Theta &= \theta, \\ \mathrm{a} &= k^2\varphi_i \mathrm{d}x^i, \\ \mathrm{A} &= \alpha_i \mathrm{d}x^i + \tfrac{\theta}{2}\Omega\mathrm{d}t, \\ \sigma &= \xi_{ij}\mathrm{d}x^i\mathrm{d}x^j, \end{aligned} \tag{4.2}$$

where the left-hand-side quantities are Riemannian (given in Eqs. (2.45), (2.46), (2.47), (2.48), (2.49)), and the right-hand-side ones Carrollian (see (3.10), (3.11), (3.14), (3.20)).

In the list (4.2), we have dealt with the first derivatives, *i.e.* connexion-related quantities. We move now to second-derivative objects and collect the tensors relevant for the derivative expansion, following the same pattern (Riemannian vs. Carrollian):

$$\mathscr{R} = \frac{1}{k^2}\xi_{ij}\xi^{ij} + 2\hat{\mathscr{K}} + 2k^2 * \varpi^2, \tag{4.3}$$

$$\omega_\mu{}^\lambda \omega_{\lambda\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = k^4 \varpi_i{}^l \varpi_{lj}\mathrm{d}x^i\mathrm{d}x^j, \tag{4.4}$$

$$\omega^{\mu\nu}\omega_{\mu\nu} = 2k^4 * \varpi^2, \tag{4.5}$$

$$\mathscr{D}_\nu \omega^\nu{}_\mu \mathrm{d}x^\mu = k^2 \hat{\mathscr{D}}_j \varpi^j{}_i \mathrm{d}x^i - 2k^4 * \varpi^2 \Omega \mathrm{d}t + 2k^4 * \varpi^2 \boldsymbol{b}. \tag{4.6}$$

Using (2.42) this leads to

$$\mathrm{S} = -\frac{k^2}{2}\left(\Omega\mathrm{d}t - \boldsymbol{b}\right)^2 \xi_{ij}\xi^{ij} + k^4 \boldsymbol{s} - 5k^6 \left(\Omega\mathrm{d}t - \boldsymbol{b}\right)^2 * \varpi^2 \tag{4.7}$$

with the Weyl-invariant tensor

$$\boldsymbol{s} = 2\left(\Omega\mathrm{d}t - \boldsymbol{b}\right)\mathrm{d}x^i \eta^j{}_i \hat{\mathscr{D}}_j * \varpi + *\varpi^2 \mathrm{d}\ell^2 - \hat{\mathscr{K}}\left(\Omega\mathrm{d}t - \boldsymbol{b}\right)^2. \tag{4.8}$$

In the derivative expansion (2.41), two explicit divergences appear at vanishing k. The first originates from the first term of S, which is the shear contribution to the Weyl-covariant

assumptions) for building up four-dimensional Ricci-flat spacetimes from a boundary Carrollian fluid, irrespective of its AdS origin. For this it is enough to assume a_{ij}, b_i and Ω k-independent (as in [52]), and use these data as fundamental blocks for the Ricci-flat reconstruction. It should be kept in mind, however, that for general Einstein spacetimes, these may depend on k with well-defined limit and subleading terms. Due to the absence of shear and to the particular structure of these solutions, the latter do not alter the Carrollian equations. This occurs for instance in Plebański–Demiański or in the Kerr–Taub–NUT sub-family, which will be discussed in Sec. 5.1. In the following, we avoid discussing this kind of sub-leading terms, hence saving further technical developments.

scalar curvature $\mathscr{R}$ of the three–dimensional AdS boundary (Eq. (4.3)).[23] The second divergence comes from the Cotton tensor and is also due to the shear. It is fortunate – and expected – that counterterms coming from equal-order (non-explicitly written) σ^2 contributions, cancel out these singular terms. This is suggestive that (2.41) is well-behaved at zero-k, showing that the reconstruction of Ricci-flat spacetimes works starting from two-dimensional Carrollian fluid data.

We will not embark here in proving finiteness at $k = 0$, but rather confine our analysis to situations without shear, as we discussed already in Sec. 2.2 for Einstein spacetimes. Vanishing σ in the pseudo-Riemannian boundary $\mathscr{I}$ implies indeed vanishing ξ_{ij} in the Carrollian (see (4.2)), and in this case, the divergent terms in S and C are absent. Of course, other divergences may occur from higher-order terms in the derivative expansion. To avoid dealing with these issues, we will focus on the resummed version of (2.41) *i.e.* (2.53), valid for algebraically special bulk geometries. This closed form is definitely smooth at zero k and reads:

$$\boxed{\mathrm{d}s^2_{\text{res. flat}} = -2\left(\Omega \mathrm{d}t - \boldsymbol{b}\right)\left(\mathrm{d}r + r\boldsymbol{\alpha} + \frac{r\theta\Omega}{2}\mathrm{d}t\right) + r^2\mathrm{d}\ell^2 + \boldsymbol{s} + \frac{\left(\Omega \mathrm{d}t - \boldsymbol{b}\right)^2}{\rho^2}\left(8\pi G\varepsilon r + c * \varpi\right).} \tag{4.9}$$

Here

$$\rho^2 = r^2 + *\varpi^2, \tag{4.10}$$

$\mathrm{d}\ell^2$, Ω, $\boldsymbol{b} = b_i \mathrm{d}x^i$, $\boldsymbol{\alpha} = \alpha_i \mathrm{d}x^i$, θ and $*\varpi$ are the Carrollian geometric objects introduced earlier, while c and ε are the zero-k (finite) limits of the corresponding relativistic functions. Expression (4.9) will grant by construction an exact Ricci-flat spacetime provided the conditions under which (2.53) was Einstein are fulfilled in the zero-k limit. These conditions are the set of Carrollian hydrodynamic equations (3.49), (3.50), (3.51) and (3.52), and the integrability conditions, as they emerge from (2.56) and (2.58) at vanishing k. Making the latter explicit is the scope of next section.

Notice eventually that the Ricci-flat line element (4.9) inherits Weyl invariance from its relativistic ancestor. The set of transformations (3.24), (3.25) and (3.27), supplemented with $*\varpi \to \mathcal{B} * \varpi$, $\varepsilon \to \mathcal{B}^3\varepsilon$ and $c \to \mathcal{B}^3 c$, can indeed be absorbed by setting $r \to \mathcal{B}r$ ($\boldsymbol{s}$ is Weyl invariant), resulting thus in the invariance of (4.9). In the relativistic case this invariance was due to the AdS conformal boundary. In the case at hand, this is rooted to the location of the two-dimensional spatial boundary $\mathscr{S}$ at null infinity $\mathscr{I}^+$.

[23] This divergence is traced back in the Gauss–Codazzi equation relating the intrinsic and extrinsic curvatures of an embedded surface, to the intrinsic curvature of the host. When the size of a fiber shrinks, the extrinsic-curvature contribution diverges.

4.2 Resummation of the Ricci-flat derivative expansion

The Cotton tensor in Carrollian geometry

The Cotton tensor monitors from the boundary the global asymptotic structure of the bulk four-dimensional Einstein spacetime (for higher dimensions, the boundary Weyl tensor is also involved, see footnote 11). In order to proceed with our resummability analysis, we need to describe the zero-k limit of the Cotton tensor (2.32) and of its conservation equation (2.33).

As already mentioned, at vanishing k divergences do generally appear for some components of the Cotton tensor. These divergences are no longer present when (2.54) is satisfied (see footnote 23), *i.e.* in the absence of shear, which is precisely the assumption under which we are working with (4.9). Every piece of the three-dimensional relativistic Cotton tensor appearing in (2.34) has thus a well-defined limit. We therefore introduce

$$\chi_i = \lim_{k\to 0} c_i, \quad \psi_i = \lim_{k\to 0} \tfrac{1}{k^2}\left(c_i - \chi_i\right), \tag{4.11}$$

$$X_{ij} = \lim_{k\to 0} c_{ij}, \quad \Psi_{ij} = \lim_{k\to 0} \tfrac{1}{k^2}\left(c_{ij} - X_{ij}\right). \tag{4.12}$$

The time components c_0, c_{00} and $c_{0i} = c_{i0}$ vanish already at finite k (due to (2.36)), and χ_i, ψ_i, X_{ij} and Ψ_{ij} are thus genuine Carrollian tensors transforming covariantly under Carrollian diffeomorphisms. Actually, in the absence of shear the Cotton current and stress tensor are given exactly (*i.e.* for finite k) by $c_i = \chi_i + k^2\psi_i$ and $c_{ij} = X_{ij} + k^2\Psi_{ij}$.

The scalar $c(t,\mathbf{x})$ is Weyl-covariant of weight 3 (like the energy density). As expected, it is expressed in terms of geometric Carrollian objects built on third-derivatives of the two-dimensional metric $\mathrm{d}\ell^2$, b_i and Ω:

$$c = \left(\hat{\mathscr{D}}_l \hat{\mathscr{D}}^l + 2\hat{\mathscr{K}}\right) * \varpi. \tag{4.13}$$

Similarly, the forms χ_i and ψ_i, of weight 2, are

$$\chi_j = \frac{1}{2}\eta^l{}_j \hat{\mathscr{D}}_l \hat{\mathscr{K}} + \frac{1}{2}\hat{\mathscr{D}}_j \hat{\mathscr{A}} - 2 * \varpi \hat{\mathscr{R}}_j, \tag{4.14}$$

$$\psi_j = 3\eta^l{}_j \hat{\mathscr{D}}_l * \varpi^2. \tag{4.15}$$

Finally, the weight-1 symmetric and traceless rank-two tensors read:

$$X_{ij} = \frac{1}{2}\eta^l{}_j \hat{\mathscr{D}}_l \hat{\mathscr{R}}_i + \frac{1}{2}\eta^l{}_i \hat{\mathscr{D}}_j \hat{\mathscr{R}}_l, \tag{4.16}$$

$$\Psi_{ij} = \hat{\mathscr{D}}_i \hat{\mathscr{D}}_j * \varpi - \frac{1}{2}a_{ij}\hat{\mathscr{D}}_l \hat{\mathscr{D}}^l * \varpi - \eta_{ij}\frac{1}{\Omega}\hat{\mathscr{D}}_t * \varpi^2. \tag{4.17}$$

Observe that c and the subleading terms ψ_i and Ψ_{ij} are present only when the vorticity is non-vanishing ($*\varpi \neq 0$). All these are of gravito-magnetic nature.

The tensors c, χ_i, ψ_i, X_{ij} and Ψ_{ij} should be considered as the two-dimensional Carrollian resurgence of the three-dimensional Riemannian Cotton tensor. They should be referred to as Cotton descendants (there is no Cotton tensor in two dimensions anyway), and obey identities inherited at zero k from its conservation equation.[24] These are similar to the hydrodynamic equations (3.49), (3.50), (3.51) and (3.52), satisfied by the different pieces of the energy–momentum tensor ε, Q_i, π_i, Σ_{ij} and Ξ_{ij}, and translating its conservation. In the case at hand, the absence of shear trivializes (3.50) and discards the last term in the other three equations:

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t c + \hat{\mathscr{D}}_i \chi^i = 0, \tag{4.18}$$

$$\frac{1}{2}\hat{\mathscr{D}}_j c + 2\chi^i \varpi_{ij} + \frac{1}{\Omega}\hat{\mathscr{D}}_t \psi_j - \hat{\mathscr{D}}_i \Psi^i{}_j = 0, \tag{4.19}$$

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t \chi_j - \hat{\mathscr{D}}_i X^i{}_j = 0. \tag{4.20}$$

One appreciates from these equations why it is important to keep the subleading corrections at vanishing k, both in the Cotton current c_μ and in the Cotton stress tensor $c_{\mu\nu}$. As for the energy–momentum tensor, ignoring them would simply lead to wrong Carrollian dynamics.

The resummability conditions

We are now ready to address the problem of resummability in Carrollian framework, for Ricci-flat spacetimes. In the relativistic case, where one describes relativistic hydrodynamics on the pseudo-Riemannian boundary of an asymptotically locally AdS spacetime, resummability – or integrability – equations are Eqs. (2.56) and (2.58). These determine the friction components of the fluid energy–momentum tensor in terms of geometric data, captured by the Cotton tensor (current and stress components), via a sort of gravitational electric–magnetic duality, transverse to the fluid congruence. Equipped with those, the fluid equations (2.1) guarantee that the bulk is Einstein, *i.e.* that bulk Einstein equations are satisfied.

Correspondingly, using (3.46), (3.47), (4.11) and (4.12), the zero-k limit of Eq. (2.56) sets up a duality relationship among the Carrollian-fluid heat current Q_i and the Carrollian-geometry third-derivative vector χ_i:

$$\boxed{Q_i = \frac{1}{8\pi G}\eta^j{}_i \chi_j = -\frac{1}{16\pi G}\left(\hat{\mathscr{D}}_i \mathscr{K} - \eta^j{}_i \hat{\mathscr{D}}_j \mathscr{A} + 4 * \varpi \eta^j{}_i \hat{\mathscr{R}}_j\right),} \tag{4.21}$$

while Eqs. (2.58) allow to relate the Carrollian-fluid quantities Σ_{ij} and Ξ_{ij}, to the Carrollian-

[24] Observe that the Cotton tensor enters in Eq. (2.60) with an extra factor $1/k$, the origin of which is explained in footnote 9. Hence, the advisable prescription is to analyze the small-k limit of $\frac{1}{k}\nabla^\mu C_{\mu\nu} = 0$.

geometry ones X_{ij} and Ψ_{ij}:

$$\boxed{\Sigma_{ij} = \frac{1}{8\pi G}\eta^l{}_i X_{lj} = \frac{1}{16\pi G}\left(\eta^k{}_j\eta^l{}_i\hat{\mathscr{D}}_k\hat{\mathscr{R}}_l - \hat{\mathscr{D}}_j\hat{\mathscr{R}}_i\right),} \tag{4.22}$$

and

$$\boxed{\Xi_{ij} = \frac{1}{8\pi G}\eta^l{}_i\Psi_{lj} = \frac{1}{8\pi G}\left(\eta^l{}_i\hat{\mathscr{D}}_l\hat{\mathscr{D}}_j * \varpi + \frac{1}{2}\eta_{ij}\hat{\mathscr{D}}_l\hat{\mathscr{D}}^l * \varpi - a_{ij}\frac{1}{\Omega}\hat{\mathscr{D}}_t * \varpi^2\right).} \tag{4.23}$$

One readily shows that (3.48) is satisfied as a consequence of the symmetry and tracelessness of X_{ij} and Ψ_{ij}.

One can finally recast the Carrollian hydrodynamic equations (3.49), (3.50), (3.51) and (3.52) for the fluid under consideration. Recalling that the shear is assumed to vanish,

$$\xi_{ij} = \frac{1}{2\Omega}\left(\partial_t a_{ij} - a_{ij}\partial_t \ln\sqrt{a}\right) = 0, \tag{4.24}$$

Eq. (3.50) is trivialized. Furthermore, Eq. (3.52) is automatically satisfied with Q_j and $\Sigma^i{}_j$ given above, thanks also to Eq. (4.20). We are therefore left with two equations for the energy density ε and the heat current π_i:

- one scalar equation from (3.49):

$$\boxed{-\frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon + \frac{1}{16\pi G}\hat{\mathscr{D}}^i\left(\hat{\mathscr{D}}_i\hat{\mathscr{K}} - \eta^j{}_i\hat{\mathscr{D}}_j\hat{\mathscr{A}} + 4 * \varpi\eta^j{}_i\hat{\mathscr{R}}_j\right) = 0;} \tag{4.25}$$

- one vector equation from (3.51):

$$\boxed{\hat{\mathscr{D}}_j\varepsilon + 4 * \varpi\eta^i{}_j Q_i + \frac{2}{\Omega}\hat{\mathscr{D}}_t\pi_j - 2\hat{\mathscr{D}}_i\Xi^i{}_j = 0} \tag{4.26}$$

  with Q_i and $\Xi^i{}_j$ given in (4.21) and (4.23).

These last two equations are Carrollian equations, describing time and space evolution of the fluid energy and heat current, as a consequence of transport phenomena like heat conduction and friction. These phenomena have been identified by duality to geometric quantities, and one recognizes distinct gravito-electric (like $\hat{\mathscr{K}}$) and gravito-magnetic contributions (like $\hat{\mathscr{A}}$). It should also be stressed that not all the terms are independent and one can reshuffle them using identities relating the Carrollian curvature elements. In the absence of shear, (3.23) holds and all information about $\hat{\mathscr{R}}_{ij}$ in (3.39) is stored in $\hat{\mathscr{K}}$ and $\hat{\mathscr{A}}$, while other geometrical data are supplied by $\hat{\mathscr{R}}_i$ in (3.38). All these obey

$$\begin{aligned} \frac{2}{\Omega}\hat{\mathscr{D}}_t * \varpi + \hat{\mathscr{A}} &= 0, \\ \frac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\mathscr{K}} - a^{ij}\hat{\mathscr{D}}_i\hat{\mathscr{R}}_j &= 0, \\ \frac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\mathscr{A}} + \eta^{ij}\hat{\mathscr{D}}_i\hat{\mathscr{R}}_j &= 0, \end{aligned} \tag{4.27}$$

which originate from three-dimensional Riemannian Bianchi identities and emerge along the k-to-zero limit.

Summarizing

Our analysis of the zero-k limit in the derivative expansion (2.53), valid assuming the absence of shear, has the following salient features.

- As the general derivative expansion (2.41), this limit reveals a two-dimensional spatial boundary $\mathscr{S}$ located at $\mathscr{I}^+$. It is endowed with a Carrollian geometry, encoded in a_{ij}, b_i and Ω, all functions of t and $\mathbf{x}$. This is inherited from the conformal three-dimensional pseudo-Riemannian boundary $\mathscr{I}$ of the original Einstein space.

- The Carrollian boundary $\mathscr{S}$ is the host of a Carrollian fluid, obtained as the limit of a relativistic fluid, and described in terms of its energy density ε, and its friction tensors Q_i, π_i, Σ_{ij} and Ξ_{ij}.

- When the friction tensors Q_i, Σ_{ij} and Ξ_{ij} of the Carrollian fluid are given in terms of the geometric objects χ_i, X_{ij} and Ψ_{ij} using (4.21), (4.22) and (4.23), and when the energy density ε and the current π_i obey the hydrodynamic equations (4.25) and (4.26), the limiting resummed derivative expansion (4.9) is an exact Ricci-flat spacetime.

- The bulk spacetime is in general asymptotically locally flat. This property is encoded in the zero-k limit of the Cotton tensor, *i.e.* in the Cotton Carrollian descendants c, χ_i and X_{ij}.

The bulk Ricci-flat spacetime obtained following the above procedure is algebraically special. We indeed observe that the bulk congruence ∂_r is null. Moreover, it is geodesic and shear-free. To prove this last statement, we rewrite the metric (4.9) in terms of a null tetrad $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$:

$$\mathrm{d}s^2_{\text{res. flat}} = -2\mathbf{k}\mathbf{l} + 2\mathbf{m}\bar{\mathbf{m}}, \quad \mathbf{k}\cdot\mathbf{l} = -1, \quad \mathbf{m}\cdot\bar{\mathbf{m}} = 1, \tag{4.28}$$

where $\mathbf{k} = -(\Omega\mathrm{d}t - \boldsymbol{b})$ is the dual of ∂_r and

$$\mathbf{l} = -\mathrm{d}r - r\boldsymbol{\alpha} - \frac{r\theta\Omega}{2}\mathrm{d}t + \frac{\boldsymbol{\psi}}{6 * \varpi} + \frac{\Omega\mathrm{d}t - \boldsymbol{b}}{2\rho^2}\left(8\pi G\varepsilon r + c * \varpi - \rho^2\hat{\mathscr{K}}\right), \tag{4.29}$$

(here $\boldsymbol{\psi} = \psi_i \mathrm{d}x^i$), along with

$$2\mathbf{m}\bar{\mathbf{m}} = \rho^2 \mathrm{d}\ell^2 . \tag{4.30}$$

Using the above results and repeating the analysis of App. A.2 in [13], we find that ∂_r is shear-free due to (4.24).

According to the Goldberg–Sachs theorem, the bulk spacetime (4.9) is therefore of Petrov type II, III, D, N or O. The precise type is encoded in the Carrollian tensors $\varepsilon^{\pm}$, $Q_i^{\pm}$ and $\Sigma_{ij}^{\pm}$

$$\begin{aligned} \varepsilon^{\pm} &= \varepsilon \pm \tfrac{\mathrm{i}}{8\pi G} c, \\ Q_i^{\pm} &= Q_i \pm \tfrac{\mathrm{i}}{8\pi G} \chi_i, \\ \Sigma_{ij}^{\pm} &= \Sigma_{ij} \pm \tfrac{\mathrm{i}}{8\pi G} X_{ij}. \end{aligned} \tag{4.31}$$

Working again in holomorphic coordinates, we find the compact result

$$\boldsymbol{Q}^{+} = \frac{\mathrm{i}}{4\pi G} \chi_\zeta \mathrm{d}\zeta, \tag{4.32}$$

$$\boldsymbol{\Sigma}^{+} = \frac{\mathrm{i}}{4\pi G} X_{\zeta\zeta} \mathrm{d}\zeta^2, \tag{4.33}$$

and their complex-conjugates $\boldsymbol{Q}^{-}$ and $\boldsymbol{\Sigma}^{-}$. These Carrollian geometric tensors encompass the information on the canonical complex functions describing the Weyl-tensor decomposition in terms of principal null directions – usually referred to as $\Psi_a, a = 0, \ldots, 4$.

5 Examples

There is a plethora of Carrollian fluids that can be studied. We will analyze here the class of *perfect conformal fluids*, and will complete the discussion of Sec. 3.2 on the *Carrollian Robinson–Trautman fluid*. In each case, assuming the integrability conditions (4.21), (4.22) and (4.23) are fulfilled and the hydrodynamic equations (4.25) and (4.26) are obeyed, a Ricci-flat spacetime is reconstructed from the Carrollian spatial boundary $\mathscr{S}$ at $\mathscr{I}^{+}$. More examples exist like the Plebański–Demiański or the Weyl axisymmetric solutions, assuming extra symmetries (but not necessarily stationarity) for a viscous Carrollian fluid. These would require a more involved presentation.

5.1 Stationary Carrollian perfect fluids and Ricci-flat Kerr–Taub–NUT families

We would like to illustrate our findings and reconstruct from purely Carrollian fluid dynamics the family of Kerr–Taub–NUT stationary Ricci-flat black holes. We pick for that the following geometric data: $a_{ij}(\mathbf{x})$, $b_i(\mathbf{x})$ and $\Omega = 1$. Stationarity is implemented in these fluids by requiring that all the quantities involved are time independent.

Under this assumption, the Carrollian shear ξ_{ij} vanishes together with the Carrollian expansion θ, whereas constant Ω makes the Carrollian acceleration φ_i vanish as well (Eq. (3.10)). Consequently

$$\hat{\mathscr{A}} = 0, \quad \hat{\mathscr{R}}_i = 0, \tag{5.1}$$

and we are left with non-trivial curvature and vorticity:

$$\hat{\mathscr{K}} = \hat{K} = K, \quad \varpi_{ij} = \partial_{[i} b_{j]} = \eta_{ij} * \varpi. \tag{5.2}$$

The Weyl–Carroll spatial covariant derivative $\hat{\mathscr{D}}_i$ reduces to the ordinary covariant derivative ∇_i, whereas the action of the Weyl–Carroll temporal covariant derivative $\hat{\mathscr{D}}_t$ vanishes.

We further assume that the Carrollian fluid is perfect: Q_i, π_i, Σ_{ij} and Ξ_{ij} vanish. This assumption is made according to the pattern of Ref. [10], where the asymptotically AdS Kerr–Taub–NUT spacetimes were studied starting from relativistic perfect fluids. Due to the duality relationships (4.21), (4.22) and (4.23) among the friction tensors of the Carrollian fluid and the geometric quantities χ_i, X_{ij} and Ψ_{ij}, the latter must also vanish. Using (4.14), (4.16) and (4.17), this sets the following simple geometric constraints:

$$\chi_i = 0 \Leftrightarrow \partial_i K = 0, \tag{5.3}$$

and

$$\Psi_{ij} = 0 \Leftrightarrow \left(\nabla_i \nabla_j - \frac{1}{2} a_{ij} \nabla_l \nabla^l \right) * \varpi = 0, \tag{5.4}$$

whereas X_{ij} vanishes identically without bringing any further restriction. These are equations for the metric $a_{ij}(\mathbf{x})$ and the scalar vorticity $*\varpi$, from which we can extract $b_i(\mathbf{x})$. Using (4.13), we also learn that

$$c = (\Delta + 2K) * \varpi, \tag{5.5}$$

where $\Delta = \nabla_l \nabla^l$ is the ordinary Laplacian operator on $\mathscr{S}$. The last piece of the geometrical data, (4.15), it is non-vanishing and reads:

$$\psi_j = 3\eta^l{}_j \partial_l * \varpi^2. \tag{5.6}$$

Finally, we must impose the fluid equations (4.25) and (4.26), leading to

$$\partial_t \varepsilon = 0, \quad \partial_i \varepsilon = 0. \tag{5.7}$$

The energy density ε of the Carrollian fluid is therefore a constant, which will be identified to the bulk mass parameter $M = 4\pi G \varepsilon$.

Every stationary Carrollian geometry encoded in $a_{ij}(\mathbf{x})$ and $b_i(\mathbf{x})$ with constant scalar curvature K hosts a conformal Carrollian perfect fluid with constant energy density, and is associated with the following exact Ricci-flat spacetime:

$$\mathrm{d}s^2_{\text{perf. fl.}} = -2\left(\mathrm{d}t - \boldsymbol{b}\right)\mathrm{d}r + \frac{2Mr + c * \varpi - K\rho^2}{\rho^2}\left(\mathrm{d}t - \boldsymbol{b}\right)^2 + \left(\mathrm{d}t - \boldsymbol{b}\right)\frac{\boldsymbol{\psi}}{3 * \varpi} + \rho^2 \mathrm{d}\ell^2, \tag{5.8}$$

where $\rho^2 = r^2 + *\varpi^2$. The vorticity $*\varpi$ is determined by Eq. (5.4), solved on a constant-curvature background.

Using holomorphic coordinates (see App. A), a constant-curvature metric on $\mathscr{S}$ reads:

$$\mathrm{d}\ell^2 = \frac{2}{P^2}\mathrm{d}\zeta \mathrm{d}\bar{\zeta} \tag{5.9}$$

with

$$P = 1 + \frac{K}{2}\zeta\bar{\zeta}, \quad K = 0, \pm 1, \tag{5.10}$$

corresponding to S^2 and E_2 or H_2 (sphere and Euclidean or hyperbolic planes). Using these expressions we can integrate (5.4). The general solution depends on three real, arbitrary parameters, n, a and ℓ:

$$* \varpi = n + a - \frac{2a}{P} + \frac{\ell}{P}\left(1 - |K|\right)\zeta\bar{\zeta}. \tag{5.11}$$

The parameter ℓ is relevant in the flat case exclusively. We can further integrate (3.11) and find thus

$$\boldsymbol{b} = \frac{\mathrm{i}}{P}\left(n - \frac{a}{P} + \frac{\ell}{2P}\left(1 - |K|\right)\zeta\bar{\zeta}\right)\left(\bar{\zeta}\mathrm{d}\zeta - \zeta \mathrm{d}\bar{\zeta}\right). \tag{5.12}$$

It is straightforward to determine the last pieces entering the bulk resumed metric (5.8):

$$c = 2Kn + 2\ell\left(1 - |K|\right) \tag{5.13}$$

and

$$\frac{\boldsymbol{\psi}}{3 * \varpi} = 2\eta^j{}_i \partial_j * \varpi \mathrm{d}x^i = 2\mathrm{i}\frac{Ka + \ell\left(1 - |K|\right)}{P^2}\left(\bar{\zeta}\mathrm{d}\zeta - \zeta \mathrm{d}\bar{\zeta}\right). \tag{5.14}$$

In order to reach a more familiar form for the line element (5.8), it is convenient to trade the complex-conjugate coordinates ζ and $\bar{\zeta}$ for their modulus[25] and argument

$$\zeta = Z\mathrm{e}^{i\Phi}, \tag{5.15}$$

and move from Eddington–Finkelstein to Boyer–Lindquist by setting

$$\mathrm{d}t \to \mathrm{d}t - \frac{r^2 + (n-a)^2}{\Delta_r}\mathrm{d}r, \quad \mathrm{d}\Phi \to \mathrm{d}\Phi - \frac{Ka + \ell(1 - |K|)}{\Delta_r}\mathrm{d}r \tag{5.16}$$

[25] The modulus and its range depend on the curvature. It is commonly expressed as: $Z = \sqrt{2}\tan\frac{\Theta}{2}, 0 < \Theta < \pi$ for S^2; $Z = \frac{R}{\sqrt{2}}, 0 < R < +\infty$ for E_2; $Z = \sqrt{2}\tanh\frac{\Psi}{2}, 0 < \Psi < +\infty$ for H_2.

with

$$\Delta_r = -2Mr + K\left(r^2 + a^2 - n^2\right) + 2\ell(n-a)(|K|-1). \tag{5.17}$$

We obtain finally:

$$\begin{aligned} \mathrm{d}s^2_{\text{perf. fl.}} &= -\frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \frac{2}{P}\left(n - \frac{a}{P} + \frac{\ell}{2P}\left(1-|K|\right)Z^2\right)Z^2\mathrm{d}\Phi\right)^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 \\ &+\frac{2\rho^2}{P^2}\mathrm{d}Z^2 + \frac{2Z^2}{\rho^2P^2}\left(\left(Ka+\ell\left(1-|K|\right)\right)\mathrm{d}t - \left(r^2+(n-a)^2\right)\mathrm{d}\Phi\right)^2 \end{aligned} \tag{5.18}$$

with

$$P = 1 + \frac{K}{2}Z^2, \quad \rho^2 = r^2 + \left(n + a - \frac{2a}{P} + \frac{\ell}{P}\left(1-|K|\right)Z^2\right)^2. \tag{5.19}$$

This bulk metric is Ricci-flat for any value of the parameters M, n, a and ℓ with $K = 0, \pm 1$. For vanishing n, a and ℓ, and with $M > 0$ and $K = 1$, one recovers the standard asymptotically flat Schwarzschild solution with spherical horizon. For $K = 0$ or -1, this is no longer Schwarzschild, but rather a metric belonging to the A class (see e.g. [83]). The parameter a switches on rotation, while n is the standard nut charge. The parameter ℓ is also a rotational parameter available only in the flat-$\mathscr{S}$ case. Scanning over all these parameters, in combination with the mass and K, we recover the whole Kerr–Taub–NUT family of black holes, plus other, less familiar configurations, like the A-metric quoted above.

For the solutions at hand, the only potentially non-vanishing Carrollian boundary Cotton descendants are c and $\boldsymbol{\psi}$, displayed in (5.13) and (5.14). The first is non-vanishing for asymptotically locally flat spacetimes, and this requires non-zero n or ℓ. The second measures the bulk twist. In every case the metric (5.18) is Petrov type D.

We would like to conclude the example of Carrollian conformal perfect fluids with a comment regarding the isometries of the associated resummed Ricci-flat spacetimes with line element (5.18). For vanishing a and ℓ, there are *four* isometry generators and the field is in this case a stationary gravito-electric and/or gravito-magnetic monopole (mass and nut parameters M, n). Constant-r hypersurfaces are homogeneous spaces in this case. The number of Killing fields is reduced to *two* (∂_t and ∂_Φ) whenever any of the rotational parameters a or ℓ is non-zero. These parameters make the gravitational field dipolar.

The bulk isometries are generally inherited from the boundary symmetries, *i.e.* the symmetries of the Carrollian geometry and the Carrollian fluid. The time-like Killing field ∂_t is clearly rooted to the stationarity of the boundary data. The space-like ones have legs on ∂_Φ and ∂_Z, and are associated to further boundary symmetries. From a Riemannian viewpoint, the metric (5.9) with (5.10) on the two-dimensional boundary surface $\mathscr{S}$ admits three Killing vector fields:

$$\boldsymbol{X}_1 = \mathrm{i}\left(\zeta\partial_\zeta - \bar\zeta\partial_{\bar\zeta}\right), \tag{5.20}$$

$$\boldsymbol{X}_2 = \mathrm{i}\left(\left(1-\frac{K}{2}\zeta^2\right)\partial_\zeta - \left(1-\frac{K}{2}\bar\zeta^2\right)\partial_{\bar\zeta}\right), \tag{5.21}$$

$$\boldsymbol{X}_3 = \left(1+\frac{K}{2}\zeta^2\right)\partial_\zeta + \left(1+\frac{K}{2}\bar\zeta^2\right)\partial_{\bar\zeta}, \tag{5.22}$$

closing in $\mathfrak{so}(3)$, $\mathfrak{e}_2$ and $\mathfrak{so}(2,1)$ algebras for $K = +1, 0$ and -1 respectively. The Carrollian structure is however richer as it hinges on the set $\{a_{ij}, b_i, \Omega\}$. Hence, not all Riemannian isometries generated by a Killing field $\boldsymbol{X}$ of $\mathscr{S}$ are necessarily promoted to Carrollian symmetries. For the latter, it is natural to further require the Carrollian vorticity be invariant:

$$\mathscr{L}_{\mathrm{X}} * \varpi = \mathbf{X}(*\varpi) = 0. \tag{5.23}$$

Condition (5.23) is fulfilled for all fields $\boldsymbol{X}_A$ ($A = 1,2,3$) in (5.20), (5.21) and (5.22), only as long as $a = \ell = 0$, since $*\varpi = n$. Otherwise $*\varpi$ is non-constant and only $\boldsymbol{X}_1 = \mathrm{i}\left(\zeta\partial_\zeta - \bar\zeta\partial_{\bar\zeta}\right) = \partial_\Phi$ leaves it invariant. This is in line with the bulk isometry properties discussed earlier, while it provides a Carrollian-boundary manifestation of the rigidity theorem.

5.2 Vorticity-free Carrollian fluid and the Ricci-flat Robinson–Trautman

The zero-k limit of the relativistic Robinson–Trautman fluid presented in Sec. (3.2) (Eqs. (3.53)–(3.56)) is in agreement with the direct Carrollian approach of Sec. 4.2. Indeed, it is straightforward to check that the general formulas (4.13)–(4.17) give $c = 0$ together with

$$\chi = \frac{\mathrm{i}}{2}\left(\partial_\zeta K \mathrm{d}\zeta - \partial_{\bar\zeta} K \mathrm{d}\bar\zeta\right), \quad \boldsymbol{X} = \frac{\mathrm{i}}{P^2}\left(\partial_\zeta\left(P^2\partial_t\partial_\zeta \ln P\right)\mathrm{d}\zeta^2 - \partial_{\bar\zeta}\left(P^2\partial_t\partial_{\bar\zeta}\ln P\right)\mathrm{d}\bar\zeta^2\right), \tag{5.24}$$

while $\psi_i = 0 = \Psi_{ij}$. These expressions satisfy (4.18)–(4.20), and the duality relations (4.21), (4.22) and (4.23) lead to the friction components of the energy–momentum tensor Q_i, Σ_{ij} and Ξ_{ij}, precisely as they appear in (3.57), (3.58). The general hydrodynamic equations (4.25), (4.26), are solved with[26] $\pi_i = 0$ and $\varepsilon = \varepsilon(t)$ satisfying (3.59), *i.e.* Robinson–Trautman's (3.62).

Our goal is to present here the resummation of the derivative expansion (4.9) into a Ricci-flat spacetime dual to the fluid at hand. The basic feature of the latter is that $b_i = 0$ and $\Omega = 1$, hence it is vorticity-free – on top of being shearless. With these data, using (4.9), we find

$$\mathrm{d}s^2_{\mathrm{RT}} = -2\mathrm{d}t\left(\mathrm{d}r + H\mathrm{d}t\right) + 2\frac{r^2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar\zeta, \tag{5.25}$$

[26] Since π_i is not related to the geometry by duality as the other friction and heat tensors, it can *a priori* assume any value. It is part of the Carrollian Robinson–Trautman fluid definition to set it to zero.

where

$$2H = -2r\partial_t \ln P + K - \frac{2M(t)}{r}, \tag{5.26}$$

with $K = 2P^2\partial_{\bar{\zeta}}\partial_\zeta \ln P$ the Gaussian curvature of (3.53). This metric is Ricci-flat provided the energy density $\varepsilon(t) = {}^{M(t)}/_{4\pi G}$ and the function $P = P(t,\zeta,\bar{\zeta})$ satisfy (3.62). These are algebraically special spacetimes of all types, as opposed to the Kerr–Taub–NUT family studied earlier (Schwarzschild solution is common to these two families). Furthermore they never have twist ($\boldsymbol{\psi} = \boldsymbol{\Psi} = 0$) and are generically asymptotically locally but not globally flat due to $\boldsymbol{\chi}$ and $\mathbf{X}$.

The specific Petrov type of Robinson–Trautman solutions is determined by analyzing the tensors (4.31), or (4.32) and (4.33) in holomorphic coordinates:

$$\varepsilon^+ = \frac{M(t)}{4\pi G}, \quad \mathbf{Q}^+ = -\frac{1}{8\pi G}\partial_\zeta K \mathrm{d}\zeta, \quad \mathbf{\Sigma}^+ = -\frac{1}{4\pi G P^2}\partial_\zeta \left(P^2 \partial_t \partial_\zeta \ln P\right) \mathrm{d}\zeta^2. \tag{5.27}$$

We find the following classification (see [12]):

II generic;

III with $\varepsilon^+ = 0$ and $\nabla_i Q^{+i} = 0$;

N with $\varepsilon^+ = 0$ and $Q^+_i = 0$;

D with $2Q^+_i Q^+_j = 3\varepsilon^+ \Sigma^+_{ij}$ and vanishing traceless part of $\nabla_{(i} Q^+_{j)}$.

6 Conclusions

The main message of our work is that starting with the standard AdS holography, there is a well-defined zero-cosmological-constant limit that relates asymptotically flat spacetimes to Carrollian fluids living on their null boundaries.

In order to unravel this relationship and make it operative for studying holographic duals, we used the derivative expansion. Originally designed for asymptotically anti-de Sitter spacetimes with cosmological constant $\Lambda = -3k^2$, this expansion provides their line element in terms of the conformal boundary data: a pseudo-Riemannian metric and a relativistic fluid. It is expressed in Eddington–Finkelstein coordinates, where the zero-k limit is unambiguous: it maps the pseudo-Riemannian boundary $\mathscr{I}$ onto a Carrollian geometry $\mathbb{R} \times \mathscr{S}$, and the conformal relativistic fluid becomes Carrollian.

The emergence of the conformal Carrollian symmetry in the Ricci-flat asymptotic is not a surprise, as we have extensively discussed in the introduction. In particular, the BMS group has been used for investigating the asymptotically flat dual dynamics. What is remarkable is the efficiency of the derivative expansion to implement the limiting procedure and deliver a genuine holographic relationship between Ricci-flat spacetimes and conformal Carrollian fluids. These are defined on $\mathscr{S}$ but their dynamics is rooted in $\mathbb{R} \times \mathscr{S}$.

Even though proving that the derivative expansion is unconditionally well-behaved in the limit under consideration is still part of our agenda, we have demonstrated this property in the instance where it is resummable.

The resummability of the derivative expansion has been studied in our earlier works about anti-de Sitter fluid/gravity correspondence. It has two features:

- the shear of the fluid congruence vanishes;
- the heat current and the viscous stress tensor are determined from the Cotton current and stress tensor components via a transverse (with respect to the velocity) duality.

The first considerably simplifies the expansion. Together with the second, it ultimately dictates the structure of the bulk Weyl tensor, making the Einstein spacetime of special Petrov type. The conservation of the energy–momentum tensor is the only requirement left for the bulk be Einstein. It involves the energy density (*i.e.* the only fluid observable left undetermined) and various geometric data in the form of partial differential equations (as is the Robinson–Trautman for the vorticity-free situation).

This pattern survives the zero-k limit, taken in a frame where the relativistic fluid is at rest. The corresponding Carrollian fluid – at rest *by law* – is required to be shearless, but has otherwise acceleration, vorticity and expansion. Since the fluid is at rest, these are geometric data, as are the descendants of the Cotton tensor used again to formulate the duality that determines the dissipative components of the Carrollian fluid.

The study of the Cotton tensor and its Carrollian limit is central in our analysis. In Carrollian geometry (conformal in the case under consideration) it opens the pandora box of the classification of curvature tensors, which we have marginally discussed here. Our observation is that the Cotton tensor grants the zero-k limiting Carrollian geometry on $\mathscr{S}$ with a scalar, two vectors and two symmetric, traceless tensors, satisfying a set of identities inherited from the original conservation equation.

In a similar fashion, the relativistic energy–momentum tensor descends in a scalar (the energy density), two heat currents and two viscous stress tensors. This doubling is suggested by that of the Cotton. The physics behind it is yet to be discovered, as it requires a microscopic approach to Carrollian fluids, missing at present. Irrespective of its microscopic origin, however, this is an essential result of our work, in contrast with previous attempts. Not only we can state that the fluid holographically dual to a Ricci-flat spacetime is neither relativistic, nor Galilean, but we can also exhibit for the actually Carrollian fluid the fundamental observables and the equations they obey.[27] These are quite convoluted, and

[27] From this perspective, trying to design four-dimensional flat holography using two-dimensional conformal field theory described in terms of a conserved two-dimensional energy–momentum tensor [42–44] looks inappropriate.

whenever satisfied, the resummed metric is Ricci-flat.

Our analysis, amply illustrated by two distinct examples departing from Carrollian hydrodynamics and ending on widely used Ricci-flat spacetimes, raises many questions, which deserve a comprehensive survey.

As already acknowledged, the Cotton Carrollian descendants enter the holographic reconstruction of a Ricci-flat spacetime, along with the energy–momentum data. It would be rewarding to explore the information stored in these objects, which may carry the boundary interpretation of the Bondi news tensor as well as of the asymptotic charges one can extract from the latter.

We should stress at this point that Cotton and energy–momentum data (and the charges they transport) play dual rôles. The nut and the mass provide the best paradigm of this statement. Altogether they raise the question on the thermodynamic interpretation of magnetic charges. Although we cannot propose a definite answer to this question, the tools of fluid/gravity holography (either AdS or flat) may turn helpful. This is tangible in the case of algebraically special Einstein solutions, where the underlying integrability conditions set a deep relationship between geometry and energy–momentum *i.e.* between geometry and local thermodynamics. To make this statement more concrete, observe the heat current as constructed using the integrability conditions, Eq. (4.21):

$$Q_i = -\frac{1}{16\pi G}\left(\hat{\mathscr{D}}_i\hat{\mathscr{K}} - \eta^j{}_i\hat{\mathscr{D}}_j\hat{\mathscr{A}} + 4 * \varpi\eta^j{}_i\hat{\mathscr{R}}_j\right).$$

In the absence of magnetic charges, only the first term is present and it is tempting to set a relationship between the temperature and the gravito-electric curvature scalar $\hat{\mathscr{K}}$. This was precisely discussed in the AdS framework when studying the Robinson–Trautman relativistic fluid, in Ref. [66]. Magnetic charges switch on the other terms, exhibiting natural thermodynamic potentials, again related with curvature components ($\hat{\mathscr{A}}$ and $\hat{\mathscr{R}}_j$).

We would like to conclude with a remark. On the one hand, we have shown that the boundary fluids holographically dual to Ricci-flat spacetimes are of Carrollian nature. On the other hand, the stretched horizon in the membrane paradigm seems to be rather described in terms of Galilean hydrodynamics [17,18,84]. Whether and how these two pictures could been related is certainly worth refining.

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A Carrollian boundary geometry in holomorphic coordinates

Using Carrollian diffeomorphisms (3.2), the metric (3.1) of the Carrollian geometry on the two-dimensional surface $\mathscr{S}$ can be recast in conformally flat form,

$$\mathrm{d}\ell^2 = \frac{2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta} \tag{A.1}$$

with $P = P(t,\zeta,\bar{\zeta})$ a real function, under the necessary and sufficient condition that the Carrollian shear ξ_{ij} displayed in (3.14) vanishes. We will here assume that this holds and present a number of useful formulas for Carrollian and conformal Carrollian geometry. These geometries carry two further pieces of data: $\Omega(t,\zeta,\bar{\zeta})$ and

$$\boldsymbol{b} = b_\zeta(t,\zeta,\bar{\zeta})\,\mathrm{d}\zeta + b_{\bar{\zeta}}(t,\zeta,\bar{\zeta})\,\mathrm{d}\bar{\zeta} \tag{A.2}$$

with $b_{\bar{\zeta}}(t,\zeta,\bar{\zeta}) = \bar{b}_\zeta(t,\zeta,\bar{\zeta})$. Our choice of orientation is inherited from the one adopted for the relativistic boundary (see footnote 13) with $a_{\zeta\bar{\zeta}} = 1/P^2$ is[28]

$$\eta_{\zeta\bar{\zeta}} = -\frac{\mathrm{i}}{P^2}. \tag{A.3}$$

The first-derivative Carrollian tensors are the acceleration (3.10), the expansion (3.14) and the scalar vorticity (3.20):

$$\varphi_\zeta = \partial_t \frac{b_\zeta}{\Omega} + \hat{\partial}_\zeta \ln\Omega, \quad \varphi_{\bar{\zeta}} = \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \ln\Omega, \tag{A.4}$$

$$\theta = -\frac{2}{\Omega}\partial_t \ln P, \quad *\varpi = \frac{\mathrm{i}\Omega P^2}{2}\left(\hat{\partial}_\zeta \frac{b_{\bar{\zeta}}}{\Omega} - \hat{\partial}_{\bar{\zeta}} \frac{b_\zeta}{\Omega}\right) \tag{A.5}$$

with

$$\hat{\partial}_\zeta = \partial_\zeta + \frac{b_\zeta}{\Omega}\partial_t, \quad \hat{\partial}_{\bar{\zeta}} = \partial_{\bar{\zeta}} + \frac{b_{\bar{\zeta}}}{\Omega}\partial_t. \tag{A.6}$$

[28] This amounts to setting $\sqrt{a} = \mathrm{i}/P^2$ in coordinate frame and $\epsilon_{\zeta\bar{\zeta}} = -1$.

Curvature scalars and vector are second-derivative (see (3.19), (3.22)):[29]

$$\hat{K} = P^2 \left(\hat{\partial}_{\bar{\zeta}}\hat{\partial}_{\zeta} + \hat{\partial}_{\zeta}\hat{\partial}_{\bar{\zeta}}\right) \ln P, \quad \hat{A} = \mathrm{i}P^2 \left(\hat{\partial}_{\bar{\zeta}}\hat{\partial}_{\zeta} - \hat{\partial}_{\zeta}\hat{\partial}_{\bar{\zeta}}\right) \ln P, \tag{A.7}$$

$$\hat{r}_{\zeta} = \frac{1}{2}\hat{\partial}_{\zeta}\left(\frac{1}{\Omega}\partial_t \ln P\right), \quad \hat{r}_{\bar{\zeta}} = \frac{1}{2}\hat{\partial}_{\bar{\zeta}}\left(\frac{1}{\Omega}\partial_t \ln P\right), \tag{A.8}$$

and we also quote:

$$*\varphi = \mathrm{i}P^2\left(\hat{\partial}_{\zeta}\varphi_{\bar{\zeta}} - \hat{\partial}_{\bar{\zeta}}\varphi_{\zeta}\right), \tag{A.9}$$

$$\hat{\nabla}_k\varphi^k = P^2\left[\hat{\partial}_{\zeta}\partial_t\tfrac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}}\partial_t\tfrac{b_{\zeta}}{\Omega} + \left(\hat{\partial}_{\zeta}\hat{\partial}_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}}\hat{\partial}_{\zeta}\right)\ln\Omega\right]. \tag{A.10}$$

Regarding conformal Carrollian tensors we remind the weight-2 curvature scalars (3.40):

$$\hat{\mathscr{K}} = \hat{K} + \hat{\nabla}_k\varphi^k, \quad \hat{\mathscr{A}} = \hat{A} - *\varphi, \tag{A.11}$$

and the weight-1 curvature one-form (3.38):

$$\hat{\mathscr{R}}_{\zeta} = \frac{1}{\Omega}\partial_t\varphi_{\zeta} - \frac{1}{2}\left(\hat{\partial}_{\zeta} + \varphi_{\zeta}\right)\theta, \quad \hat{\mathscr{R}}_{\bar{\zeta}} = \frac{1}{\Omega}\partial_t\varphi_{\bar{\zeta}} - \frac{1}{2}\left(\hat{\partial}_{\bar{\zeta}} + \varphi_{\bar{\zeta}}\right)\theta. \tag{A.12}$$

The three-derivative Cotton descendants displayed in (4.13)–(4.17) are a scalar

$$c = \left(\hat{\mathscr{D}}_l\hat{\mathscr{D}}^l + 2\hat{\mathscr{K}}\right) * \varpi \tag{A.13}$$

of weight 3 ($*\varpi$ is of weght 1), two vectors

$$\chi_{\zeta} = \tfrac{\mathrm{i}}{2}\hat{\mathscr{D}}_{\zeta}\hat{\mathscr{K}} + \tfrac{1}{2}\hat{\mathscr{D}}_{\zeta}\hat{\mathscr{A}} - 2*\varpi\hat{\mathscr{R}}_{\zeta}, \quad \chi_{\bar{\zeta}} = -\tfrac{\mathrm{i}}{2}\hat{\mathscr{D}}_{\bar{\zeta}}\hat{\mathscr{K}} + \tfrac{1}{2}\hat{\mathscr{D}}_{\bar{\zeta}}\hat{\mathscr{A}} - 2*\varpi\hat{\mathscr{R}}_{\bar{\zeta}}, \tag{A.14}$$

$$\psi_{\zeta} = 3\mathrm{i}\hat{\mathscr{D}}_{\zeta} * \varpi^2, \quad \psi_{\bar{\zeta}} = -3\mathrm{i}\hat{\mathscr{D}}_{\bar{\zeta}} * \varpi^2, \tag{A.15}$$

of weight 2, and two symmetric and traceless tensors

$$X_{\zeta\zeta} = \mathrm{i}\hat{\mathscr{D}}_{\zeta}\hat{\mathscr{R}}_{\zeta}, \quad X_{\bar{\zeta}\bar{\zeta}} = -\mathrm{i}\hat{\mathscr{D}}_{\bar{\zeta}}\hat{\mathscr{R}}_{\bar{\zeta}}, \tag{A.16}$$

$$\Psi_{\zeta\zeta} = \hat{\mathscr{D}}_{\zeta}\hat{\mathscr{D}}_{\zeta} * \varpi, \quad \Psi_{\bar{\zeta}\bar{\zeta}} = \hat{\mathscr{D}}_{\bar{\zeta}}\hat{\mathscr{D}}_{\bar{\zeta}} * \varpi, \tag{A.17}$$

of weight 1. Notice that in holomorphic coordinates a symmetric and traceless tensor S_{ij} has only diagonal entries: $S_{\zeta\bar{\zeta}} = 0 = S_{\bar{\zeta}\zeta}$.

We also remind for convenience some expressions for the determination of Weyl–Carroll

[29] We also quote for completeness (useful *e.g.* in Eq. (A.11)):

$$\hat{K} = K + P^2\left[\partial_{\zeta}\frac{b_{\bar{\zeta}}}{\Omega} + \partial_{\bar{\zeta}}\frac{b_{\zeta}}{\Omega} + \partial_t\frac{b_{\zeta}b_{\bar{\zeta}}}{\Omega^2} + 2\frac{b_{\bar{\zeta}}}{\Omega}\partial_{\zeta} + 2\frac{b_{\zeta}}{\Omega}\partial_{\bar{\zeta}} + 2\frac{b_{\zeta}b_{\bar{\zeta}}}{\Omega^2}\partial_t\right]\partial_t \ln P$$

with $K = 2P^2\partial_{\bar{\zeta}}\partial_{\zeta}\ln P$ the ordinary Gaussian curvature of the two-dimensional metric (A.1).

covariant derivatives. If Φ is a weight-w scalar function

$$\hat{\mathscr{D}}_\zeta \Phi = \hat{\partial}_\zeta \Phi + w\varphi_\zeta \Phi, \quad \hat{\mathscr{D}}_{\bar{\zeta}} \Phi = \hat{\partial}_{\bar{\zeta}} \Phi + w\varphi_{\bar{\zeta}} \Phi. \tag{A.18}$$

For weight-w form components V_ζ and $V_{\bar{\zeta}}$ the Weyl–Carroll derivatives read:

$$\hat{\mathscr{D}}_\zeta V_\zeta = \hat{\nabla}_\zeta V_\zeta + (w+2)\varphi_\zeta V_\zeta, \quad \hat{\mathscr{D}}_{\bar{\zeta}} V_{\bar{\zeta}} = \hat{\nabla}_{\bar{\zeta}} V_{\bar{\zeta}} + (w+2)\varphi_{\bar{\zeta}} V_{\bar{\zeta}}, \tag{A.19}$$

$$\hat{\mathscr{D}}_\zeta V_{\bar{\zeta}} = \hat{\nabla}_\zeta V_{\bar{\zeta}} + w\varphi_\zeta V_{\bar{\zeta}}, \quad \hat{\mathscr{D}}_{\bar{\zeta}} V_\zeta = \hat{\nabla}_{\bar{\zeta}} V_\zeta + w\varphi_{\bar{\zeta}} V_\zeta, \tag{A.20}$$

while the Carrollian covariant derivatives are simply:

$$\hat{\nabla}_\zeta V_\zeta = \frac{1}{P^2}\hat{\partial}_\zeta \left(P^2 V_\zeta\right), \quad \hat{\nabla}_{\bar{\zeta}} V_{\bar{\zeta}} = \frac{1}{P^2}\hat{\partial}_{\bar{\zeta}} \left(P^2 V_{\bar{\zeta}}\right), \tag{A.21}$$

$$\hat{\nabla}_\zeta V_{\bar{\zeta}} = \hat{\partial}_\zeta V_{\bar{\zeta}}, \quad \hat{\nabla}_{\bar{\zeta}} V_\zeta = \hat{\partial}_{\bar{\zeta}} V_\zeta. \tag{A.22}$$

Finally,

$$\hat{\mathscr{D}}_k \hat{\mathscr{D}}^k \Phi = P^2 \left(\hat{\partial}_\zeta \hat{\partial}_{\bar{\zeta}} \Phi + \hat{\partial}_{\bar{\zeta}} \hat{\partial}_\zeta \Phi + w\Phi \left(\hat{\partial}_\zeta \varphi_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}} \varphi_\zeta\right) + 2w \left(\varphi_\zeta \hat{\partial}_{\bar{\zeta}} \Phi + \varphi_{\bar{\zeta}} \hat{\partial}_\zeta \Phi + w\varphi_\zeta \varphi_{\bar{\zeta}} \Phi\right)\right). \tag{A.23}$$

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Carrollian conservation laws and Ricci-flat gravity

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Abstract

We construct the Carrollian equivalent of the relativistic energy–momentum tensor, based on variation of the action with respect to the elementary fields of the Carrollian geometry. We prove that, exactly like in the relativistic case, it satisfies conservation equations that are imposed by general Carrollian covariance. In the flat case we recover the usual non-symmetric energy–momentum tensor obtained using Nœther procedure. We show how Carrollian conservation equations emerge taking the ultra-relativistic limit of the relativistic ones. We introduce Carrollian Killing vectors and build associated conserved charges. We finally apply our results to asymptotically flat gravity, where we interpret the boundary equations of motion as ultra-relativistic Carrollian conservation laws, and observe that the surface charges obtained through covariant phase-space formalism match the ones we defined earlier.

Contents

1 Introduction

The Carroll group was firstly introduced in [1] as a contraction of the Poincaré group for vanishing speed of light and this is referred to as the *ultra-relativistic* limit. The main feature is that, as opposed to the Galilean case, this group allows for boosts only in the time direction: space is absolute.

We could wonder what happens when we take the zero-c limit of a relativistic general-covariant theory. The resulting theory ends up being covariant only under a subset of the diffeormorphisms, as illustrated in [2] , the so-called *Carrollian diffeormorphisms*

$$t' = t'(t, \mathbf{x}), \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}). \tag{1.1}$$

The ultra-relativistic limit breaks the spacetime metric into three independent data, a scalar density, a connection and a spatial metric. These geometric fields are nicely interpreted as constituents of a *Carrollian geometry*, as we will show in Sec. 2. Now considering an action defined on such a geometry, covariant under (1.1), we are facing a problem in defining the energy–momentum tensor. Indeed, in general-covariant theories it is obtained as the variation of the action with respect to the metric. This requires the existence of a regular metric *i.e.* of a pseudo-Riemannian manifold, but in the Carrollian case, as we mentioned, there is no spacetime non-degenerate metric. Therefore, we must introduce new objects. The core of Sec. 2 will be dedicated to the definition of these new objects, dubbed *Carrollian momenta*, and obtained as the variation of the action with respect to the 3 geometric fields mentioned above.

General covariance usually ensures that the energy–momentum tensor is conserved. In the context of Carroll-covariant theories, we will derive similar conservation equations for the Carrollian momenta. In order to gain confidence with these new definitions, we will study a simple Carrollian action, and show that, on a flat geometrical background, the Carrollian momenta are packaged in a spacetime energy–momentum tensor which coincides with the Nœther current associated with spacetime translations. This will be done in Sec. 3.

We will further discuss the intrinsic Carrollian nature of the ultra-relativistic limit. Indeed, in Sec. 4, starting from the conservation equations of an energy–momentum tensor, covariant under all changes of coordinates, we reach conservation laws that look strikingly similar to the ones we derived for the Carrollian momenta, which are covariant only under (1.1).

In general-covariant theories, the existence of a Killing vector allows to build a conserved current by projecting the energy–momentum tensor on the Killing field. This ultimately leads to a conserved charge. After briefly introducing the notion of conserved current in the Carrollian context, we define in Sec. 5 the Carrollian Killing vectors and build their associated currents and charges.

There are by now different instances in which the Carrollian framework has found applications. For instance, it has been used in electromagnetism [3] and to discuss the so-called Carroll strings [4]. The last part of this paper is devoted to yet another application of the Carrollian framework: flat holography. The latter is a holographic correspondence between a theory of asymptotically flat gravity and a non-gravitational theory leaving on its boundary, see [5–12] for recent progresses in this direction. Asymptotically anti-de-Sitter spacetimes enjoy a timelike pseudo-Riemannian boundary and the associated metric sources its dual operator: the boundary energy–momentum tensor. For asymptotically flat spacetimes, the dual theory leaves on the null infinity $\mathcal{I}^+$. Nevertheless this surface does not carry the same geometrical structure, it is a null hypersurface thus equipped with a Carrollian geometry [9] and this will be the source for the Carrollian momenta. The conservation of the latter will be shown to correspond to the gravitational dynamics in the bulk.[1] As a cross check, it has been shown [14] that the conformal Carroll group has a particular realization which is nothing but the Bondi–Metzner–Sachs (BMS) group [15]: the symmetries associated with a Carrollian structure match the asymptotic symmetries of the bulk.

In Secs. 6.1 and 6.2 we focus on the Carrollian theory on $\mathcal{I}^+$ and its relevance for gravitational asymptotically flat duals in 3 and 4 dimensions, and in Sec. 6.3 we study explicit solutions, namely the Robinson–Trautman and the Kerr–Taub–NUT families.

2 Carrollian momenta

We start with a brief reminder on the energy–momentum tensor in the relativistic case, and then define its counterpart, that we call *Carrollian momenta*, on a general Carrollian background. This requires the study of Carrollian geometry and covariance, which will be eventually the guideline for obtaining the conservation equations of these momenta. We also extend our results for a scale invariant theory (Weyl invariant) and write the conservation equations in a Weyl-covariant way. Finally, we focus on the flat case and show how, in this case only, one can promote the Carrollian momenta to a "non-symmetric energy–momentum tensor".

2.1 A relativistic synopsis

In a relativistic theory, the energy–momentum tensor is usually defined as

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}. \tag{2.1}$$

[1]Some attention has been recently given to the interpretation of the bulk dynamics in terms of null conservation laws, see *e.g.* [13].

For a general-covariant theory, it is easy to prove that it is conserved. Indeed, considering the variation of the action under an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$, we have

$$\delta_\xi S = \int \mathrm{d}^{d+1}x \left(\frac{\delta S}{\delta g_{\mu\nu}} \delta_\xi g_{\mu\nu} + \frac{\delta S}{\delta \phi} \delta_\xi \phi \right) + \text{b. t.}, \tag{2.2}$$

where $d+1$ is the spacetime dimension and ϕ stands for the various other fields of the theory. We assume that we are on-shell so $\frac{\delta S}{\delta \phi} = 0$. Moreover, δ_ξ is the Lie derivative, which for a Levi Civita reads

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{2.3}$$

We thus obtain

$$\delta_\xi S = -\int \mathrm{d}^{d+1}x \sqrt{-g} T^{\mu\nu} \nabla_\mu \xi_\nu = \int \mathrm{d}^{d+1}x \sqrt{-g} \nabla_\mu T^{\mu\nu} \xi_\nu + \text{b. t.}. \tag{2.4}$$

If the theory is general-covariant, $\delta_\xi S = 0$ for all ξ. From this we deduce that $\nabla_\mu T^{\mu\nu}$ vanishes on shell, which is the usual conservation law of the energy–momentum tensor.

2.2 Carrollian geometry

We briefly introduce here the Carrollian geometry, as it emerges from an ultra-relativistic ($c \to 0$) limit of the relativistic metric. It has been shown in [2, 12] that the conservation equations of a relativistic energy–momentum tensor, covariant under all diffeomorphisms, lead, in the $c \to 0$ limit, to equations covariant under a subset called *Carrollian diffeomorphisms*

$$t' = t'(t, \mathbf{x}), \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}). \tag{2.5}$$

An adequate parametrization for taking this limit is the so-called Randers–Papapetrou, in which the various components transform nicely under this subset of diffeomorphisms. The metric takes the form[2]

$$g = \begin{pmatrix} -\Omega^2 & c\Omega b_i \\ c\Omega b_j & a_{ij} - c^2 b_i b_j \end{pmatrix}_{\{c\mathrm{d}t, \mathrm{d}x^i\}} \tag{2.6}$$

where $i = \{1, \ldots, d\}$. Indeed, under (2.5)

$$a'_{ij} = a_{kl} J^{-1k}{}_i J^{-1l}{}_j, \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i \right) J^{-1i}{}_k, \quad \Omega' = \frac{\Omega}{J}, \tag{2.7}$$

where $J_i^k = \frac{\partial x'^k}{\partial x^i}$, $j_i = \frac{\partial t'}{\partial x^i}$ and $J = \frac{\partial t'}{\partial t}$. In the $c \to 0$ limit the metric becomes degenerate, hence we cannot package the different metric fields in a spacetime tensor $g_{\mu\nu}$, but instead we have to treat those three fields separately: time and space decouple as (2.5) clearly suggests. We

[2]Every metric can be parametrized in this way. The alternative parametrization, known as Zermelo, turns out to be useful for the Galilean limit (see [2, 16]).

therefore trade the metric $g_{\mu\nu}$ for the time lapse $\Omega(t,\mathbf{x})$, connection $b_i(t,\mathbf{x})$ and spatial metric $a_{ij}(t,\mathbf{x})$,[3] which we refer to as Carrollian metric fields, defining a Carrollian geometry. On the derivatives, (2.5) infers

$$\partial'_t = \frac{1}{J}\partial_t, \quad \partial'_i = J_i^{-1k}\left(\partial_k - \frac{j_k}{J}\partial_t\right), \tag{2.8}$$

which implies that the spatial derivative is not a Carrollian tensor and the temporal one is a density. Therefore we introduce the Carroll-covariant derivatives $\frac{1}{\Omega}\partial_t$ and $\hat{\nabla}_i$. In the temporal one the role of Ω as a time lapse is clear, and the spatial one is defined through its action on scalars as

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega}\partial_t. \tag{2.9}$$

On Carrollian tensors, it acts as usual with the following Christoffel symbols

$$\hat{\gamma}^i_{jk} = \frac{a^{il}}{2}\left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}\right). \tag{2.10}$$

By construction, $\hat{\partial}_i$ transforms as a Carrollian tensor

$$\hat{\partial}'_i = J_i^{-1k}\hat{\partial}_k, \tag{2.11}$$

and thus we also see clearly the role of b_i as connection. Out of the Carrollian metric fields, we can build first-order derivative geometrical objects

$$\varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right), \tag{2.12}$$

$$\theta = \frac{1}{\Omega}\partial_t \ln\sqrt{a}, \tag{2.13}$$

$$\xi_{ij} = \frac{1}{\Omega}\left(\frac{1}{2}\partial_t a_{ij} - \frac{1}{d}a_{ij}\partial_t \ln\sqrt{a}\right), \tag{2.14}$$

$$\varpi_{ij} = \partial_{[i}b_{j]} + \frac{1}{\Omega}b_{[i}\partial_{j]}\Omega + \frac{1}{\Omega}b_{[i}\partial_t b_{j]}. \tag{2.15}$$

They are all Carrollian tensors and they encode the non-flatness of the Carrollian geometrical structure we are defining. They will turn out very useful in writing the conservation equations of the Carrollian momenta defined in the next section.

[3] Hence, we will use a_{ij} to raise and lower spatial indexes in the Carrollian geometry.

2.3 Carrollian momenta

We define the Carrollian equivalent of the energy–momentum tensor as the three following pieces of data:

$$\mathcal{O} = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta \Omega}, \quad \mathcal{B}^i = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta b_i} \quad \text{and} \quad \mathcal{A}^{ij} = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta a_{ij}}. \tag{2.16}$$

Here $\Omega\sqrt{a}$ is the Carrollian counterpart of the relativistic $\sqrt{-g}$ and the variations are taken with respect to the 3 fields that replace the metric in the Carrollian setting. From now on, we call (2.16) the *Carrollian momenta*. Before continuing, notice that these quantities transform under Carrollian diffeomorphisms as

$$\mathcal{O}' = J\mathcal{O} - \mathcal{B}^i j_i, \quad \mathcal{B}^{i\prime} = J^i_j \mathcal{B}^j, \quad \text{and} \quad \mathcal{A}^{ij\prime} = J^i_k J^j_l \mathcal{A}^{kl}. \tag{2.17}$$

The spatial vector $\mathcal{B}^i$ and matrix $\mathcal{A}^{ij}$ are indeed Carrollian tensors. However, $\mathcal{O}$ is not a scalar and, as we will see and use, it is wiser to introduce the scalar combination $\mathcal{E} = \Omega\mathcal{O} + b_i\mathcal{B}^i$.

Given a Carroll-covariant theory, the action is invariant under Carrollian diffeomorphisms, generated by the spacetime vector ξ

$$\delta_\xi S = 0, \quad \xi = \xi^t(t,\mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i. \tag{2.18}$$

Notice that ξ^i only depends on $\mathbf{x}$, this is the infinitesimal translation of (2.5). Under such an infinitesimal coordinate transformation we have

$$\delta_\xi S = \int \mathrm{d}^{d+1}x \left(\frac{\delta S}{\delta \Omega}\delta_\xi\Omega + \frac{\delta S}{\delta b_i}\delta_\xi b_i + \frac{\delta S}{\delta a_{ij}}\delta_\xi a_{ij} + \frac{\delta S}{\delta \phi}\delta_\xi \phi \right) + \text{b.t.}, \tag{2.19}$$

and the on-shell condition ensures $\frac{\delta S}{\delta \phi} = 0$. We need to compute $\delta_\xi\Omega$, $\delta_\xi b_i$ and $\delta_\xi a_{ij}$. In order to do so we compute the infinitesimal version of (2.7). If $x'^\mu = x^\mu - \xi^\mu$, then

$$\begin{aligned}
\delta_\xi\Omega &= \xi(\Omega) + \Omega\partial_t\xi^t, & (2.20)\\
\delta_\xi b_i &= \xi(b_i) - \Omega\partial_i\xi^t + b_j\partial_i\xi^j, & (2.21)\\
\delta_\xi a_{ij} &= \xi\left(a_{ij}\right) + \partial_i\xi^k a_{kj} + \partial_j\xi^k a_{ik}, & (2.22)
\end{aligned}$$

where $\xi(f) \equiv \xi^t\partial_t f + \xi^i\partial_i f$. We would like to write these transformations in terms of manifestly Carroll-covariant objects, so we define $X = \Omega\xi^t - b_i\xi^i$. By noticing that the components of a spacetime vector transform as

$$\xi^{t\prime} = J\xi^t + j_i\xi^i, \quad \xi^{i\prime} = J^i_k\xi^k, \tag{2.23}$$

it is straightforward to show that X is the right combination for obtaining a scalar. We thus rewrite (2.20), (2.21) and (2.22) in terms of X, ξ^i and the Carrollian geometrical tensors introduced above

$$\begin{aligned}
\delta_\xi \Omega &= \partial_t X + \Omega \varphi_j \xi^j, &(2.24)\\
\delta_\xi b_i &= -\hat{\partial}_i X + \varphi_i X - 2\varpi_{ij}\xi^j + \frac{b_i}{\Omega}\left(\partial_t X + \Omega\varphi_j \xi^j\right), &(2.25)\\
\delta_\xi a_{ij} &= \hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i + \frac{X}{\Omega}\partial_t a_{ij}. &(2.26)
\end{aligned}$$

This rewriting hints toward Carrollian covariance, as it replaces ξ^t with X. Therefore, we obtain $\delta_\xi S = \delta_X S + \delta_{\xi^i} S$ with

$$\begin{aligned}
\delta_X S &= \int \mathrm{d}^{d+1}x \Omega \sqrt{a}\left(\mathcal{O}\partial_t X - \mathcal{B}^i \hat{\partial}_i X + \mathcal{B}^i \varphi_i X + \mathcal{B}^i \frac{b_i}{\Omega}\partial_t X + \mathcal{A}^{ij}\frac{X}{\Omega}\partial_t a_{ij}\right), &(2.27)\\
\delta_{\xi^i} S &= \int \mathrm{d}^{d+1}x \Omega \sqrt{a}\left(\mathcal{O}\Omega\varphi_j \xi^j - 2\mathcal{B}^i \varpi_{ij}\xi^j + \mathcal{B}^i b_i \varphi_j \xi^j + 2\mathcal{A}^{ij}\hat{\nabla}_i \xi_j\right). &(2.28)
\end{aligned}$$

Finally, demanding $\delta_X S$ and $\delta_{\xi^i} S$ be zero separately and manipulating them, we obtain two conservation equations which are manifestly Carroll-covariant:[4]

$$\begin{aligned}
\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{E} - \left(\hat{\nabla}_i + 2\varphi_i\right)\mathcal{B}^i - \mathcal{A}^{ij}\frac{1}{\Omega}\partial_t a_{ij} &= 0, &(2.29)\\
2\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{A}^i_j + 2\mathcal{B}^i \varpi_{ij} - \mathcal{E}\varphi_j &= 0, &(2.30)
\end{aligned}$$

where we used the already introduced scalar combination $\mathcal{E} = \Omega\mathcal{O} + b_i \mathcal{B}^i$.

Let us briefly summarize. By strict comparison with the relativistic situation, we have defined the momenta of our Carrollian theory to be the variation of the action under the geometrical set of data that characterizes the background. Exploiting the underlying Carrollian symmetry we reached a set of two equations which encode the conservation properties of the momenta. As expected, these equations are fully Carroll-covariant.

2.4 Weyl covariance

At the relativistic level, Weyl invariance merges when the theory is invariant under a rescaling $g_{\mu\nu} \to \frac{g_{\mu\nu}}{\mathcal{B}^2}$ for any $\mathcal{B}$ function of spacetime coordinates.[5] The transformations of Ω, b_i and a_{ij} under Weyl rescaling are deduced from the relativistic Randers–Papapetrou metric (2.6)

$$\Omega \to \frac{\Omega}{\mathcal{B}}, \quad b_i \to \frac{b_i}{\mathcal{B}} \quad \text{and} \quad a_{ij} \to \frac{a_{ij}}{\mathcal{B}^2}. \tag{2.31}$$

[4] A useful relation is $\mathcal{B}^i \hat{\partial}_i X = -X\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{B}^i$, valid up to total derivatives and for any scalar X and vector $\mathcal{B}^i$.

[5] This conformal symmetry has important consequences in hydrodynamical holographic theories, [17, 18].

If the action is invariant under such transformations,

$$\delta_\lambda S = \int \mathrm{d}^{d+1}x\Omega\sqrt{a}\left(\mathcal{O}\delta_\lambda\Omega + \mathcal{B}^i\delta_\lambda b_i + \mathcal{A}^{ij}\delta_\lambda a_{ij}\right) = \int \mathrm{d}^{d+1}x\Omega\sqrt{a}\lambda\left(\mathcal{O}\Omega + \mathcal{B}^i b_i + 2\mathcal{A}^{ij}a_{ij}\right) \tag{2.32}$$

has to vanish for every $\lambda(t,\mathbf{x})$. Therefore

$$\delta_\lambda S = 0 \quad \Rightarrow \quad \mathcal{E} = -2\mathcal{A}_i^i. \tag{2.33}$$

We will refer to this condition as the *conformal state equation*, it is the equivalent of the tracelessness of the energy–momentum tensor in the relativistic case. From (2.31) we deduce the following transformations of the Carrollian momenta

$$\mathcal{O} \to \mathcal{B}^{d+2}\mathcal{O}, \quad \mathcal{B}^i \to \mathcal{B}^{d+2}\mathcal{B}^i \quad \text{and} \quad \mathcal{A}^{ij} \to \mathcal{B}^{d+3}\mathcal{A}^{ij}. \tag{2.34}$$

This implies also $\mathcal{E} \to \mathcal{B}^{d+1}\mathcal{E}$.

We would like to write the conservation equations in a manifestly Weyl-covariant form. To do so, we decompose $\mathcal{A}^{ij} = -\frac{1}{2}\left(\mathcal{P}a^{ij} - \Xi^{ij}\right)$ with Ξ^{ij} traceless, such that the constraint (2.33) becomes $\mathcal{E} = d\mathcal{P}$. This enable us rewriting (2.29) and (2.30) as

$$\begin{aligned}
\left(\frac{1}{\Omega}\partial_t + \frac{d+1}{d}\theta\right)\mathcal{E} - \left(\hat{\nabla}_i + 2\varphi_i\right)\mathcal{B}^i - \Xi^{ij}\xi_{ij} &= 0, && (2.35)\\
\left(\hat{\nabla}_i + \varphi_i\right)\Xi^i_j - \frac{1}{d}\left(\hat{\partial}_j + (d+1)\varphi_j\right)\mathcal{E} + 2\mathcal{B}^i\varpi_{ij} &= 0. && (2.36)
\end{aligned}$$

The Carrollian derivatives are not covariant under Weyl rescaling, since the latter brings extra shift terms. In order to reach manifestly Weyl-Carroll-covariant equations, we can upgrade the Carroll derivatives to Weyl-Carroll ones. Among the Carrollian first derivative tensors introduced above, φ_i and θ are Weyl connections as

$$\varphi_i \to \varphi_i - \hat{\partial}_i \ln\mathcal{B}, \quad \theta \to \mathcal{B}\theta - \frac{d}{\Omega}\partial_t\mathcal{B}. \tag{2.37}$$

Therefore, they can be used for defining the Weyl-Carroll derivative. For a weight-w scalar function Φ, *i.e.* a function scaling with $\mathcal{B}^w$ under Weyl, and a weight-w vector, the Weyl-Carroll spatial and temporal derivatives are defined as

$$\begin{aligned}
\hat{\mathscr{D}}_j\Phi &= \hat{\partial}_j\Phi + w\varphi_j\Phi, && (2.38)\\
\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi &= \frac{1}{\Omega}\partial_t\Phi + \frac{w}{d}\theta\Phi, && (2.39)\\
\hat{\mathscr{D}}_j V^l &= \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta^l_j V^i\varphi_i, && (2.40)\\
\frac{1}{\Omega}\hat{\mathscr{D}}_t V^l &= \frac{1}{\Omega}\partial_t V^l + \frac{w}{d}\theta V^l + \xi^l{}_i V^i, && (2.41)
\end{aligned}$$

such that under a Weyl transformation

$$
\begin{aligned}
\hat{\mathscr{D}}_j\Phi &\rightarrow \mathcal{B}^w\hat{\mathscr{D}}_j\Phi, && (2.42)\\
\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi &\rightarrow \mathcal{B}^{w+1}\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi, && (2.43)\\
\hat{\mathscr{D}}_jV^l &\rightarrow \mathcal{B}^w\hat{\mathscr{D}}_jV^l, && (2.44)\\
\frac{1}{\Omega}\hat{\mathscr{D}}_tV^l &\rightarrow \mathcal{B}^{w+1}\frac{1}{\Omega}\hat{\mathscr{D}}_tV^l. && (2.45)
\end{aligned}
$$

The action on any other tensor is obtained using the Leibniz rule.

Eventually, we can write (2.35) and (2.36) using these derivatives as

$$
\begin{aligned}
\frac{1}{\Omega}\hat{\mathscr{D}}_t\mathcal{E} - \hat{\mathscr{D}}_i\mathcal{B}^i - \Xi^{ij}\xi_{ij} &= 0, && (2.46)\\
-\frac{1}{d}\hat{\mathscr{D}}_j\mathcal{E} + 2\mathcal{B}^i\varpi_{ij} + \hat{\mathscr{D}}_i\Xi^i_j &= 0. && (2.47)
\end{aligned}
$$

Not only these equations are now very compact, they are also manifestly Weyl-Carroll-covariant.

2.5 The flat case

So far we have worked on general Carrollian geometry, *i.e.* we did not impose any particular value of Ω, b_i and a_{ij}. We now restrict our attention to the flat Carrollian background.[6]

At the relativistic level, the Poincaré group is defined as the set of coordinate transformations that leave the Minkowski metric invariant. By strict analogy, the Carroll group is defined as the set of transformations that preserve the Carrollian flatness, [16]. Therefore, the Carroll group corresponds to the transformations satisfying

$$
\partial_t \rightarrow \partial_t, \quad \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \rightarrow \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad b_{0i} \rightarrow R^j_i\left(b_{0j} + \beta_j\right), \tag{2.48}
$$

with b_{0i} constant. The resulting change of coordinates is

$$
t' = t + \beta_i x^i + t_0, \quad x'^i = R^i_j x^j + x^i_0, \tag{2.49}
$$

where $t_0 \in \mathbb{R}$, $\{x^i_0, \beta_i\} \in \mathbb{R}^d$ and $R^i_j \in \mathrm{O}(d)$. This group is known in the literature as the Carroll group. [7]

[6]We refer here to flat Carrollian geometry as the geometry for which the Carroll group is an isometry, see *e.g.* [16].

[7]The Carroll group was already shown to be the symmetry group of flat zero signature geometries in the precursory work [19].

Recasting (2.29) and (2.30) for $a_{ij}(t,\mathbf{x}) = \delta_{ij}$, $\Omega(t,\mathbf{x}) = 1$ and $b_i(t,\mathbf{x}) = b_{0i}$, we obtain

$$\partial_t \mathcal{O} - \partial_i \mathcal{B}^i = 0, \tag{2.50}$$

$$2\partial_i \mathcal{A}^i_j + 2b_{0i}\partial_t \mathcal{A}^i_j = 0. \tag{2.51}$$

The momenta appearing in these two equations can be packaged in a spacetime energy–momentum tensor (where spacetime does not mean relativistic)

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{O} & -2b_{0k}\mathcal{A}^{ki} \\ -\mathcal{B}^j & -2\mathcal{A}^{ij} \end{pmatrix}. \tag{2.52}$$

The usual conservation of this tensor $\partial_\mu T^{\mu\nu} = 0$ is ensured by the conservation equations of the momenta, namely (2.50) and (2.51). This tensor is not symmetric, but this should not come as a surprise: it is not defined throughout the variation of the action with respect to the spacetime metric (symmetric by construction), instead it is defined using the Carrollian metric fields.[8] Finally notice that this spacetime lifting procedure was possible here due to the flatness of the Carrollian geometry. In general backgrounds, this is not possible, and the very concept of spacetime energy–momentum tensor is ambiguous–whereas the Carrollian momenta are by construction well suited.

As a conclusive remark notice that the Carroll group contains spacetime translations, so if a theory is invariant under this group, there will be a set of $d+1$ Nœther currents associated with spacetime translations. Packaging them in a $d+1$-dimensional kind of Nœther energy–momentum tensor, enables us comparing it with (2.52), as we do in the next section.

3 A Carrollian scalar-field action

In order to probe our results, we start with the example of a single scalar field $\phi(t,\mathbf{x})$. We begin the study on a general Carrollian background and show that the momenta are conserved. Then, we restrict the geometry to the flat case, where spacetime translational invariance of the theory allows us to compare our energy–momentum tensor (defined only in the flat case, as in Sec. 2.5) to the conserved current computed using Nœther procedure. The two energy–momentum tensors will turn out to be equivalent up to divergence-free terms.

In order to ensure Carroll invariance of the scalar-field action, we need to trade the usual derivatives for the Carrollian ones. So we consider the action

$$S[\phi] = \frac{1}{2}\int \mathrm{d}^{d+1}x\Omega\sqrt{a}\,a^{ij}\hat{\partial}_i\phi\hat{\partial}_j\phi = \int \mathrm{d}^{d+1}x\mathcal{L}, \tag{3.1}$$

[8] Although the construction is different, another example of non-symmetric Carrollian energy–momentum tensor can be found in [20].

which is manifestly covariant. The equations of motion are readily determined

$$\left(\hat{\nabla}_i + \varphi_i\right)\hat{\partial}^i\phi = 0. \tag{3.2}$$

The Carrollian momenta are

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}\hat{\partial}_i\phi\hat{\partial}^i\phi, &(3.3)\\
\mathcal{B}^i &= \frac{1}{\Omega}\partial_t\phi\hat{\partial}^i\phi, &(3.4)\\
\mathcal{A}^{ij} &= \frac{1}{2}\left(\frac{1}{2}a^{ij}\hat{\partial}^k\phi\hat{\partial}_k\phi - \hat{\partial}^i\phi\hat{\partial}^j\phi\right). &(3.5)
\end{aligned}$$

These momenta are conserved on shell since the conservation equations (2.29) and (2.30) are automatically satisfied given the equations of motion (3.2). This last result shows unambiguously the relevance of these objects. Notice moreover that these momenta satisfy the conformal state equation (2.33) only for $d = 1$. In fact this action can be recovered from an ultra-relativistic limit of the free relativistic scalar theory, which is known to be conformal only in 2 spacetime dimensions.

We now impose the Carrollian background to be flat. In this case, the action (3.1) becomes

$$S[\phi] = \int \mathrm{d}^{d+1}x\mathcal{L} = \frac{1}{2}\int \mathrm{d}^{d+1}x\delta^{ij}\left(\partial_i + b_{0i}\partial_t\right)\phi\left(\partial_j + b_{0j}\partial_t\right)\phi, \tag{3.6}$$

which is invariant under spacetime translations. In the flat case, we can lift the Carrollian momenta into a spacetime energy–momentum tensor (2.52), which here takes the form

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{2}\hat{\partial}_i\phi\hat{\partial}^i\phi - b_{0i}\partial_t\phi\hat{\partial}^i\phi & -\frac{b_0^i}{2}\hat{\partial}^k\phi\hat{\partial}_k\phi + b_{0k}\hat{\partial}^k\phi\hat{\partial}^i\phi \\ -\partial_t\phi\hat{\partial}^i\phi & -\frac{1}{2}a^{ij}\hat{\partial}^k\phi\hat{\partial}_k\phi + \hat{\partial}^i\phi\hat{\partial}^j\phi \end{pmatrix}, \tag{3.7}$$

and it is conserved.

The action (3.6) is invariant under spacetime translations. As stated in the previous section, we therefore have $d+1$ associated Nœther currents

$$\hat{T}^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}, \tag{3.8}$$

which explicitly read:

$$\begin{aligned}
\hat{T}^{tt} &= \frac{1}{2}\hat{\partial}_i\phi\hat{\partial}^i\phi - b_{0i}\hat{\partial}^i\phi\partial_t\phi, &(3.9)\\
\hat{T}^{it} &= -\hat{\partial}^i\phi\partial_t\phi, &(3.10)\\
\hat{T}^{ti} &= b_{0j}\hat{\partial}^j\phi\partial^i\phi, &(3.11)\\
\hat{T}^{ij} &= \hat{\partial}^i\phi\partial^j\phi - \frac{1}{2}\delta^{ij}\hat{\partial}^k\phi\hat{\partial}_k\phi. &(3.12)
\end{aligned}$$

The conservation $\partial_\mu \hat{T}^{\mu\nu} = 0$, is achieved thanks to the equations of motion (3.2) for flat geometry $\hat{\partial}^i \hat{\partial}_i \phi = 0$.

We can now compare the energy–momentum tensor (3.7) with (3.9), (3.10), (3.11) and (3.12). We obtain

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + B^{\mu\nu}, \tag{3.13}$$

with

$$\begin{aligned} B^{tt} &= 0, & (3.14)\\ B^{it} &= 0, & (3.15)\\ B^{ti} &= -b_0^i b_0^j \hat{\partial}_j \phi \partial_t \phi + \frac{1}{2} b_0^i \hat{\partial}_k \phi \hat{\partial}^k \phi, & (3.16)\\ B^{ij} &= -b_0^j \partial_t \phi \hat{\partial}^i \phi. & (3.17) \end{aligned}$$

As anticipated, the tensor $B^{\mu\nu}$ is divergenceless on-shell $\partial_\mu B^{\mu\nu} = 0$, which implies that the two energy–momentum tensors carry the same physical information on the theory.

4 Ultra-relativistic limit: the emergence of Carrollian physics

In the previous sections, we have intrinsically defined the Carrollian momenta starting from the metric fields of a Carrollian geometry. The Carrollian geometry was inspired by an ultra-relativistic contraction of the relativistic metric. We will see now how the ultra-relativistic limit can be directly taken at the level of the conservation equation of the relativistic energy–momentum tensor. This limit provides a richer structure, with more equations and fields. This is neither surprising nor contradictory. It is suggested by the dual Galilean limit, [12]. Indeed, in the non-relativistic case, on top of the momentum and energy conservation, an extra equation arises, which is ultimately identified with the continuity equation. A similar phenomenon occurs in the Carrollian case: additional fields and equations survive in the limit, and this is controlled by our choice of c-dependence of the fields.

Given a vector field u^μ, normalized as $u^2 = -c^2$ with respect to the relativistic metric (2.6), the energy–momentum tensor can always be decomposed as[9]

$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P}) \frac{u^\mu u^\nu}{c^2} + \mathcal{P} g^{\mu\nu} + \tau^{\mu\nu} + \frac{q^\mu u^\nu}{c^2} + \frac{q^\nu u^\mu}{c^2}. \tag{4.1}$$

In the hydrodynamic interpretation, $\mathcal{E}$ and $\mathcal{P}$ are the energy density and pressure of the fluid, $g^{\mu\nu}$ is the spacetime metric and $\tau^{\mu\nu}$ and q^μ are the transverse dissipative tensors, named viscous stress tensor and heat current. We choose to adapt the velocity to the geometry $u^\mu = \left(\frac{c}{\Omega}, 0\right)$: the fluid is at rest. The advantage of this choice is that the dissipative tensors,

[9] Reminder of the conventions: $x^\mu = (x^0, x^i) = (ct, x^i)$.

since transverse, have only spatial independent components. Inspired by flat holography [12], we choose a particular scaling of these tensors in c, namely

$$\tau^{ij} = -\frac{\Sigma^{ij}}{c^2} - \Xi^{ij} \quad \text{and} \quad q^i = -\mathcal{B}^i + c^2\pi^i. \tag{4.2}$$

A more general dependence could have been considered. This would add new fields and new equations to the resulting Carrollian theory, whereas the present choice will be sufficient for the examples we want to analyze. Notice that the c-independent situation is recovered for $\Sigma^{ij} = 0 = \pi^i$. We now perform the zero-c limit of $\nabla_\mu T^{\mu\nu} = 0$. Defining again $\mathcal{A}^{ij} = -\frac{1}{2}\left(\mathcal{P}a^{ij} - \Xi^{ij}\right)$, we obtain the following set of equations[10]

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{E} - \left(\hat{\nabla}_i + 2\varphi_i\right)\mathcal{B}^i - \mathcal{A}^{ij}\frac{1}{\Omega}\partial_t a_{ij} = 0, \tag{4.3}$$

$$2\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{A}^i_j + 2\mathcal{B}^i\varpi_{ij} - \mathcal{E}\varphi_j - \left(\frac{1}{\Omega}\partial_t + \theta\right)\pi_j = 0, \tag{4.4}$$

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{B}_j + \left(\hat{\nabla}_i + \varphi_i\right)\Sigma^i_j = 0, \tag{4.5}$$

$$\Sigma^{ij}\xi_{ij} + \frac{\theta}{d}\Sigma^i_i = 0. \tag{4.6}$$

As advertised, we immediately recognize (4.5) and (4.6) as the Carrollian counterpart of the continuity equation: these are two consistency equations of the limit. Notice moreover how these equations reduce to the Carrollian equations (2.29) and (2.30) when the dissipative terms have no c-dependence, $\Sigma^{ij} = 0 = \pi^i$, together with the additional constraint $\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{B}_j = 0$. This result undoubtedly shows the nature of the ultra-relativistic limit: it is a Carrollian limit. Conversely, this analysis gives credit to our intrinsic Carrollian construction of the previous sections.

Summarizing, we have shown how the ultra-relativistic expansion gives rise to a leading Carrollian behavior. Furthermore, we have analyzed the extra inputs this limit brings and the associated conservation equations. It is remarkable how the Carrollian momenta intrinsically defined using Carrollian geometry match the ultra-relativistic limit.

We conclude with an aside important remark: we have taken the ultra-relativistic limit of the conservation equations because it would have been inconsistent to compute directly the limit of the energy–momentum tensor itself. Indeed we would have lost information on the fields which survive and the conservation equations they satisfy. This confirms that we have to give up the concept of spacetime energy–momentum tensor on general Carrollian backgrounds, as anticipated in [2] but sometimes disregarded in the current literature.

[10]This limit is performed using the decomposition (4.1) and the Randers–Papapetrou parametrization of the spacetime metric. For the detailed derivation of these equations, see [2].

5 Charges

This section is dedicated to the definition of charges in the Carrollian framework. Charges are conserved quantities associated with a symmetry of the theory. Relativistically, the latter can be generated by a Killing vector field. By projecting the energy–momentum tensor on the Killing vector, we obtain a conserved current. We will show here how to implement this procedure in the Carrollian case. In order to do so, we firstly derive charges starting from a conserved Carrollian current. Secondly, we define Carrollian Killing and conformal Killing vectors. Thirdly, we build conserved charges associated with conformal Killing vectors. This will be useful for the forthcoming examples involving asymptotically flat gravity. Finally, we give another example of Carrollian action and compute the charges to illustrate our results.

5.1 Conserved Carrollian current and associated charge

We show here a way to define a conserved charge starting from a conserved current. In this derivation we never impose the current to be associated with a Killing vector, therefore our construction is very general. Whenever we have a scalar $\mathcal{J}$ and a vector $\mathcal{J}^i$ satisfying

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{J} + \left(\hat{\nabla}_i + \varphi_i\right)\mathcal{J}^i = 0, \tag{5.1}$$

we can build the conserved charge

$$\mathcal{Q} = \int_{\Sigma_t} \mathrm{d}^d x\, \sqrt{a}\left(\mathcal{J} + b_i\mathcal{J}^i\right), \tag{5.2}$$

where Σ_t is a constant-time slice. A way to derive this formula is to start from the relativistic level: consider a conserved current J^μ, the charge is then

$$Q = \int_{\Sigma_t} \mathrm{d}^d x\, \sqrt{\sigma} n_\mu J^\mu. \tag{5.3}$$

Here n_μ is the unit vector normal to Σ_t and $\sigma_{\mu\nu}$ is the induced metric on Σ_t. In order to perform the zero-c limit, we decompose J^μ in an already Carroll-covariant basis

$$J = \mathcal{J}\left(\frac{c}{\Omega}\partial_0\right) + \mathcal{J}^i\left(\partial_i + \frac{cb_i}{\Omega}\partial_0\right). \tag{5.4}$$

Then, using the Randers–Papapetrou parametrization for the relativistic spacetime metric $\mathrm{d}s^2 = -c^2(\Omega\mathrm{d}t - b_i\mathrm{d}x^i)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j$, we obtain

$$\sqrt{\sigma} = \sqrt{a} + \mathcal{O}\left(c^2\right), \quad n_0 = c\Omega + \mathcal{O}\left(c^3\right), \quad J^0 = \frac{c}{\Omega}\left(\mathcal{J} + b_i\mathcal{J}^i\right). \tag{5.5}$$

Therefore, we find $Q \underset{c\to 0}{\to} c^2\mathcal{Q}$, showing the relevance of the proposed Carrollian charge (5.2).

5.2 Carrollian Killing vectors and associated conserved currents

A Killing vector is usually defined as a vector field that preserves the metric. Analogously, we define the Carrollian Killing vector ξ to be the vector satisfying[11]

$$\delta_\xi \Omega = 0 = \delta_\xi a_{ij}, \tag{5.6}$$

where δ_ξ is the Lie derivative. This gives rise to two Killing equations on ξ, which are exactly (2.24) and (2.26),[12]

$$\begin{aligned} \partial_t X + \Omega \varphi_j \xi^j &= 0, && (5.7) \\ \hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i + \frac{X}{\Omega} \partial_t a_{ij} &= 0, && (5.8) \end{aligned}$$

where we recall $X = \Omega \xi^t - b_i \xi^i$. Notice that these equations do not actually depend on b_i.

The generalization to conformal Carrollian Killing vectors is straightforward. We call ξ a conformal Carrollian Killing vector if

$$\delta_\xi \Omega = \lambda \Omega \quad \text{and} \quad \delta_\xi a_{ij} = 2\lambda a_{ij}. \tag{5.9}$$

It obeys the following conformal Killing equations:

$$\begin{aligned} \partial_t X + \Omega \varphi_j \xi^j &= \lambda \Omega, && (5.10) \\ \hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i + \frac{X}{\Omega} \partial_t a_{ij} &= 2\lambda a_{ij}. && (5.11) \end{aligned}$$

In particular from the last equation we obtain $\lambda = \frac{1}{d}\left(\hat{\nabla}_i \xi^i + \frac{X}{\Omega}\partial_t \ln \sqrt{a}\right)$. This general construction is very useful, as we will shortly confirm.

We now build a conserved current by projecting the Carrollian momenta on a Carrollian Killing vector, exactly like in the relativistic case. Indeed consider the following Carrollian current:

$$\mathcal{J} = \xi_i \mathcal{B}^i, \quad \mathcal{J}^i = \xi_j \Sigma^{ij}. \tag{5.12}$$

It is conserved provided ξ satisfies (5.8), and the Carrollian conservation equations (4.5) and (4.6) are verified. According to Sec. 5.1, the corresponding conserved charge is

$$\mathcal{Q}_\xi = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\, \xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji}\right), \tag{5.13}$$

This charge is also conserved when ξ satisfies (5.11), if we further impose the condition $\Sigma^i_i = 0$.

[11] This is the translation in our language of $\mathcal{L}_X g = 0$ and $\mathcal{L}_X \xi = 0$ of (III.6) in [16].
[12] On top of these equations, a Carrollian Killing vector has a time independent spatial part, *i.e.* $\partial_t \xi^i = 0$.

5.3 Charges for $\mathcal{B}^i = 0$

We will show in Sec. 6 that the equations describing the dynamics of asymptotically flat spacetimes in 3 and 4 dimensions can be related to Carrollian conservation laws for $\mathcal{B}^i = 0$. For this reason we focus here on this particular case and build other conserved currents associated with conformal Killing vectors. In Sec. 6 we will observe that the corresponding charges match the surface charges obtained through covariant phase-space formalism.

The Carrollian conservation equations obtained from the ultra-relativistic limit (4.3) and (4.4), for $\mathcal{B}^i = 0$, become

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\mathcal{E} - \mathcal{A}^{ij}\frac{1}{\Omega}\partial_t a_{ij} = 0, \tag{5.14}$$

$$2\left(\hat{\nabla}_i + \varphi_i\right)\mathcal{A}^i_j - \mathcal{E}\varphi_j - \left(\frac{1}{\Omega}\partial_t + \theta\right)\pi_j = 0. \tag{5.15}$$

We could have also reported the two equations on Σ^{ij}, (4.5) and (4.6), but they are immaterial here. Consider a Killing vector ξ, the following charge, up to boundary terms, is conserved

$$\mathcal{C}_\xi = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\left(X\mathcal{E} - \xi^i\pi_i + 2b_i\xi^j\mathcal{A}^i_j\right), \tag{5.16}$$

assuming only (5.14) and (5.15). This charge is also conserved when ξ is a conformal Killing vector, if we further impose the conformal state equation $\mathcal{E} = -2\mathcal{A}^i_i$. According to Sec. 5.1, the corresponding conserved current reads[13]

$$\mathcal{J} = X\mathcal{E} - \xi^i\pi_i, \quad \mathcal{J}^i = 2\xi^j\mathcal{A}^i_j. \tag{5.17}$$

It is interesting to investigate the flat case $a_{ij}(t,\mathbf{x}) = \delta_{ij}$, $\Omega(t,\mathbf{x}) = 1$ and $b_i(t,\mathbf{x}) = b_{0i}$. Here, (5.14) and (5.15) can be written as $\partial_\mu T^{\mu\nu} = 0$ with[14]

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{O} & -2b_{0k}\mathcal{A}^{ki} + \pi^i \\ 0 & -2\mathcal{A}^{ij} \end{pmatrix}, \tag{5.18}$$

and we notice that the charge, up to a divergenceless term, takes the usual form

$$\mathcal{C}^{\text{Flat}}_\xi = \int_{\Sigma_t} \mathrm{d}^d x \left(\xi^t\mathcal{O} - \xi^i b_{0i}\mathcal{O} - \xi^i\pi_i + 2b_{0i}\xi^j\mathcal{A}^i_j\right) = -\int_{\Sigma_t}\mathrm{d}^d x T^{0\mu}\xi_\mu + \tilde{\mathcal{C}}_{\xi^i}, \tag{5.19}$$

with $\tilde{\mathcal{C}}_{\xi^i} = -\int_{\Sigma_t}\mathrm{d}^d x\xi^i b_{0i}\mathcal{O}$, which is separately conserved.

[13] Its conservation (5.1) is ensured thanks to the Killing equations together with (5.14) and (5.15).
[14] We recall that for $\mathcal{B}^i = 0$, $\mathcal{E} = \Omega\mathcal{O}$. Thus in the flat case $\mathcal{E} = \mathcal{O}$.

For ξ and η Killing vectors, we define the brackets

$$\begin{aligned} \{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} &\equiv \int_{\Sigma_t} \mathrm{d}^d x \delta_\eta \left[\sqrt{a}\xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji} \right) \right], \\ \{\mathcal{C}_\xi, \mathcal{C}_\eta\} &\equiv \int_{\Sigma_t} \mathrm{d}^d x \delta_\eta \left[\sqrt{a} \left(X\mathcal{E} - \xi^i \pi_i + 2 b_i \xi^j \mathcal{A}^i_j \right) \right]. \end{aligned} \tag{5.20}$$

Here δ_η is the Lie derivative acting on the metric fields and the momenta, but not on ξ^t and ξ^i. A lengthly computation (see appendix A) shows that the charges $\mathcal{Q}_\xi$ and $\mathcal{C}_\xi$ equipped with these brackets form two representations of the Carrollian Killing algebra:

$$\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} = \mathcal{Q}_{[\xi,\eta]} \quad \text{and} \quad \{\mathcal{C}_\xi, \mathcal{C}_\eta\} = \mathcal{C}_{[\xi,\eta]}. \tag{5.21}$$

We can extend these results to the conformal Killing algebra when imposing the conformal state equation $\mathcal{E} = -2\mathcal{A}^i_i$ for the charge $\mathcal{C}_\xi$ and the condition $\Sigma^i_i = 0$ for the charge $\mathcal{Q}_\xi$.

5.4 Application to the scalar field

We close this section with an example of scalar-field action whose Carrollian momenta reproduce exactly the conservation equations described in Sec. 5.3. Consider a scalar field $\phi(t, \mathbf{x})$ and the following Carroll-covariant action:

$$S[\phi] = \frac{1}{2} \int \mathrm{d}^{d+1} x \sqrt{a} \frac{\dot{\phi}^2}{\Omega} = \int \mathrm{d}^{d+1} x \mathcal{L}, \tag{5.22}$$

where $\dot{\phi} = \partial_t \phi$. The equation of motion reads

$$\left(\frac{1}{\Omega} \partial_t + \theta \right) \left(\frac{\dot{\phi}}{\Omega} \right) = 0, \tag{5.23}$$

and we find the following Carrollian momenta through the variational definition (2.16)

$$\mathcal{E} = -\frac{1}{2\Omega^2} \dot{\phi}^2, \quad \mathcal{B}^i = 0 \quad \text{and} \quad \mathcal{A}^{ij} = \frac{1}{4\Omega^2} \dot{\phi}^2 a^{ij}. \tag{5.24}$$

Carrollian conservation equations of the type (5.14) and (5.15) are satisfied provided $\pi_i = \frac{1}{\Omega} \dot{\phi} \hat{\partial}_i \phi$. In the flat case the energy–momentum tensor (5.18) computed earlier becomes:

$$T^{\mu\nu} = \begin{pmatrix} -\frac{1}{2}\dot{\phi}^2 & \frac{1}{2} b^i_0 \dot{\phi}^2 + \dot{\phi} \partial^i \phi \\ 0 & -\frac{1}{2} \dot{\phi}^2 \delta^{ij} \end{pmatrix}. \tag{5.25}$$

As in the other example of scalar-field action (Sec. 3), this object coincides with the Nœther current associated with spacetime translations, up to a divergenceless term.

We can now focus on the charges in the Hamiltonian formalism. Defining the conjugate momentum $\psi = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dfrac{\sqrt{a}}{\Omega}\dot{\phi}$, and writing the Carrollian momenta in terms of ϕ and ψ, we obtain

$$\mathcal{E} = -\frac{1}{2}\left(\frac{\psi}{\sqrt{a}}\right)^2, \quad \pi_i = \frac{\psi}{\sqrt{a}}\left(\partial_i \phi + b_i \frac{\psi}{\sqrt{a}}\right) \quad \text{and} \quad \mathcal{A}^{ij} = \frac{1}{4}\left(\frac{\psi}{\sqrt{a}}\right)^2 a^{ij}. \tag{5.26}$$

Therefore, the charges (5.16) become

$$\mathcal{C}_\xi = -\int_{\Sigma_t} \mathrm{d}^d x \left(\frac{\xi^t}{2}\frac{\Omega}{\sqrt{a}}\psi^2 + \xi^i \psi \partial_i \phi\right). \tag{5.27}$$

These charges are expressed in Hamiltonian formalism. They are indeed conserved thanks to the equation of motion and together with the Poisson bracket they realize a representation of the Carrollian Killing algebra:

$$\{\mathcal{C}_\xi, \mathcal{C}_\eta\}_{\text{Poisson}} = \int_{\Sigma_t} \mathrm{d}^d x \left[\frac{\delta \mathcal{C}_\eta}{\delta \phi}\frac{\delta \mathcal{C}_\xi}{\delta \psi} - \frac{\delta \mathcal{C}_\xi}{\delta \phi}\frac{\delta \mathcal{C}_\eta}{\delta \psi}\right] = \mathcal{C}_{[\xi,\eta]}. \tag{5.28}$$

This result confirms that the charges (5.16) previously introduced are the correct ones. Finally, we notice that when $d = 1$ the conformal state equation (2.33) is satisfied and the representation can be extended to conformal Killing vectors.

6 Carrollian conservation laws in Ricci-flat gravity

We will now turn our attention to Ricci-flat gravity. When the bulk metric is expressed in an appropriate gauge, usually given by imposing the radial coordinate be null, Einstein equations can reduce in some instances to equations defined on null infinity $\mathscr{I}^+$.[15] Its null nature makes it a natural host for a Carrollian geometry and the gravitational dynamics will be shown to match with Carrollian conservation laws. This section can be considered as a precursor of a full asymptotically flat holographic scheme. Indeed, the putative dual boundary theory would be Carrollian and live on $\mathscr{I}^+$. This theory would be coupled to a Carrollian geometry and satisfy Carrollian conservation laws that we map here to the gravitational dynamics. In gravity, the covariant phase-space formalism allows to compute surface charges, those will be shown to be given exactly or partially by the conserved charges defined in Sec. 5.3, depending whether the gravitational solution has radiation or not. To compute the charges explicitly, we use the code [21].

[15] It will be the case for the three families of solutions we study in this section: the 3-dimensional asymptotically flat spacetimes, the weak field approximation of 4-dimensional asymptotically flat spacetimes in Bondi gauge and the Robinson Trautman solutions. The reduction of Einstein equations to equations on $\mathscr{I}^+$ would not be true, for example, for non-linearized 4-dimensional asymptotically flat gravity in Bondi gauge.

6.1 Asymptotically flat spacetimes in three dimensions

Three-dimensional asymptotically flat spacetimes are often studied in the Bondi gauge which, as we will shortly describe, imposes by definition the corresponding two-dimensional Carrollian manifold be flat. Here we want to show that we can source the geometric boundary fields, in order to create a general Carrollian structure [10].

Consider the following bulk metric

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -2\mathrm{u}\left(\mathrm{d}r + r\left(\varphi_x\mathrm{d}x + \theta\mathrm{u}\right)\right) + r^2 a_{xx}\mathrm{d}x^2 + 8\pi G\mathrm{u}\left(\mathcal{E}\mathrm{u} - \pi_x\mathrm{d}x\right). \tag{6.1}$$

The bulk coordinates are $\{u, r, x \in \mathbf{S}^1\}$, $\mathrm{u} = \Omega\mathrm{d}u - b_x\mathrm{d}x$, a_{xx} is the one-dimensional boundary spatial metric, $\mathcal{E}$ and π_x are the Carrollian momenta and θ and φ_x correspond to (2.13) and (2.12) defined earlier:

$$\theta = \frac{1}{\Omega}\partial_u \ln\sqrt{a_{xx}} \quad \text{and} \quad \varphi_x = \frac{1}{\Omega}\left(\partial_x\Omega + \partial_u b_x\right). \tag{6.2}$$

All the fields appearing in the bulk metric depend only on u and x. From this metric we can extract the corresponding Carrollian geometry on $\mathcal{I}^+ = \{r \to \infty\}$. The following procedure is general but we will use the specific case of three-dimensional asymptotically flat spacetimes as an illustration. Consider the conformal extension of (6.1)

$$\mathrm{d}\tilde{s}^2 = r^{-2}\mathrm{d}s^2, \tag{6.3}$$

the factor r^{-2} is present to regularize the metric on $\mathcal{I}^+$. We perform the change of variable $\omega = r^{-1}$ in the conformal metric, it becomes[16]

$$\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b = -2\mathrm{u}\left(-\mathrm{d}\omega + \omega\left(\varphi_x\mathrm{d}x + \theta\mathrm{u}\right)\right) + a_{xx}\mathrm{d}x^2 + 8\pi G\omega^2\mathrm{u}\left(\mathcal{E}\mathrm{u} - \pi_x\mathrm{d}x\right). \tag{6.4}$$

We can deduce the Carrollian geometry on $\mathcal{I}^+$

$$\tilde{g}^{-1}\left(.,\mathrm{d}\omega\right)_{|\mathcal{I}^+} = \frac{1}{\Omega}\partial_u, \quad \mathrm{d}\tilde{s}^2_{|\mathcal{I}^+} = a_{xx}\mathrm{d}x^2 \quad \text{and} \quad \tilde{g}\left(.,\partial_\omega\right)_{|\mathcal{I}^+} = \Omega\mathrm{d}u - b_i\mathrm{d}x^i. \tag{6.5}$$

We now move to the dynamics. In the following, we restrict our attention to the bulk line element (6.1) with the additional geometrical constraint

$$\hat{\mathscr{D}}_x s^x = \hat{\nabla}_x s^x + 2\varphi_x s^x = 0, \tag{6.6}$$

where $s_x = \frac{1}{\Omega}\partial_u\varphi_x - \theta\varphi_x - \hat{\partial}_x\theta$ is a Weyl-weight 1 two-derivative object. The Carrollian momenta do not appear in this equation, it is just a constraint on the boundary geometrical background as it involves only the Carrollian metric fields. Using this ansatz, Einstein

[16] The null asymptote is thus $\mathcal{I}^+ = \{\omega \to 0\}$.

equations reduce to

$$\left(\frac{1}{\Omega}\partial_u + 2\theta\right)\mathcal{E} = 0, \tag{6.7}$$

$$\left(\hat{\partial}_x + 2\varphi_x\right)\mathcal{E} + \left(\frac{1}{\Omega}\partial_u + \theta\right)\pi_x = 0. \tag{6.8}$$

We interpret them as the Carrollian conservation equations (4.3), (4.4), (4.5) and (4.6) for $\Sigma^{xx} = \mathcal{B}^x = 0$ and $\mathcal{E} = \mathcal{P}$ (conformal case). Furthermore Ξ^{xx} is automatically zero due to its tracelessness. Therefore, the gravitational dynamics of this metric ansatz coincides with the Carrollian conservation equations that fall into the case described in Sec. 5.3.[17]

We would like at this point to obtain the surface charges. We thus compute the asymptotic Killing vectors of $\mathrm{d}s^2$ whose leading orders in r^{-1} are

$$\hat{\xi}^r = -r\lambda(u,x) + \mathcal{O}(1), \quad \hat{\xi}^u = \xi^u(u,x) + \mathcal{O}(r^{-1}) \quad \text{and} \quad \hat{\xi}^x = \xi^x(x) + \mathcal{O}(r^{-1}). \tag{6.9}$$

Here $\lambda = \hat{\nabla}_x\xi^x + \frac{X}{\Omega}\partial_u \ln\sqrt{a_{xx}}$ and $\xi = \xi^u\partial_u + \xi^x\partial_x$ is a conformal Killing vector (*i.e.* satisfying (5.10) and (5.11)) of the corresponding Carrollian geometry $\{\Omega, a_{xx}, b_x\}$. We calculate the associated surface charge through covariant phase-space formalism and obtain that they are integrable and have exactly the same expression as the conserved charges defined in Sec. 5.3 out of purely Carrollian considerations

$$Q_{\hat{\xi}}[\mathrm{d}s^2] = \int_{\mathbf{S}^1} \mathrm{d}x\,\sqrt{a_{xx}}\left(\left(\Omega\xi^u - 2b_x\xi^x\right)\mathcal{E} - \xi^x\pi_x\right) = \mathcal{C}_\xi. \tag{6.10}$$

There is no gravitational radiation in three dimensions, the charges are thus conserved. We will see that things are slightly different in four dimensions, where we have to consider the radiation at null infinity.

If we restrict our attention to the case $\Omega = 1$, $a_{xx} = 1$ and $b_x = 0$, we recover the usual Bondi gauge for asymptotically flat spacetimes and Carrollian conservation becomes

$$\partial_u\mathcal{E} = 0, \tag{6.11}$$

$$\partial_x\mathcal{E} = -\partial_u\pi_x. \tag{6.12}$$

This set-up was extensively studied for instance in [22]. Here, the solutions to the Carrollian Killing equations are exactly the $\mathfrak{bms}_3$ algebra vectors $\xi = \xi^u\partial_u + \xi^x\partial_x$ with $\xi^u = \partial_x\xi^x u + \alpha$, for any smooth functions $\xi^x(x)$ and $\alpha(x)$ on $\mathbf{S}^1$. Moreover the solutions to (6.11) and (6.12) are

$$\mathcal{E}(u,x) = \mathcal{E}_0(x) \quad \text{and} \quad \pi_x(u,x) = -\partial_x\mathcal{E}_0 u + \pi_0(x). \tag{6.13}$$

[17] With respect to Sec. 5.3, we trade here t with u, to empathize that it is a retarded time.

Hence, the charges become the usual ones

$$\mathcal{C}_{\xi}^{\text{Bondi}} = \int_{S^1} \mathrm{d}x \left(\alpha \mathcal{E}_0 - \xi^x \pi_0\right), \tag{6.14}$$

which are manifestly conserved. These were obtained in [6, 22]. [18]

6.2 Linearized gravity in four dimensions

We can perform the same kind of analysis in the case of asymptotically flat spacetimes in four dimensions, where asymptotic charges have been computed. We show that the boundary equations of motion, which are the linearized Einstein equations after gauge fixing, can be interpreted as a Carrollian conservation, and that the asymptotic charges are also charges associated with conformal Carrollian Killing vectors.

The bulk metric is $g_{ab} = \eta_{ab} + h_{ab}$ with

$$\begin{aligned}
&\eta = -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j, \\
&h_{uu} = \frac{2}{r}m_B + \mathcal{O}\left(r^{-2}\right), \\
&h_{uj} = \frac{1}{2}\nabla^i C_{ij} + \frac{1}{r}N_j + \mathcal{O}\left(r^{-2}\right), \\
&h_{ij} = rC_{ij} + \mathcal{O}(1), \\
&h_{ra} = 0.
\end{aligned} \tag{6.15}$$

where $a = \{r, \mu\} = \{r, u, x^i\}$, $i = 1,2$. The perturbation h_{ab} is traceless, so $\gamma^{ij}C_{ij} = 0$, where γ^{ij} is the metric of the two-sphere and ∇_i the associated covariant derivative. We recognize the mass aspect m_B, the angular momentum aspect N_i and the gravitational wave aspect C_{ij}, all depending on u and x^i. In this gauge, the linearized Einstein equations become:[19]

$$\partial_u m_B = \frac{1}{4}\partial_u \nabla^i\nabla^j C_{ij}, \tag{6.16}$$

$$\partial_u N_i = \frac{2}{3}\partial_i m_B - \frac{1}{6}\left[(\Delta - 1)\nabla^j C_{ji} - \nabla_i \nabla^k \nabla^j C_{jk}\right]. \tag{6.17}$$

We first consider the case

$$\nabla^i\nabla^j C_{ij} = 0. \tag{6.18}$$

Then (6.16) and (6.17) admit a Carrollian interpretation and are recovered from (2.29) and

[18] To compare, we have to identify $\phi = x$, $\Xi(\phi) = -4\pi G \pi_0(x)$, $\Theta(\phi) = 8\pi G \mathcal{E}_0(x)$, $Y(\phi) = \xi^x(x)$ and $T(\phi) = \alpha(x)$.

[19] Solving empty linearized Einstein equations order by order in r^{-1} allows to express the various subleading coefficients in terms of m_B, C_{ij} and N_i. The only residual equations are then the ones that we present here.

(2.30) with the following metric data

$$\Omega = 1, \quad b_i = 0, \quad a_{ij} = \gamma_{ij}, \tag{6.19}$$

and Carrollian momenta

$$\Sigma^{ij} = \mathcal{B}^i = \Xi^i_i = 0, \tag{6.20}$$

$$\mathcal{E} = 4m_B, \quad \mathcal{A}^{ij} = -\frac{1}{2}\left(\frac{\mathcal{E}}{2}a^{ij} - \Xi^{ij}\right), \quad \pi^i = -3N^i, \quad \Xi^i_j = \frac{1}{2}(\Delta - 4)\,C^i_j, \tag{6.21}$$

where $\mathcal{E} = -2\mathcal{A}^i_i$ and $\Xi^i_i = 0$–we are in the conformal case. We obtain the following conservation equations:

$$\begin{aligned} \partial_u \mathcal{E} &= 0, & (6.22)\\ \partial_u \pi_i + \nabla_j\left(\frac{\mathcal{E}}{2}\gamma^j_i - \Xi^j_i\right) &= 0. & (6.23) \end{aligned}$$

This type of Carrollian conservation falls again into the class described in Sec. 5.3.

The asymptotic Killing vectors $\hat{\xi} = \hat{\xi}^r\partial_r + \hat{\xi}^u\partial_u + \hat{\xi}^i\partial_i$ associated with the gauge (6.15) have the following leading order in r^{-1}

$$\hat{\xi}^r = -\lambda(\mathbf{x})r + \mathcal{O}(1), \quad \hat{\xi}^u = \xi^u(t,\mathbf{x}) + \mathcal{O}(r^{-1}) \quad \text{and} \quad \hat{\xi}^i = \xi^i(\mathbf{x}) + \mathcal{O}(r^{-1}), \tag{6.24}$$

where $\xi = \xi^u\partial_u + \xi^i\partial_i$ is a conformal Killing vector (*i.e.* satisfying (5.10) and (5.11)) of the Carrollian geometry given by $\{\Omega = 1, a_{ij} = \gamma_{ij}, b_i = 0\}$ and λ is the conformal factor. The solutions to the corresponding conformal Killing equations reproduce exactly the $\mathfrak{bms}_4$ algebra: $\xi^u = \frac{u}{2}\nabla_i\xi^i + \alpha(\mathbf{x})$, α being any function on $\mathbf{S}^2$, ξ^i a conformal Killing of $\mathbf{S}^2$ and $\lambda = \frac{1}{2}\nabla_i\xi^i$. We compute the corresponding surface charges. When $\nabla^i\nabla^j C_{ij} = 0$ they take the form

$$Q_{\hat{\xi}}[g] = \int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\left(\xi^u\mathcal{E} - \xi^i\pi_i\right) = C_\xi, \tag{6.25}$$

with $\mathcal{E}$ and π_i given by (6.21). We recognize again the charges defined from purely Carrollian considerations in Sec. 5.3, associated with the data (6.19), (6.20) and (6.21). These charges are automatically conserved. Physically, this is due to the fact that part of the effect of gravitational radiation has suppressed by demanding $\nabla^i\nabla^j C_{ij} = 0$. We will find shortly that relaxing this condition has an effect on the charge conservation.

Integrating (6.22) and (6.23) we obtain

$$\mathcal{E} = \mathcal{E}_0(\mathbf{x}), \quad \pi_i = -\frac{1}{2}\partial_i\mathcal{E}_0 u + \int \mathrm{d}u'\nabla_j\Xi^j_i + \pi_{0i}(\mathbf{x}). \tag{6.26}$$

The charges become

$$
\begin{aligned}
\mathcal{C}_\xi &= \int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\left(\left(\frac{\nabla_i\xi^i}{2}u+\alpha\right)\mathcal{E}_0-\xi^i\left(-\frac{1}{2}\partial_i\mathcal{E}_0 u+\int \mathrm{d}u'\nabla_j\Xi_i^j+\pi_{0i}\right)\right)\\
&= u\int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\left(\frac{1}{2}\nabla_i(\xi^i\mathcal{E}_0)\right)+\int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\left(\alpha\mathcal{E}_0-\xi^i\left(\int \mathrm{d}u'\nabla_j\Xi_i^j+\pi_{0i}\right)\right)\\
&= \int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\left(\alpha\mathcal{E}_0-\xi^i\pi_{0i}\right)-\int \mathrm{d}u'\int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\xi^i\nabla_j\Xi_i^j+\text{b.t.}\\
&= \int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\left(\alpha\mathcal{E}_0-\xi^i\pi_{0i}\right)+\text{b.t.}.
\end{aligned}
\tag{6.27}
$$

The last step follows from the fact that ξ^i is a conformal Killing vector on $\mathbf{S}^2$ and Ξ_j^i is traceless. We observe that $\mathcal{C}_\xi$ is now manifestly conserved.

When $\nabla^i\nabla^j C_{ij}\neq 0$, on the gravity side the radiation affects the surface charges and spoils their conservation. Therefore, these charges do not match those we defined earlier. This situation can be further investigated and recast in Carrollian language. To this end, we define $\sigma=\nabla^i\nabla^j C_{ij}$ and rewrite (6.16) and (6.17)

$$
\partial_u\mathcal{E} = 0, \tag{6.28}
$$

$$
\partial_u\pi_i+\nabla_j\left(\mathcal{P}\gamma_i^j-\Xi_i^j\right) = 0. \tag{6.29}
$$

Here, the metric fields are

$$
\Omega=1,\quad b_i=0,\quad a_{ij}=\gamma_{ij}, \tag{6.30}
$$

together with the Carrollian momenta

$$
\Sigma^{ij}=\mathcal{B}^i=0, \tag{6.31}
$$

$$
\mathcal{E}=4m_B-\sigma,\quad \mathcal{P}=\frac{\mathcal{E}}{2}+\sigma,\quad \pi^i=-3N^i,\quad \Xi_j^i=\frac{1}{2}(\Delta-4)C_j^i. \tag{6.32}
$$

Hence turning on σ can be interpreted as spoiling the conformal state equation: $\mathcal{E}=-2\left(\mathcal{A}_i^i+\sigma\right)$. It appears as a sort of *conformal anomaly* in the boundary theory. The surface charges become

$$
Q_{\hat{\xi}}[g](u)=\int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\left(\xi^u(\mathcal{E}+\sigma)-\xi^i\pi_i\right), \tag{6.33}
$$

and, as already stated, they are no longer conserved

$$
\partial_u Q_{\hat{\xi}}[g]=\int_{\mathbf{S}^2} \mathrm{d}^2x\,\sqrt{\gamma}\,(\delta_\xi+\lambda)\,\sigma, \tag{6.34}
$$

where δ_ξ is the usual Lie derivative and $\lambda=\frac{1}{2}\nabla_i\xi^i$ the conformal factor. These charges were obtained in [23].[20] For non linear gravity see [24], where the charges are now non-integrable.

[20] See the $n=2$ case of Sec. 3. Their charges coincide with (6.33) with $\alpha=T$, $\xi^i=v^i$, $\mathcal{E}_0=4\mathcal{M}$ and $\pi_0^i=-3\mathcal{N}^i$.

6.3 Black hole solutions: Robinson–Trautman and Kerr–Taub–NUT

For asymptotically AdS solutions, Einstein equations lead to the conservation of an energy–momentum tensor on the timelike boundary with the cosmological constant playing the role of the velocity of light [12]. Taking the flat limit in the bulk therefore corresponds to an ultra-relativistic limit on the boundary, and this is how Carrollian dynamics emerges. We illustrate this for the specific examples of Robinson–Trautman and Kerr–Taub–NUT, and analyze their charges.

Robinson–Trautman

The Robinson–Trautman ansatz is

$$\mathrm{d}s^2 = \frac{2r^2}{P^2}\mathrm{d}z\mathrm{d}\bar{z} - 2\mathrm{d}u\mathrm{d}r - \left(\Delta \ln P - 2r\partial_u \ln P - \frac{2m}{r}\right)\mathrm{d}u^2, \tag{6.35}$$

where m and P depend on the boundary coordinates $\{u, z, \bar{z}\}$. This metric is Ricci-flat provided the Robinson–Trautman equations are satisfied:

$$\Delta\Delta \ln P + 12M\partial_u \ln P - 4\partial_u M = 0, \tag{6.36}$$

$$\partial_z M = 0, \tag{6.37}$$

$$\partial_{\bar{z}} M = 0, \tag{6.38}$$

where we have defined $\Delta = \nabla^i\nabla_i$, for $i = \{z, \bar{z}\}$, and ∇_i is the Levi Civita covariant derivative of the spatial metric $a = \frac{2}{P^2}\mathrm{d}z\mathrm{d}\bar{z}$. These equations can be interpreted as Carrollian conservation laws (4.3), (4.4), (4.5) and (4.6) with the metric data $\Omega = 1$, $b_i = 0$ and $a = \frac{2}{P(u,z,\bar{z})^2}\mathrm{d}z\mathrm{d}\bar{z}$ and the Carrollian momenta

$$\Xi^{ij} = \pi^i = \Sigma_i^i = 0, \tag{6.39}$$

$$\mathcal{E} = 4M, \quad \mathcal{B}^i = \nabla^i K, \quad \mathcal{A}^{ij} = -Ma^{ij}, \quad \Sigma^{ij} = \nabla^i\nabla^j\theta - \frac{1}{2}a^{ij}\nabla^k\nabla_k\theta. \tag{6.40}$$

Here we have introduced the Gaussian curvature $K = \Delta \ln P$. Weyl covariance is ensured by the conformal state equation $\mathcal{E} = -2\mathcal{A}_i^i$, together with $\Sigma_i^i = 0$. With this set of data, the conservation equations are

$$\left(\partial_u + \frac{3\theta}{2}\right)\mathcal{E} - \nabla_i\mathcal{B}^i = 0, \tag{6.41}$$

$$\partial_j\mathcal{E} = 0, \tag{6.42}$$

$$(\partial_u + \theta)\,\mathcal{B}_j + \nabla_i\Sigma_j^i = 0, \tag{6.43}$$

$$\Sigma^{ij}\xi_{ij} + \frac{\theta}{d}\Sigma_i^i = 0. \tag{6.44}$$

Equations (4.5) and (4.6) do not appear in the Robinson–Trautman equations because they are geometrical constraints on the spatial metric, which are automatically satisfied when imposing $a = \frac{2}{P^2} \mathrm{d}z\mathrm{d}\bar{z}$.

We want to interpret the charges we have introduced in Secs. 5.2 and 5.3 for the Robinson–Trautman spacetime. To this end, we introduce a conformal Carrollian Killing vector ξ, with (5.10) and (5.11) here given by

$$\begin{aligned} \partial_u \xi^u &= \lambda, & (6.45) \\ \nabla_i \xi_j + \nabla_j \xi_i + \xi^u \partial_u a_{ij} &= 2\lambda a_{ij}. & (6.46) \end{aligned}$$

The solution is the following vector[21]

$$\xi = \left(\sqrt{a}\right)^{\frac{1}{2}} \left(\alpha(\mathbf{x}) + \frac{1}{2} \int \mathrm{d}u \left(\sqrt{a}\right)^{-\frac{1}{2}} \nabla_i \xi^i\right) \partial_u + \xi^i(\mathbf{x}) \partial_i, \tag{6.47}$$

where ξ^i is a spatial conformal Killing vector, *i.e.* it satisfies

$$\nabla_i \xi_j + \nabla_j \xi_i = \nabla_k \xi^k a_{ij}. \tag{6.48}$$

The associated charges (5.13) become

$$\mathcal{Q}_\xi = \int_{S^2} \mathrm{d}^2 z \sqrt{a} \xi_j \mathcal{B}^j = \int_{S^2} \mathrm{d}^2 z P^{-2} \left(\xi^z \partial_z K + \xi^{\bar{z}} \partial_{\bar{z}} K\right). \tag{6.49}$$

They are conserved by construction.

Even though the second family of charges (5.16) were defined only for $\mathcal{B}^i = 0$, we can nevertheless study what their expression is for the solution at hand. We find

$$\mathcal{C}_\xi = \int_{S^2} \mathrm{d}^2 z \sqrt{a} \xi^u \mathcal{E} = \int_{S^2} \mathrm{d}^2 z P^{-3} \left(\alpha(z, \bar{z}) + \frac{1}{2} \int \mathrm{d}u P \nabla_i \xi^i\right) 4M. \tag{6.50}$$

As expected, they are not generically conserved, and using (6.41) we find

$$\partial_u \mathcal{C}_\xi = -\int_{S^2} \mathrm{d}^2 z \sqrt{a} \partial_i \xi^u \mathcal{B}^i. \tag{6.51}$$

Their conservation holds in two instances. The first, expected by construction, is when $\mathcal{B}_i = \partial_i K = 0$, and corresponds to a uniform curvature of the boundary sphere at all times. The second, which is a new condition, occurs when the conformal Killing vectors satisfy also $\partial_i \xi^u = 0$. This can be written as

$$\delta_\xi b_i = 0, \tag{6.52}$$

[21] The metric (6.35) is not in the Bondi gauge unless P is time independent. Therefore, the conformal Killing vector ξ does not satisfy the usual bms_4 algebra, but a generalized version of it.

when considering the Robinson–Trautman Carrollian geometry $\Omega = 1$, $b_i = 0$ and $a = \frac{2}{P^2}\mathrm{d}z\mathrm{d}\bar{z}$.[22]

Kerr–Taub–NUT family

The interesting feature of the Kerr–Taub–NUT family is that, although stationary, it has a non-trivial metric field b_i. Its line element, in $\{t, r, \theta, \phi\}$ coordinates, is given by

$$\mathrm{d}s^2 = -\frac{\Delta_r}{\rho^2}\left(\mathrm{d}t - \mathrm{b}\right)^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \rho^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) + \frac{\sin^2\theta}{\rho^2}\left(\alpha\mathrm{d}t - \left(r^2 + (n-\alpha)^2\right)\mathrm{d}\phi\right)^2, \tag{6.53}$$

where

$$\Delta_r = -2Mr + r^2 + \alpha^2 - n^2, \tag{6.54}$$

$$\rho^2 = r^2 + (n - \alpha\cos\theta)^2, \tag{6.55}$$

$$\mathrm{b} = \left(2n(\cos\theta - 1) + \alpha\sin^2\theta\right)\mathrm{d}\phi. \tag{6.56}$$

In this solution, M is interpreted as the black hole mass, α its angular parameter and n its NUT charge. The Carrollian geometrical data are $\Omega = 1$, b_i as in (6.56) and $a = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$. The bulk Einstein equations are satisfied for a constant mass. We can interpret this result as given by the following Carrollian data

$$\Xi^{ij} = \pi^i = \Sigma^{ij} = \mathcal{B}^i = 0 \quad \mathcal{E} = M \quad \mathcal{A}^{ij} = -\frac{M}{4}a^{ij}, \tag{6.57}$$

such that Carrollian conservation equations give straightforwardly M constant. From the hydrodynamical viewpoint, these data describe a perfect fluid.

The conformal Carrollian Killing equations can be solved with the result

$$\xi = \left(T(\mathbf{x}) + \frac{1}{2}t\nabla_i\xi^i\right)\partial_t + \xi^i(\mathbf{x})\partial_i. \tag{6.58}$$

where T is any smooth function on $\mathbf{S}^2$ and ξ^i a Killing vector of the sphere. This is precisely the bms_4 generator. The charges (5.13) are identically zero in this case. Conversely, the charges (5.16) are non-trivial

$$\mathcal{C}_\xi = M\int_{\mathbf{S}^2}\mathrm{d}\theta\mathrm{d}\phi\sin\theta\left(T - \frac{3}{2}\xi^i b_i\right). \tag{6.59}$$

They explicitly depend on the Kerr–Taub–NUT parameters thanks to the presence of the metric field b_i, and they are manifestly conserved.

[22] Actually, it is possible to show that, even when $\mathcal{B}^i \neq 0$, the charges (5.16) are generically conserved if the vectors ξ satisfy $\delta_\xi a_{ij} = 0$, $\delta_\xi\Omega = 0$ *and* $\delta_\xi b_i = 0$.

7 Conclusions

We are now ready to summarize our achievements.

In the framework of Carrollian dynamics we have defined Carrollian momenta as the variation of the action with respect to the Carrollian metric fields Ω, b_i, a_{ij}. These momenta obey conservation laws ensuing the invariance of the action under Carrollian diffeomorphisms. We have carefully stressed that this set of Carrollian momenta plays the role the energy–momentum tensor has in relativistic theories, since such an object cannot be defined in general Carrollian dynamics. In the very particular instance of flat Carrollian geometry, due to the existence of global symmetries, the on-shell Carrollian momenta are indistinguishable from the Nœther conserved currents. In this case they can be packaged in a non-symmetric spacetime energy–momentum tensor.

We have proven that the general conservation equations of the set of Carrollian momenta are recovered as the ultra-relativistic limit of the relativistic energy–momentum tensor conservation equations. This is expected and shows in passing that the Carrollian limit of the energy–momentum tensor outside its conservation equations is non sensible.

As usual in theories with local symmetries, volume conserved charges cannot be defined from plain conserved momenta. Killing fields are needed, in order to construct conserved currents and extract conserved charges, which encode the physical information stored in the fields at hand. We performed all these steps in a general Carrollian geometry, starting with the definition of the Killing vectors and proceeding with currents (projections of the Carrollian momenta) and charges.

All these concepts and techniques have been finally illustrated in concrete examples inspired from flat holography. Indeed, the null infinity of an asymptotically flat spacetime is a natural host for Carrollian geometry, and Carrollian conservation equations on $\mathcal{I}^+$ emerge as part of the bulk Einstein dynamics. More specifically, we have shown that in three bulk dimensions the Carrollian charges match the surface charges obtained from standard bulk methods. However, in four-dimensional linearized gravity, the presence of gravitational radiation spoils the conservation of surface charges. At the level of the Carrollian conservation equations, this is interpreted as a conformal anomaly, the radiation sourcing the anomalous factor. The subsequent analysis of the Robinson–Trautman and Kerr–Taub–NUT exact solutions nicely confirms these expectations and the interplay among the bulk and the boundary dynamics.

Our analysis triggers many questions. Among others, the two examples of exact Ricci-flat spacetimes treated here suggest to further investigate the Carrollian interpretation of four-dimensional gravity in full generality, *i.e.* without assuming linearity. More generally, this work may help in paving the road toward the Carrollian understanding of flat holography, already discussed in several instances in the literature.

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A Carrollian Charges algebra

We have defined two types of conserved charges in 5.2 and 5.3, $\mathcal{Q}_\xi$ and $\mathcal{C}_\xi$. The first one is conserved for any type of Carrollian conservation laws given by (4.3), (4.4), (4.5) and (4.6), while the second is conserved only when the Carrollian momenta $\mathcal{B}^i$ vanishes. We recall their expression:

$$\mathcal{Q}_\xi = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\, \xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji}\right) \quad \text{and} \quad \mathcal{C}_\xi = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a} \left(X\mathcal{E} - \xi^i \pi_i + 2b_i \xi^j \mathcal{A}^i_j\right). \tag{A.1}$$

In this appendix we show that both of them are also representations of the (conformal) Carrollian Killing algebra.

Consider two Carrollian Killing vectors ξ and η. It is possible to decompose them in a coordinate basis,

$$\xi = \xi^t(t,\mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i \quad \text{and} \quad \eta = \eta^t(t,\mathbf{x})\partial_t + \eta^i(\mathbf{x})\partial_i, \tag{A.2}$$

or in a Carroll-covariant one,

$$\xi = \frac{X}{\Omega}\partial_t + \xi^i \hat{\partial}_i \quad \text{and} \quad \eta = \frac{Y}{\Omega}\partial_t + \eta^i \hat{\partial}_i, \tag{A.3}$$

where $X = \Omega\xi^t - b_i\xi^i$, $Y = \Omega\eta^t - b_i\eta^i$ and $\hat{\partial}_i$ is the Carroll-covariant spatial derivative defined in 2.2. The commutator of ξ and η is given by

$$\lambda \equiv [\xi,\eta] = \left(\xi^t\partial_t\eta^t - \eta^t\partial_t\xi^t + \xi^k\partial_k\eta^t - \eta^k\partial_k\xi^t\right)\partial_t + \left(\xi^k\partial_k\eta^i - \eta^k\partial_k\xi^i\right)\partial_i = \frac{L}{\Omega}\partial_t + \lambda^i\hat{\partial}_i. \tag{A.4}$$

For ξ and η Carrollian Killing vectors, we define the two following quantities

$$
\begin{aligned}
\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} &\equiv \int_{\Sigma_t} \mathrm{d}^d x \delta_\eta \left[\sqrt{a}\xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji}\right)\right], \\
\{\mathcal{C}_\xi, \mathcal{C}_\eta\} &\equiv \int_{\Sigma_t} \mathrm{d}^d x \delta_\eta \left[\sqrt{a} \left(X\mathcal{E} - \xi^i \pi_i + 2 b_i \xi^j \mathcal{A}^i_j \right)\right],
\end{aligned}
\tag{A.5}
$$

where δ_η is the Lie derivative w.r.t. η acting on the metric fields and the momenta, but not on ξ^t and ξ^i. We want to show that, up to boundary terms,

$$
\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} = \mathcal{Q}_{[\xi,\eta]} \quad \text{and} \quad \{\mathcal{C}_\xi, \mathcal{C}_\eta\} = \mathcal{C}_{[\xi,\eta]}, \tag{A.6}
$$

the first result being true for any type of Carrollian conservation laws while the second one holds only when $\mathcal{B}^i = 0$.

We start with the first one, we have

$$
\begin{aligned}
\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} = \int_{\Sigma_t} \mathrm{d}^d x \Big[&\delta_\eta \sqrt{a}\xi_i \left(\mathcal{B}^i + b_j \Sigma^{ji}\right) + \sqrt{a} (\delta_\eta a_{ik}) \xi^k \left(\mathcal{B}^i + b_j \Sigma^{ji}\right) \\
&+ \sqrt{a}\xi_i \left(\delta_\eta \mathcal{B}^i + \delta_\eta b_j \Sigma^{ji} + b_j \delta_\eta \Sigma^{ji}\right)\Big].
\end{aligned}
\tag{A.7}
$$

We compute the infinitesimal variations of the geometric fields and the Carrollian momenta:

$$
\delta_\eta a_{ik} = \eta^t \partial_t a_{ik} + \eta^j \partial_j a_{ik} + \partial_i \eta^j a_{kj} + \partial_k \eta^j a_{ij} = 0, \tag{A.8}
$$

$$
\delta_\eta \sqrt{a} = \eta^i \partial_i \sqrt{a} + \eta^t \partial_t \sqrt{a} + \partial_i \eta^i \sqrt{a} = 0, \tag{A.9}
$$

$$
\delta_\eta b_i = \eta^t \partial_t b_i + \eta^j \partial_j b_i - \Omega \partial_i \eta^t + b_j \partial_i \eta^j, \tag{A.10}
$$

$$
\delta_\eta \mathcal{B}^i = \eta^t \partial_t \mathcal{B}^i + \eta^j \partial_j \mathcal{B}^i - \mathcal{B}^j \partial_j \eta^i, \tag{A.11}
$$

$$
\delta_\eta \Sigma^{ij} = \eta^t \partial_t \Sigma^{ij} + \eta^k \partial_k \Sigma^{ij} - \Sigma^{kj} \partial_k \eta^i - \Sigma^{ik} \partial_k \eta^j. \tag{A.12}
$$

The variation of a_{ik} and $\sqrt{a}$ vanish because η is a Carrollian Killing vector. Then we eliminate every temporal derivative of the Carrollian momenta using the conservation laws (4.5) and (4.6). Finally performing integration by parts and using properties of the Carrollian Killing vectors (5.7) and (5.8), we suppress every spatial derivative of the Carrollian momenta to obtain:

$$
\{\mathcal{Q}_\xi, \mathcal{Q}_\eta\} = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\lambda_i \left(\mathcal{B}^i + b_j \Sigma^{ji}\right) + \text{b.t.} = \mathcal{Q}_\lambda + \text{b.t.}. \tag{A.13}
$$

This proves that the charges $\mathcal{Q}_\xi$ form a representation of the Carrollian Killing algebra.

We now prove the second relation. We have

$$\{\mathcal{C}_\xi,\mathcal{C}_\eta\} = \int_{\Sigma_t} \mathrm{d}^d x \Big[\delta_\eta \sqrt{a}\left((\Omega\xi^t - b_i\xi^i)\mathcal{E} - \xi^i\pi_i + 2b_i\xi^j\mathcal{A}^i_j\right)$$
$$+ \sqrt{a}\left((\delta_\eta\Omega\xi^t - \delta_\eta b_i\xi^i)\mathcal{E} + (\Omega\xi^t - b_i\xi^i)\delta_\eta\mathcal{E} - \xi^i\delta_\eta\pi_i + 2\delta_\eta b_i\xi^j\mathcal{A}^i_j + 2b_i\xi^j\delta_\eta\mathcal{A}^i_j\right)\Big]. \tag{A.14}$$

We compute the infinitesimal variations of the geometric fields and the Carrollian momenta:

$$\begin{aligned}
\delta_\eta\Omega &= \eta^t\partial_t\Omega + \eta^i\partial_i\Omega + \Omega\partial_t\eta^t = 0, &\text{(A.15)}\\
\delta_\eta\sqrt{a} &= \eta^i\partial_i\sqrt{a} + \eta^t\partial_t\sqrt{a} + \partial_i\eta^i\sqrt{a} = 0, &\text{(A.16)}\\
\delta_\eta b_i &= \eta^t\partial_t b_i + \eta^j\partial_j b_i - \Omega\partial_i\eta^t + b_j\partial_i\eta^j, &\text{(A.17)}\\
\delta_\eta\mathcal{E} &= \eta^i\partial_i\mathcal{E} + \eta^t\partial_t\mathcal{E}, &\text{(A.18)}\\
\delta_\eta\pi_i &= \eta^t\partial_t\pi_i + \eta^j\partial_j\pi_i + \pi_j\partial_i\eta^j, &\text{(A.19)}\\
\delta_\eta\mathcal{A}^i_j &= \eta^t\partial_t\mathcal{A}^i_j + \eta^k\partial_k\mathcal{A}^i_j - \mathcal{A}^k_j\partial_k\eta^i + \mathcal{A}^i_k\partial_j\eta^k. &\text{(A.20)}
\end{aligned}$$

The variations of Ω and $\sqrt{a}$ are vanishing because η is a Carrollian Killing vector. Then we eliminate every temporal derivative of the Carrollian momenta using the conservation laws (5.14) and (5.15). Finally performing integration by parts and using properties of the Carrollian Killings, (5.7) and (5.8), we suppress every spatial derivative of the Carrollian momenta to obtain:

$$\{\mathcal{C}_\xi,\mathcal{C}_\eta\} = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\Big[\left(\Omega(\xi^t\partial_t\eta^t - \eta^t\partial_t\xi^t + \xi^k\partial_k\eta^t - \eta^k\partial_k\xi^t) - b_i(\xi^k\partial_k\eta^i - \eta^k\partial_k\xi^i)\right)\mathcal{E}$$
$$- (\xi^k\partial_k\eta^i - \eta^k\partial_k\xi^i)\pi_i + 2b_i(\xi^k\partial_k\eta^j - \eta^k\partial_k\xi^j)\mathcal{A}^i_j\Big] + \text{b.t.}, \tag{A.21}$$

which corresponds to

$$\{\mathcal{C}_\xi,\mathcal{C}_\eta\} = \int_{\Sigma_t} \mathrm{d}^d x \sqrt{a}\left(L\mathcal{E} - \xi^i\pi_i + 2b_i\lambda^j\mathcal{A}^i_j\right) + \text{b.t.} = \mathcal{C}_\lambda + \text{b.t.}. \tag{A.22}$$

Therefore, up to boundary terms, the charges $\mathcal{C}_\xi$ form a representation of the Carrollian Killing algebra.

We can extend the previous results to the conformal Carrollian Killing algebra when imposing $\Sigma^i_i = 0$ and the conformal state equation $\mathcal{E} = -2\mathcal{A}^i_i$.

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Two-dimensional fluids and their holographic duals

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Abstract

We describe the dynamics of two-dimensional relativistic and Carrollian fluids. These are mapped holographically to three-dimensional locally anti-de Sitter and locally Minkowski spacetimes, respectively. To this end, we use Eddington–Finkelstein coordinates, and grant general curved two-dimensional geometries as hosts for hydrodynamics. This requires to handle the conformal anomaly, and the expressions obtained for the reconstructed bulk metrics incorporate non-conformal-fluid data. We also analyze the freedom of choosing arbitrarily the hydrodynamic frame for the description of relativistic fluids, and propose an invariant entropy current compatible with classical and extended irreversible thermodynamics. This local freedom breaks down in the dual gravitational picture, and fluid/gravity correspondence turns out to be sensitive to dissipation processes: the fluid heat current is a necessary ingredient for reconstructing all Bañados asymptotically anti-de Sitter solutions. The same feature emerges for Carrollian fluids, which enjoy a residual frame invariance, and their Barnich–Troessaert locally Minkowski duals. These statements are proven by computing the algebra of surface conserved charges in the fluid-reconstructed bulk three-dimensional spacetimes.

Contents

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1 Introduction

Fluid/gravity correspondence is a macroscopic spin-off of holography, originally mapping relativistic fluid configurations onto Einstein spacetimes, *i.e.* spacetimes whose Ricci tensor is proportional to the metric. These are obtained in the form of a derivative expansion [1–4], inspired from the fluid homonymous expansion (see *e.g.* [5,6]). An alternative reconstruction of Einstein spacetimes from boundary data is based on the Fefferman–Graham theorem [7,8], which provides an expansion in powers of a radial space-like coordinate in the so-called Fefferman–Graham gauge.

Compared to the radial Fefferman–Graham expansion, the derivative expansion has several distinctive features listed hereafter.

- The boundary data in the Fefferman–Graham expansion are the first and second fundamental forms, interpreted as the boundary metric and the boundary fluid energy–momentum tensor. For the derivative expansion, the boundary data include also a vector congruence, whose derivatives set the order of the expansion. This congruence is interpreted as the boundary fluid velocity field.

- The derivative expansion is not built along a spatial but rather a null radial coordinate, whose differential form is the dual of the fluid velocity vector. It is implemented in Eddington–Finkelstein coordinates, and provides radial fall-offs which are slightly less restrictive than those of the Bondi gauge [9,10].

- The derivative expansion is well behaved in the Ricci-flat limit (vanishing bulk scalar curvature, *i.e.* cosmological constant).

The last property has recently allowed to set up a derivative expansion for asymptotically flat spacetimes, establishing thereby, at least macroscopically, a holographic correspondence among Ricci-flat bulk solutions and boundary Carrollian hydrodynamics [11], which is the ultra-relativistic (vanishing velocity of light) limit of fluid dynamics. The derivative expansion in Eddington–Finkelstein coordinates has been instrumental in reaching this result, because the Fefferman–Graham expansion is ill-defined in the limit of vanishing cosmological constant.

The first of the above three features raises another important question, regarding the role played by the boundary fluid congruence. In this respect, we remind that the velocity field of a relativistic fluid can be chosen freely, altering neither the energy–momentum tensor nor the entropy current, but only transforming the various pieces that enter the decomposition of these quantities with respect to its longitudinal and transverse directions [12]. This is usually referred to as the hydrodynamic-frame invariance.

The fluid congruence appears explicitly in the derivative expansion, as we will discuss in the following. Conforming to the above fluid-dynamics logic, one could consider another fluid frame. This would leave the boundary metric and energy–momentum tensor unchanged, and the corresponding reconstructed bulk metric would be amenable to its former expression by an appropriate bulk diffeomorphism. Still, this diffeomorphism might be large, in which case the two boundary hydrodynamic frames would lead to definitely distinct dual spacetimes with different global properties.

Analyzing the role of the velocity field in the fluid/gravity derivative expansion is not an easy task. Generically this derivative expansion is organized in the form of a series, whose order is set by the derivatives of the velocity field, and which is designed to comply with Weyl covariance. Furthermore, in the original works [1–4], this series was expressed using a specific hydrodynamic frame known as Landau–Lifshitz. In this context it is difficult to investigate the global behaviour under a congruence transformation, since typically only the first few orders in the expansion are available. In some more specific classes, it is possible to resum the derivative expansion (see [13–17]), which could help circumventing the latter difficulty. In order to resum the expansion, one needs to abandon the Landau–Lifshitz frame, and impose integrability conditions relating the heat current and stress tensor (*i.e.* the non-perfect components of the energy–momentum tensor) to the boundary geometry. The integrability conditions, however, are not covariant under changes of fluid congruence. Hence, the benefit of adopting resummed expressions is tempered when coming to the point of hydrodynamic-frame transformations.

Substantial simplifications occur in three bulk dimensions. On the one hand, all expansions, Fefferman–Graham or derivative, are naturally truncated to a finite number of terms.

On the other hand, asymptotically anti-de Sitter spacetimes are locally anti-de Sitter. As a consequence the distinction among Einstein solutions is exclusively encoded in their global properties, labeled unambiguously by their conserved surface charges, as *e.g.* in Bañados solutions [18]. Probing the fluid/gravity hydrodynamic-frame invariance amounts therefore to analyze the conserved charges and their algebra in different fluid frames. This is one of the aims of the present work, and we will show that contrary to the naive expectation,[1] changing fluid frame can alter the global properties of the reconstructed Einstein spacetime.

As already mentioned, the derivative expansion in Eddington–Finkelstein coordinates admits a well-defined limit of vanishing cosmological constant. This limit generalizes the customary fluid/gravity correspondence to a duality between Ricci-flat spacetimes and Carrollian hydrodynamics emerging at null infinity [19]. In some instances, Carrollian fluids possess a residual frame invariance involving a kinematical parameter reminiscent of the relativistic velocity field. The latter enters the flat derivative expansion, and it is legitimate to ask the same questions about the role of frame invariance as for anti-de Sitter spacetimes. Again, answering is possible in three dimensions, where the derivative expansion admits a finite number of terms, and all Ricci-flat spaces are locally Minkowskian. These are globally distinguishable by conserved surface charges, as *e.g.* for the family obtained in [20] with appropriate fall-off conditions that will be referred to as *Barnich–Troessaert* solutions.

In order to undertake the above analysis we will set up the fluid/gravity derivative expansions in three dimensions.[2] In other words, we will obtain expressions providing the bulk dual (Einstein or Ricci-flat) of an arbitrary fluid, hosted by any two-dimensional geometry. Such expressions were not available in full generality for the relativistic fluids, and were unknown for Carrollian (*i.e.* ultra-relativistic) fluids.

In the relativistic case, we exhibit a universal resummation formula, which turns out to be a BMS-like (Bondi–Metzner–Sachs, [9,10]) alternative to the existing Fefferman–Graham expression [20,21]. The prime virtue of our practice is to accommodate the conformal anomaly arising from the curvature of the boundary, which has been ignored in earlier fluid/gravity literature [2,3] and has a detectable counterpart in the Carrollian situation. For the latter, our fluid reconstruction of flat spacetimes resembles the general formulas given in BMS gauge in [20].

After having settled the derivative expansions, we express the asymptotic charges[3] of the reconstructed spacetimes in terms of the fluid data and we prove that the choice of frame may affect the global properties of the solutions. Indeed, we show that the holographic

[1] The question of global versus local properties of bulk solutions in relation with the dual boundary fluid was mentioned in the Appendix B of Ref. [3]. This discussion is not conclusive though, in particular because of the absence of any charge computation, which would have allowed to make concrete statements about the landscape of locally anti-de Sitter spacetimes and their dual fluids.

[2] *Expansion* is an abuse of terminology in three dimensions because there, it is naturally truncated. We will often make it, and use the word *resummation* for simple sums.

[3] Useful references for the analysis of asymptotic charges are *e.g.* [22,23]. Our surface-charge computations have been performed with the package [24], built using the conventions of the papers just quoted.

reconstruction of all *Bañados* and *Barnich–Troessaert* solutions requires the boundary fluid (relativistic or Carrollian) have a non-vanishing heat current. In this instance, the charge algebra is either Virasoro or BMS with the expected central charges. Setting the heat current to zero, the solutions carry surface charges obeying algebras of the same type, where the central charges can be trivially reabsorbed though.

In Sec. 2 we review two-dimensional relativistic conformal fluid dynamics, and expand its Carrollian limit, insisting on the hydrodynamic-frame invariance. Section 3 is devoted to the general method of holographic reconstruction of asymptotically AdS and flat spacetimes. This method is applied in Sec. 4 for flat two-dimensional boundary metrics, without loosing generality, and followed by the computation of charges, which enables us to reach a clear image of the solutions under investigation.

Before moving to the main part of the paper, we should add that Sec. 2.1 includes a part dedicated to the entropy current of relativistic two-dimensional conformal fluids. Contrary to the energy–momentum tensor the entropy current has no general microscopic definition for systems that are only at local thermodynamic equilibrium. It is usually constructed phenomenologically, in a given hydrodynamic frame, order by order in the velocity and temperature derivative expansion, and subject to several physical conditions. We propose here an entropy current, which fulfills all known criteria, has a closed form that can be expanded in a non-trivial infinite series, and is explicitly hydrodynamic-frame invariant. This last feature is the backbone of fluid frame invariance.

2 Two-dimensional fluids

2.1 Relativistic fluids

General properties

We consider a two-dimensional geometry $\mathscr{M}$ equipped with a metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. The dynamics of a relativistic fluid is captured by the energy–momentum tensor $\mathrm{T} = T_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, which is symmetric ($T_{\mu\nu} = T_{\nu\mu}$) and generally obeys:

$$\nabla^\mu T_{\mu\nu} = f_\nu, \tag{2.1}$$

where f_ν is an external force density. Together with the equation of state (local thermodynamic equilibrium is assumed), this set of equations provide the hydrodynamic equations of motion. Normalizing the velocity congruence u as $\|\mathrm{u}\|^2 = -k^2$ (k plays the role of velocity of light), we can in general decompose the energy–momentum tensor as

$$T_{\mu\nu} = (\varepsilon + p)\frac{u_\mu u_\nu}{k^2} + p g_{\mu\nu} + \tau_{\mu\nu} + \frac{u_\mu q_\nu}{k^2} + \frac{u_\nu q_\mu}{k^2} \tag{2.2}$$

with p the local pressure and ε the local energy density:

$$\varepsilon = \frac{1}{k^2} T_{\mu\nu} u^\mu u^\nu. \tag{2.3}$$

The symmetric viscous stress tensor $\tau_{\mu\nu}$ and the heat current q_μ are purely transverse:

$$u^\mu \tau_{\mu\nu} = 0, \quad u^\mu q_\mu = 0, \quad q_\nu = -\varepsilon u_\nu - u^\mu T_{\mu\nu}. \tag{2.4}$$

In two dimensions, the transverse direction with respect to u is entirely supported by the Hodge-dual $*$u:[4]

$$* u_\rho = u^\sigma \eta_{\sigma\rho}. \tag{2.5}$$

This dual congruence is space-like and normalized as $\| * \mathrm{u} \|^2 = k^2$. Therefore

$$\mathrm{q} = \chi * \mathrm{u} \quad \text{with} \quad \chi = -\frac{1}{k^2} * u^\mu T_{\mu\nu} u^\nu, \tag{2.6}$$

the local *heat density*, appearing here as the magnetic dual of the energy density. Similarly, the viscous stress tensor has a unique component encoded in the *viscous stress scalar* τ:[5]

$$\tau_{\mu\nu} = \tau h_{\mu\nu} \quad \text{with} \quad h_{\mu\nu} = \frac{1}{k^2} * u_\mu * u_\nu \tag{2.7}$$

the projector onto the space transverse to the velocity field. The trace reads: $T^\mu{}_\mu = p - \varepsilon + \tau$.

The pressure p and the viscous stress scalar τ appear in the fully transverse component of the energy–momentum tensor. Their sum is therefore the total stress. If the system is free and at *global* equilibrium, τ vanishes and the stress is given by the thermodynamic pressure p alone. Hence, the viscous stress scalar τ is usually expressed as an expansion in temperature and velocity gradients, and this distinguishes it from p. The same holds for the heat current q. The coefficients of these expansions characterize the transport phenomena occurring in the fluid.

The shear and the vorticity vanish identically in two spacetime dimensions. The only non-vanishing first-derivative tensors of the velocity are the acceleration and the expansion

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \tag{2.8}$$

and one defines similarly the expansion of the dual congruence as[6]

$$\Theta^* = \nabla_\mu * u^\mu, \tag{2.9}$$

[4] Our conventions are: $\eta_{\sigma\rho} = \sqrt{g}\epsilon_{\sigma\rho}$ with $\epsilon_{01} = +1$. Hence $\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu_\nu$.

[5] This component of the energy–momentum tensor is also referred to as the *viscous bulk pressure*, or the *dynamic pressure*, or else the *non-equilibrium pressure*.

[6] The hodge-dual of a scalar is a two-form and would spell with a suffix star. Instead, Θ^* is just another scalar.

which enables us expressing the acceleration:

$$a_\mu = \Theta^* * u_\mu. \tag{2.10}$$

In first-order hydrodynamics[7]

$$\begin{aligned} \tau_{(1)} &= -\zeta\Theta, & (2.11)\\ \chi_{(1)} &= -\frac{\kappa}{k^2}\left(*\mathrm{u}(T) + T\Theta^*\right). & (2.12)\end{aligned}$$

As usual, ζ is the bulk viscosity and κ is the thermal conductivity – assumed constant in this expression.

It is convenient to use the orthonormal Cartan frame $\{{}^{\mathrm{u}}/_{k}, {}^{*\mathrm{u}}/_{k}\}$. Then the metric reads:

$$\mathrm{d}s^2 = \frac{1}{k^2}\left(-\mathrm{u}^2 + *\mathrm{u}^2\right), \tag{2.13}$$

while the energy–momentum tensor takes the form:

$$\mathrm{T} = \frac{1}{2k^2}\left((\varepsilon + \chi)\left(\mathrm{u} + *\mathrm{u}\right)^2 + (\varepsilon - \chi)\left(\mathrm{u} - *\mathrm{u}\right)^2\right) + \frac{1}{k^2}(p - \varepsilon + \tau) * \mathrm{u}^2. \tag{2.14}$$

In holographic systems, the boundary enjoys remarkable conformal properties as it defines a conformal class, rather than a specific metric. Under Weyl transformations

$$\mathrm{d}s^2 \to \frac{\mathrm{d}s^2}{\mathcal{B}^2}, \tag{2.15}$$

the velocity form components u_μ are traded for ${}^{u_\mu}/_{\mathcal{B}}$, the energy and heat densities have weight 2, and the local-equilibrium equation of state is conformal

$$\varepsilon = p, \tag{2.16}$$

which is accompanied by Stefan's law (σ is the Stefan–Boltzmann constant):

$$\varepsilon = \sigma T^2. \tag{2.17}$$

Hence, the trace of the energy–momentum tensor is τ. In the absence of anomalies it vanishes and $T_{\mu\nu}$ is invariant under (2.15). If τ is non-vanishing, the fluid is not conformal and τ is an anomalous weight-2 quantity.

Covariantization with respect to rescalings requires to introduce a Weyl connection one-

[7] For any vector v and a function f, $\mathrm{v}(f)$ stands for $v^\mu\partial_\mu f$. We remind the following identities: $\mathrm{d}^\dagger\mathrm{d}f = -\Box f$ with $\mathrm{d}^\dagger\mathrm{w} = *\mathrm{d} * \mathrm{w} = -\nabla^\mu w_\mu$ and $\mathrm{d}f = \frac{1}{k^2}\left(*\mathrm{u}(f) * \mathrm{u} - \mathrm{u}(f)\mathrm{u}\right)$, $*\mathrm{d}f = \frac{1}{k^2}\left(*\mathrm{u}(f)\mathrm{u} - \mathrm{u}(f) * \mathrm{u}\right)$.

form [26,27], see also Appendix D of [28]:[8]

$$\mathrm{A} = \frac{1}{k^2}\left(\mathrm{a} - \Theta\mathrm{u}\right) = \frac{1}{k^2}\left(\Theta^* * \mathrm{u} - \Theta\mathrm{u}\right), \tag{2.18}$$

which transforms as $\mathrm{A} \to \mathrm{A} - \mathrm{d}\ln\mathcal{B}$. Ordinary covariant derivatives ∇ are thus traded for the Weyl covariant combination $\mathscr{D} = \nabla + w\,\mathrm{A}$, w being the conformal weight of the tensor under consideration. We provide for concreteness the Weyl covariant derivative of a form v_μ and of a scalar function Φ, both of weight w:

$$\begin{aligned} &\mathscr{D}_\nu v_\mu = \nabla_\nu v_\mu + (w+1)A_\nu v_\mu + A_\mu v_\nu - g_{\mu\nu}A^\rho v_\rho, \\ &\mathscr{D}_\nu \Phi = \partial_\nu \Phi + w A_\nu \Phi. \end{aligned} \tag{2.19}$$

The Weyl covariant derivative is metric-compatible with effective torsion:

$$\mathscr{D}_\rho g_{\mu\nu} = 0, \tag{2.20}$$

$$\left(\mathscr{D}_\mu\mathscr{D}_\nu - \mathscr{D}_\nu\mathscr{D}_\mu\right)\Phi = w\Phi F_{\mu\nu}, \tag{2.21}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.22}$$

is the Weyl-invariant field strength. Its dual

$$F = *\mathrm{d}\mathrm{A} = \eta^{\mu\nu}\partial_\mu A_\nu = \frac{1}{k^2}\left(*\mathrm{u}(\Theta) - \mathrm{u}(\Theta^*)\right) \tag{2.23}$$

is a weight-2 scalar.

Commuting the Weyl-covariant derivatives acting on vectors, one defines the Weyl covariant Riemann tensor

$$\left(\mathscr{D}_\mu\mathscr{D}_\nu - \mathscr{D}_\nu\mathscr{D}_\mu\right)V^\rho = \mathscr{R}^\rho{}_{\sigma\mu\nu}V^\sigma + wF_{\mu\nu}V^\rho \tag{2.24}$$

(V^ρ are weight-w) and the usual subsequent quantities. In two spacetime dimensions, the covariant Ricci tensor (weight-0) and the scalar (weight-2) curvatures read:

$$\mathscr{R}_{\mu\nu} = R_{\mu\nu} + g_{\mu\nu}\nabla_\lambda A^\lambda - F_{\mu\nu}, \tag{2.25}$$

$$\mathscr{R} = R + 2\nabla_\mu A^\mu. \tag{2.26}$$

It turns out that $R_{\mu\nu} + g_{\mu\nu}\nabla_\lambda A^\lambda$ vanishes identically. Hence

$$\mathscr{R} = 0 \Leftrightarrow R = 2\mathrm{d}^\dagger\mathrm{A} \quad \text{and} \quad \mathscr{R}_{\mu\nu} = -F_{\mu\nu}. \tag{2.27}$$

[8]The explicit form of A is obtained by demanding $\mathscr{D}_\mu u^\mu = 0$ and $u^\lambda\mathscr{D}_\lambda u_\mu = 0$.

The ordinary scalar curvature has a weight-2 anomalous transformation

$$R \to \mathcal{B}^2 \left(R + 2\Box \ln \mathcal{B}\right) \tag{2.28}$$

(the box operator is here referring to the metric before the Weyl transformation).

Hydrodynamic equations and the hydrodynamic-frame covariance

Using the above tools as well as the identity

$$\nabla^\mu T_{\mu\nu} = \mathscr{D}^\mu T_{\mu\nu} - A_\nu T^\mu{}_\mu, \tag{2.29}$$

(based on Eqs. (2.19) and Leibniz rule, for a weight-0, rank-2 symmetric tensor), the general fluid equations (2.1) with $\varepsilon = p$, projected on the light-cone directions $\mathrm{u} \pm *\mathrm{u}$ read:[9]

$$\begin{cases} (u^\mu + *u^\mu)\mathscr{D}_\mu(\varepsilon + \chi) + (u^\mu - *u^\mu) f_\mu = -\Theta\tau - \Theta^*\tau - *\mathrm{u}(\tau), \\ (u^\mu - *u^\mu)\mathscr{D}_\mu(\varepsilon - \chi) + (u^\mu + *u^\mu) f_\mu = -\Theta\tau + \Theta^*\tau + *\mathrm{u}(\tau). \end{cases} \tag{2.30}$$

Equivalently, these equations are expressed as

$$\begin{cases} \mathrm{d}\left(\sqrt{\varepsilon + \chi + \tau/2}(\mathrm{u} + *\mathrm{u})\right) + \dfrac{1}{2\sqrt{\varepsilon + \chi + \tau/2}}(\mathrm{u} - *\mathrm{u}) \wedge *\left(\mathrm{f} - \frac{1}{2}\mathrm{d}\tau\right) = 0, \\ \mathrm{d}\left(\sqrt{\varepsilon - \chi + \tau/2}(\mathrm{u} - *\mathrm{u})\right) - \dfrac{1}{2\sqrt{\varepsilon - \chi + \tau/2}}(\mathrm{u} + *\mathrm{u}) \wedge *\left(\mathrm{f} - \frac{1}{2}\mathrm{d}\tau\right) = 0. \end{cases} \tag{2.31}$$

Changing hydrodynamic frame, *i.e.* the fluid velocity field, amounts to perform an arbitrary local Lorentz transformation on the Cartan mobile frame

$$\begin{pmatrix} \mathrm{u}' \\ *\mathrm{u}' \end{pmatrix} = \begin{pmatrix} \cosh\psi(x) & \sinh\psi(x) \\ \sinh\psi(x) & \cosh\psi(x) \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ *\mathrm{u} \end{pmatrix}, \tag{2.32}$$

or for the null directions $\mathrm{u}' \pm *\mathrm{u}' = (\mathrm{u} \pm *\mathrm{u})\, \mathrm{e}^{\pm\psi}$. This affects the Weyl connection and Weyl curvature scalar as follows

$$\mathrm{A}' = \mathrm{A} - *\mathrm{d}\psi \tag{2.33}$$

$$F' = F + \Box\psi. \tag{2.34}$$

The transformation (2.32) keeps the energy–momentum tensor invariant provided the energy density and the heat density transform appropriately. Imposing that in the new frame

[9]Notice that any congruence with $w = -1$ in two dimensions obeys $\mathscr{D}_\mu u_\nu = \nabla_\mu u_\nu + \frac{1}{k^2} u_\mu a_\nu - \Theta h_{\mu\nu} = 0$ due to the absence of shear and vorticity, and similarly $\mathscr{D}_\mu * u_\nu = 0$.

(2.16) holds, *i.e.* $\varepsilon' = p'$, we conclude that

$$\begin{pmatrix} \varepsilon' \\ \chi' \end{pmatrix} = \begin{pmatrix} \cosh 2\psi(x) & -\sinh 2\psi(x) \\ -\sinh 2\psi(x) & \cosh 2\psi(x) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \chi \end{pmatrix} + \tau \sinh \psi(x) \begin{pmatrix} \sinh \psi(x) \\ -\cosh \psi(x) \end{pmatrix}, \tag{2.35}$$

while, due to the invariance of the trace,

$$\tau' = \tau. \tag{2.36}$$

Equivalently one can use $\sqrt{\left(\varepsilon' \pm \chi' + \frac{\tau'}{2}\right)} = \sqrt{\left(\varepsilon \pm \chi + \frac{\tau}{2}\right)}\, \mathrm{e}^{\mp\psi}$.

The energy–momentum tensor can be diagonalized with a specific local Lorentz transformation. By definition, the corresponding hydrodynamic frame is the Landau–Lifshitz frame, where the heat current χ_{LL} is vanishing. We find

$$\mathrm{T} = \frac{\varepsilon_{\mathrm{LL}}}{k^2} \mathrm{u}_{\mathrm{LL}}^2 + \frac{\varepsilon_{\mathrm{LL}} + \tau}{k^2} * \mathrm{u}_{\mathrm{LL}}^2 \tag{2.37}$$

since $\tau_{\mathrm{LL}} = \tau$ and $\chi_{\mathrm{LL}} = 0$. The latter condition allows to find the local boost towards the Landau–Lifshitz frame

$$\mathrm{e}^{4\psi_{\mathrm{LL}}} = \frac{\varepsilon + \chi + \tau/2}{\varepsilon - \chi + \tau/2}. \tag{2.38}$$

With this, the eigenvalues are easily computed. One finds the Landau–Lifshitz energy density

$$\varepsilon_{\mathrm{LL}} = \sqrt{\left(\varepsilon + \chi + \frac{\tau}{2}\right)\left(\varepsilon - \chi + \frac{\tau}{2}\right)} - \frac{\tau}{2}. \tag{2.39}$$

It exhibits an upper bound for χ^2, $\chi^2_{\max} = (\varepsilon + \tau/2)^2$, which translates causality and unitarity properties of the underlying microscopic field theory. The eigenvalue[10] $\varepsilon_{\mathrm{LL}}$ is supported by the time-like eigenvector

$$\mathrm{u}_{\mathrm{LL}} = \frac{1}{2}\left(\left(\frac{\varepsilon + \chi + \tau/2}{\varepsilon - \chi + \tau/2}\right)^{1/4} (\mathrm{u} + *\mathrm{u}) + \left(\frac{\varepsilon - \chi + \tau/2}{\varepsilon + \chi + \tau/2}\right)^{1/4} (\mathrm{u} - *\mathrm{u})\right), \tag{2.40}$$

whereas

$$\varepsilon^*_{\mathrm{LL}} = \varepsilon_{\mathrm{LL}} + \tau = \sqrt{\left(\varepsilon + \chi + \frac{\tau}{2}\right)\left(\varepsilon - \chi + \frac{\tau}{2}\right)} + \frac{\tau}{2} \tag{2.41}$$

is the eigenvalue along the space-like eigenvector $*\mathrm{u}_{\mathrm{LL}}$. Using the above expressions in the

[10] We make the reasonable assumption that the fluid energy density is positive. This is generically true, although some exceptions exist. One of those is global AdS_3, indeed realized with a negative-energy dual fluid, whereas the conventional zero-energy fluid reconstructs one Poincaré patch of AdS_3.

Landau–Lifshitz frame, the fluid equations (2.31) are recast as follows

$$\begin{cases} 2\sqrt{\varepsilon_{\mathrm{LL}}}\,\mathrm{d}^{\dagger}\left(\sqrt{\varepsilon_{\mathrm{LL}}}\,\mathrm{u}_{\mathrm{LL}}\right) - \mathrm{u}_{\mathrm{LL}}\cdot \mathrm{f} - \Theta_{\mathrm{LL}}\tau = 0\,, \\ 2\sqrt{\varepsilon^{*}_{\mathrm{LL}}}\,\mathrm{d}^{\dagger}\left(\sqrt{\varepsilon^{*}_{\mathrm{LL}}} * \mathrm{u}_{\mathrm{LL}}\right) + *\mathrm{u}_{\mathrm{LL}}\cdot \mathrm{f} + \Theta^{*}_{\mathrm{LL}}\tau = 0\,. \end{cases} \tag{2.42}$$

A non-anomalous conformal fluid in two dimensions is defined through the relations (2.16), (2.17) and

$$\tau = 0. \tag{2.43}$$

Under these assumptions, the last term of (2.14) drops, whereas following the fluid equations (2.31) at zero external force ($\mathrm{f} = f_\mu \mathrm{d}x^\mu = 0$), the forms $\sqrt{\varepsilon \pm \chi}\,(\mathrm{u} \pm *\mathrm{u})$ are *closed*, and can be used to define a privileged light-cone coordinate system, adapted to the fluid configuration. In this specific case, the on-shell Weyl scalar curvature reads

$$F = -\frac{1}{2}\Box \ln \sqrt{\frac{\varepsilon + \chi}{\varepsilon - \chi}}. \tag{2.44}$$

For conformal fluids, the hydrodynamic-frame transformation (2.32) acts on the energy and heat densities as a spin-two electric–magnetic boost, the energy being electric and the heat magnetic.

The entropy current

We would like to close this overview on two-dimensional conformal fluids with the entropy current. The entropy appears in Gibbs–Duhem equation

$$Ts = p + \varepsilon, \tag{2.45}$$

and is easily computed for conformal fluids in terms of the energy density, using Eq. (2.16) and Stefan's law (2.17):

$$s = 2\sqrt{\sigma\varepsilon}. \tag{2.46}$$

The entropy current is an involved concept because, among other reasons, no microscopic definition is available for out-of-global-equilibrium systems. In arbitrary dimension, there is no generic and closed expression in terms of the dissipative tensors for this current, which is generally constructed order by order as a derivative expansion (see [29]). Whether this expansion can be hydrodynamic-frame invariant, and at the same time compatible with the underlying already quoted microscopic laws (unitarity and causality) as well as with the second law of thermodynamics is not known in full generality, although this is in principle part of the rationale behind frame invariance.

In two dimensions, the ingredients for building a hydrodynamic-frame-invariant entropy current are the time-like invariant vector u_{LL} (given in (2.40)) and its space-like dual

$*\mathrm{u}_{\mathrm{LL}}$, plus the invariant scalars $\varepsilon_{\mathrm{LL}}$ and $\varepsilon^*_{\mathrm{LL}}$ (or any combination, see (2.39) and (2.41)). The entropy current should have non-negative divergence, vanishing for a free (*i.e.* at zero external force) perfect fluid. In the case at hand, a perfect fluid is necessarily conformal since it must have vanishing τ.

A good candidate for a hydrodynamic-frame-invariant entropy current is

$$\mathrm{S}_{\mathbf{0}} = s_{\mathrm{LL}}\mathrm{u}_{\mathrm{LL}} = 2\sqrt{\sigma\varepsilon_{\mathrm{LL}}}\,\mathrm{u}_{\mathrm{LL}}, \tag{2.47}$$

which can be expressed in any frame using Eqs. (2.39) and (2.40). This is usually adopted as the entropy current of a perfect fluid, and in that case it is divergence-free when external forces vanish. Here, it obeys (see (2.42))

$$\nabla\cdot\mathrm{S}_{\mathbf{0}} = -\sqrt{\frac{\sigma}{\varepsilon_{\mathrm{LL}}}}\left(\Theta_{\mathrm{LL}}\tau + \mathrm{u}_{\mathrm{LL}}\cdot\mathrm{f}\right) = -\frac{1}{T_{\mathrm{LL}}}\left(\Theta_{\mathrm{LL}}\tau + \mathrm{u}_{\mathrm{LL}}\cdot\mathrm{f}\right), \tag{2.48}$$

which can be recast in terms of arbitrary-frame data using the already quoted (2.39), (2.40) and the divergence of the latter. Expanding this result up to first order for $\chi,\tau \ll \varepsilon$, we find for a free fluid

$$\nabla\cdot\mathrm{S}_{\mathbf{0}(1)} = -\frac{1}{T}\Theta\tau = \frac{\zeta}{T}\Theta^2, \tag{2.49}$$

where we have used in the last equality the first-order derivative expansion of τ, given in (2.11). For this to be positive one finds the usual requirement $\zeta > 0$. From this perspective, the current $\mathrm{S}_{\mathbf{0}}$ seems fine.

The expansion of $\mathrm{S}_{\mathbf{0}}$ up to second order in $\chi,\tau \ll \varepsilon$,

$$\mathrm{S}_{\mathbf{0}} = 2\sqrt{\sigma\varepsilon}\,\mathrm{u} + \chi\sqrt{\frac{\sigma}{\varepsilon}} * \mathrm{u} - \frac{\chi^2}{4\varepsilon}\sqrt{\frac{\sigma}{\varepsilon}}\mathrm{u} - \frac{\tau\chi}{2\varepsilon}\sqrt{\frac{\sigma}{\varepsilon}} * \mathrm{u} + \cdots = s\mathrm{u} + \frac{\mathrm{q}}{T} - \frac{\chi^2}{4\varepsilon T}\mathrm{u} - \frac{\tau}{2\varepsilon T}\mathrm{q} + \cdots, \tag{2.50}$$

is in agreement with the usual expectations dictated by *extended irreversible thermodynamics* (completing the first-order *classical irreversible thermodynamics*) [29]. These can be summarized as follows, the order referring to the dissipative expansion:

1. free perfect limit: $\mathrm{S}|_{\chi=\tau=0} = \mathrm{S}_{(0)} = s\mathrm{u} = 2\sqrt{\sigma\varepsilon}\,\mathrm{u}$;
2. stability $\left.\frac{\partial \mathrm{S}\cdot\mathrm{u}}{\partial\tau}\right|_{\chi=\tau=0} = 0$;
3. first-order (CIT) correction: $\mathrm{S}_{(1)} = \frac{\mathrm{q}}{T}$;
4. second-order (EIT) corrections: $\mathrm{S}_{(2)}$ might contain $\frac{\tau^2}{\varepsilon T}\mathrm{u}$, $\frac{\chi^2}{\varepsilon T}\mathrm{u}$ and $\frac{\tau}{\varepsilon T}\mathrm{q}$;
5. second law: $\nabla\cdot\mathrm{S} \geqslant 0$.

Other invariant terms may be considered in the definition of S as long as the above requirements are satisfied. In the absence of a concrete proposal for selecting other terms, we will not pursue the argument any further. Related discussions can be found in [30–33].[11]

Light-cone versus Randers–Papapetrou frames

Light-cone frame Every two-dimensional metric is amenable by diffeomorphisms to a conformally flat form. This suggests to use:[12]

$$\mathrm{d}s^2 = \mathrm{e}^{-2\omega}\mathrm{d}x^+\mathrm{d}x^- \tag{2.51}$$

(with usual time and space coordinates defined as $x^\pm = x \pm kt$), where ω is an arbitrary function of x^+ and x^-.

Any normalized congruence has the following form:

$$\mathrm{u} = u_+\mathrm{d}x^+ + u_-\mathrm{d}x^- \quad \Leftrightarrow \quad *\mathrm{u} = -u_+\mathrm{d}x^+ + u_-\mathrm{d}x^-, \tag{2.52}$$

where $u_\pm$, functions of x^+ and x^-, are related by the normalization condition

$$u_+u_- = -\frac{k^2}{4}\mathrm{e}^{-2\omega}. \tag{2.53}$$

We can parameterize the velocity field as

$$u_+ = -\frac{k}{2}\mathrm{e}^{-\omega}\sqrt{\xi}, \quad u_- = \frac{k}{2}\mathrm{e}^{-\omega}\frac{1}{\sqrt{\xi}}, \tag{2.54}$$

where $\xi = \xi(x^+, x^-)$ is defined as the ratio

$$\xi = -\frac{u_+}{u_-}. \tag{2.55}$$

The choice $\xi = 1$ corresponds to a comoving fluid because in this case $\mathrm{u} = -k^2\mathrm{e}^{-\omega}\mathrm{d}t$.

For the congruence at hand

$$\Theta \pm \Theta^* = \pm 2k\mathrm{e}^{2\omega}\partial_\pm \mathrm{e}^{-\left(\omega \pm \ln\sqrt{\xi}\right)}. \tag{2.56}$$

We can also determine the Weyl connection and field strength:

$$\mathrm{A} = -\mathrm{d}\omega + *\mathrm{d}\ln\sqrt{\xi} \quad \text{and} \quad F = -\Box \ln\sqrt{\xi} = -2\mathrm{e}^{2\omega}\partial_+\partial_-\ln\xi, \tag{2.57}$$

whereas the ordinary (non Weyl-covariant) scalar curvature reads (see (2.27))

[11] It should be quoted that S as defined in (2.47) does not coincide with the entropy current proposed in Ref. [33]. Hydrodynamic-frame invariance and CIT/EIT arguments were not part of the agenda in this work, based essentially on the second law of thermodynamics.

[12] With this choice, $g_{+-} = {}^1\!/_2\mathrm{e}^{-2\omega}$, $\eta_{+-} = {}^1\!/_2\mathrm{e}^{-2\omega}$, $\eta^{+-} = -2\mathrm{e}^{2\omega}$, $\eta_+{}^+ = 1$, $\eta_-{}^- = -1$. Notice also that $*\left(\mathrm{d}x^+ \wedge \mathrm{d}x^-\right) = \eta^{+-} = -2\mathrm{e}^{2\omega}$.

$$R = 2\Box\omega = 8\mathrm{e}^{2\omega}\partial_+\partial_-\omega. \tag{2.58}$$

In the present light-cone frame $\{\mathrm{d}x^+, \mathrm{d}x^-\}$, a general energy–momentum tensor with $\epsilon = p$ has components

$$\begin{gathered} T_{++} = \frac{\zeta}{2}\left(\varepsilon - \chi + \frac{\tau}{2}\right)\mathrm{e}^{-2\omega}, \quad T_{--} = \frac{1}{2\zeta}\left(\varepsilon + \chi + \frac{\tau}{2}\right)\mathrm{e}^{-2\omega}, \\ T_{+-} = T_{-+} = \frac{\tau}{4}\mathrm{e}^{-2\omega}. \end{gathered} \tag{2.59}$$

For a conformal fluid Eqs. (2.43) lead to $T_{+-} = T_{-+} = 0$ and

$$(\varepsilon + \chi)(\varepsilon - \chi) = 4\mathrm{e}^{4\omega}T_{++}T_{--}, \quad \frac{\varepsilon + \chi}{\varepsilon - \chi} = \frac{T_{--}}{T_{++}}\zeta^2. \tag{2.60}$$

In the latter case, and in the absence of external forces, the forms (2.31) are closed, which in light-cone coordinates implies that $(\varepsilon - \chi)\mathrm{e}^{-2\omega}\zeta$ is locally a function of x^+, and $(\varepsilon + \chi)\frac{\mathrm{e}^{-2\omega}}{\zeta}$ a function of x^-. Observe that in the Landau–Lifshitz frame ($\chi_{\mathrm{LL}} = 0$)

$$\zeta^2_{\mathrm{LL}} = \frac{T_{++}}{T_{--}}, \quad \varepsilon^2_{\mathrm{LL}} = 4\mathrm{e}^{4\omega}T_{++}T_{--}. \tag{2.61}$$

In this frame, on-shell, F vanishes. Moving from a given hydrodynamic frame to another by a local Lorentz boost, amounts to perform the following transformation on the function ζ

$$\zeta(x^+, x^-) \to \zeta'(x^+, x^-) = \mathrm{e}^{-2\psi(x^+, x^-)}\zeta(x^+, x^-). \tag{2.62}$$

Randers–Papapetrou frame The light-cone frame is not well suited for the Carrollian limit, which is the ultra-relativistic limit reached at vanishing k, and emerging at the null-infinity conformal boundary of a flat spacetime (subject of next section). As discussed in [19], Carrollian fluid dynamics is elegantly reached in the Randers–Papapetrou frame, where

$$\mathrm{d}s^2 = -k^2\left(\Omega\mathrm{d}t - b_x\mathrm{d}x\right)^2 + a\mathrm{d}x^2 \tag{2.63}$$

with all three functions of the coordinates t and x.

A generic velocity vector field u reads:

$$\mathrm{u} = \gamma\left(\partial_t + v^x\partial_x\right). \tag{2.64}$$

It is convenient to parametrize the velocity v^x (see [19]) as[13]

$$v^x = \frac{k^2\Omega\beta^x}{1+k^2\boldsymbol{\beta}\cdot\boldsymbol{b}} \Leftrightarrow \beta^x = \frac{v^x}{k^2\Omega\left(1-\frac{v^x b_x}{\Omega}\right)} \tag{2.65}$$

with Lorentz factor

$$\gamma = \frac{1+k^2\boldsymbol{\beta}\cdot\boldsymbol{b}}{\Omega\sqrt{1-k^2\boldsymbol{\beta}^2}}. \tag{2.66}$$

The velocity form and its Hodge-dual read:

$$\mathrm{u} = -\frac{k^2}{\sqrt{1-k^2\boldsymbol{\beta}^2}}\left(\Omega \mathrm{d}t - (b_x+\beta_x)\,\mathrm{d}x\right), \quad *\mathrm{u} = k\sqrt{a}\,\Omega\gamma\left(\mathrm{d}x - v^x\mathrm{d}t\right), \tag{2.67}$$

while the corresponding vector is

$$*\,\mathrm{u} = \frac{k}{\sqrt{a}\sqrt{1-k^2\boldsymbol{\beta}^2}}\left(\frac{b_x+\beta_x}{\Omega}\partial_t + \partial_x\right). \tag{2.68}$$

We can determine the form of the heat current q, which must be proportional to $*$u, in terms of a single component q_x. We find

$$\chi = \frac{q_x}{k\sqrt{a}\,\Omega\gamma} = \frac{q^x\sqrt{a}\sqrt{1-k^2\boldsymbol{\beta}^2}}{k}. \tag{2.69}$$

Similarly, for the viscous stress tensor

$$\tau = \frac{\tau_{xx}}{a\Omega^2\gamma^2} = \tau^{xx}a\left(1-k^2\boldsymbol{\beta}^2\right). \tag{2.70}$$

Performing a local Lorentz boost (2.32) on the hydrodynamic frame does not affect the geometric objects Ω, b_x or a, and is thus entirely captured by the transformation of the vector $\boldsymbol{\beta}$. Parameterizing the boost in terms of a Carrollian vector $\boldsymbol{B} = B^x\partial_x$ as

$$\cosh\psi = \Gamma = \frac{1}{\sqrt{1-k^2\boldsymbol{B}^2}}, \quad \sinh\psi = \Gamma k\sqrt{a}B^x = \frac{k\sqrt{a}B^x}{\sqrt{1-k^2\boldsymbol{B}^2}}, \tag{2.71}$$

we get:

$$\boldsymbol{\beta}' = \frac{\boldsymbol{\beta}+\boldsymbol{B}}{1+k^2\boldsymbol{\beta}\cdot\boldsymbol{B}}, \tag{2.72}$$

as expected from the velocity rule composition in special relativity. Using (2.35), we also

[13] With these definitions, β^x transforms as the component of a genuine Carrollian vector $\boldsymbol{\beta} = \beta^x\partial_x$, when considering the flat limit of the bulk spacetime. Notice that $\beta_x + b_x = -\frac{\Omega u_x}{k u_0}$. We define as usual $b^x = a^{xx}b_x$, $\beta_x = a_{xx}\beta^x$, $v_x = a_{xx}v^x$ with $a_{xx} = 1/a^{xx} = a$, $\boldsymbol{b}^2 = b_x b^x$, $\boldsymbol{\beta}^2 = \boldsymbol{\beta}\cdot\boldsymbol{\beta} = \beta_x\beta^x$ and $\boldsymbol{b}\cdot\boldsymbol{\beta} = b_x\beta^x$.

obtain

$$\varepsilon' = \frac{1}{1-k^2\boldsymbol{B}^2}\left(\left(1+k^2\boldsymbol{B}^2\right)\varepsilon - k\sqrt{a}B^x 2\chi + k^2\boldsymbol{B}^2\tau\right), \tag{2.73}$$

$$\chi' = \frac{1}{1-k^2\boldsymbol{B}^2}\left(\left(1+k^2\boldsymbol{B}^2\right)\chi - k\sqrt{a}B^x(2\varepsilon+\tau)\right), \tag{2.74}$$

accompanying (2.36). Together with (2.69) and (2.70), we finally reach:

$$\frac{q'_x}{\sqrt{a}} = \left(\left(1+k^2\boldsymbol{B}^2\right)\chi - k\sqrt{a}B^x(2\varepsilon+\tau)\right)k\frac{\left(1+k^2\left(\boldsymbol{\beta}\cdot\boldsymbol{B}+(\boldsymbol{\beta}+\boldsymbol{B})\cdot\boldsymbol{b}\right)\right)}{\left(1-k^2\boldsymbol{\beta}^2\right)^{1/2}\left(1-k^2\boldsymbol{B}^2\right)^{3/2}}, \tag{2.75}$$

$$\frac{\tau'_{xx}}{a} = \tau\frac{\left(1+k^2\left(\boldsymbol{\beta}\cdot\boldsymbol{B}+(\boldsymbol{\beta}+\boldsymbol{B})\cdot\boldsymbol{b}\right)\right)^2}{\left(1-k^2\boldsymbol{\beta}^2\right)\left(1-k^2\boldsymbol{B}^2\right)}. \tag{2.76}$$

2.2 Carrollian fluids

The Carrollian geometry

The Carrollian geometry $\mathbb{R}\times\mathscr{S}$ is obtained as the vanishing-k limit of the two-dimensional pseudo-Riemannian geometry $\mathscr{M}$ equipped with metric (2.63). In this limit, the line $\mathscr{S}$ inherits a metric[14]

$$\mathrm{d}\ell^2 = a\mathrm{d}x^2, \tag{2.77}$$

and $t\in\mathbb{R}$ is the Carrollian time. Much like a Galilean space is observed from a spatial frame moving with respect to a local inertial frame with velocity $\mathbf{w}$, a Carrollian frame is described by a form $\boldsymbol{b} = b_x(t,x)\,\mathrm{d}x$. The latter is *not* a velocity because in Carrollian spacetimes motion is forbidden. It is rather an inverse velocity, describing a "temporal frame" and plays a dual role. A scalar $\Omega(t,x)$ also remains in the $k\to 0$ limit (as in the Galilean case, see [19] – this reference will be useful along the present section).

We define the Carrollian diffeomorphisms as

$$t' = t'(t,x) \quad \text{and} \quad x' = x'(x). \tag{2.78}$$

The ordinary exterior derivative of a scalar function does not transform as a form. To overcome this issue, it is desirable to introduce a Carrollian derivative as

$$\hat{\partial}_x = \partial_x + \frac{b_x}{\Omega}\partial_t, \tag{2.79}$$

transforming as a form. With this derivative we can proceed and define a Carrollian covariant derivative $\hat{\nabla}_x$, based on Levi–Civita–Carroll connection

$$\hat{\gamma}^x_{xx} = \hat{\partial}_x \ln\sqrt{a}. \tag{2.80}$$

[14] This metric lowers all x indices.

As we will see in 3.2, in the framework of flat holography, the spatial surface $\mathscr{S}$ emerges as the null infinity $\mathscr{I}^+$ of the Ricci-flat geometry. The geometry of $\mathscr{I}^+$ is equipped with a conformal class of metrics rather than with a metric. From a representative of this class, we must be able to explore others by Weyl transformations, and this amounts to study conformal Carrollian geometry as opposed to plain Carrollian geometry (see [34]).

The action of Weyl transformations on the elements of the Carrollian geometry on a surface $\mathscr{S}$ is inherited from (2.15)

$$a \to \frac{a}{\mathcal{B}^2}, \quad b_x \to \frac{b_x}{\mathcal{B}}, \quad \Omega \to \frac{\Omega}{\mathcal{B}}, \quad \beta_x \to \frac{\beta_x}{\mathcal{B}}, \tag{2.81}$$

where $\mathcal{B} = \mathcal{B}(t, x)$ is an arbitrary function. However, the Levi–Civita–Carroll covariant derivatives are not covariant under (2.81). Following [19], they must be replaced with Weyl–Carroll covariant spatial and time metric-compatible derivatives built on the Carrollian acceleration φ_x and the Carrollian expansion θ,

$$\varphi_x = \frac{1}{\Omega}\left(\partial_t b_x + \partial_x \Omega\right) = \partial_t \frac{b_x}{\Omega} + \hat{\partial}_x \ln \Omega, \tag{2.82}$$

$$\theta = \frac{1}{\Omega}\partial_t \ln \sqrt{a}, \tag{2.83}$$

which transform as connections:

$$\varphi_x \to \varphi_x - \hat{\partial}_x \ln \mathcal{B}, \quad \theta \to \mathcal{B}\theta - \frac{1}{\Omega}\partial_t \mathcal{B}. \tag{2.84}$$

In particular, these can be combined in[15]

$$\alpha_x = \varphi_x - \theta b_x, \tag{2.85}$$

transforming under Weyl rescaling as

$$\alpha_x \to \alpha_x - \partial_x \ln \mathcal{B}. \tag{2.86}$$

The spatial Weyl–Carrol derivative is

$$\hat{\mathscr{D}}_x \Phi = \hat{\partial}_x \Phi + w \varphi_x \Phi, \tag{2.87}$$

for a weight-w scalar function Φ, and

$$\hat{\mathscr{D}}_x V^x = \hat{\nabla}_x V^x + (w-1)\varphi_x V^x, \tag{2.88}$$

[15] Contrary to φ_x, α_x is not a Carrollian one-form, *i.e.* it does not transform covariantly under Carrollian diffeomorphisms (2.78).

for a vector with weight-w component V^x. It does not alter the conformal weight, and is generalized to any tensor by Leibniz rule.

Similarly we define the temporal Weyl–Carroll derivative by its action on a weight-w function Φ

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\Phi = \frac{1}{\Omega}\partial_t\Phi + w\theta\Phi, \tag{2.89}$$

which is a scalar of weight $w+1$ under (2.81). Accordingly, the action of the Weyl–Carroll time derivative on a weight-w vector is

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t V^x = \frac{1}{\Omega}\partial_t V^x + w\theta V^x. \tag{2.90}$$

This is the component of a genuine Carrollian vector of weight $w+1$, and Leibniz rule allows to generalize this action to any tensor.

The Weyl–Carroll connections have curvature. Here, the only non-vanishing piece is the curvature one-form resulting from the commutation of $\hat{\mathscr{D}}_x$ and $\frac{1}{\Omega}\hat{\mathscr{D}}_t$, which has weight 1:

$$\mathscr{R}_x = \frac{1}{\Omega}\left(\partial_t\alpha_x - \partial_x(\theta\Omega)\right) = \frac{1}{\Omega}\partial_t\varphi_x - \theta\varphi_x - \hat{\partial}_x\theta. \tag{2.91}$$

Carrollian fluid observables

A relativistic fluid satisfying Eq. (2.1) will obey Carrollian dynamics in the ultra-relativistic limit, reached at vanishing k. The original relativistic fluid is not at rest, but has a velocity parametrized with $\boldsymbol{\beta} = \beta_x \mathrm{d}x$ (see (2.65)), which remains in the Carrollian limit as the kinematical "inverse-velocity" variable. We will keep calling it abusively "velocity". This variable transforms as a Carrollian vector and allows to define further kinematical objects.

- We introduce the acceleration $\boldsymbol{\gamma} = \gamma_x \mathrm{d}x$

$$\gamma_x = \frac{1}{\Omega}\partial_t\beta_x. \tag{2.92}$$

  This is not Weyl-covariant, as opposed to

$$\delta_x = \frac{1}{\Omega}\hat{\mathscr{D}}_t\beta_x = \gamma_x - \theta\beta_x = \frac{\sqrt{a}}{\Omega}\partial_t\frac{\beta_x}{\sqrt{a}}, \tag{2.93}$$

  which has weight 0.

- The suracceleration is the weight-1 conformal Carrollian one-form

$$\mathscr{A}_x = \frac{1}{\Omega}\hat{\mathscr{D}}_t\frac{1}{\Omega}\hat{\mathscr{D}}_t\beta_x = \frac{1}{\Omega}\partial_t\left(\frac{1}{\Omega}\partial_t\beta_x - \theta\beta_x\right). \tag{2.94}$$

It can be combined with the curvature (2.91), which has equal weight,

$$s_x = \mathscr{A}_x + \mathscr{R}_x = \frac{1}{\Omega}\partial_t\left(\frac{1}{\Omega}\partial_t\beta_x - \theta\beta_x\right) + \frac{1}{\Omega}\partial_t\varphi_x - \theta\varphi_x - \hat{\partial}_x\theta. \tag{2.95}$$

This appears as a conformal Carrollian total (*i.e.* kinematical plus geometric) suracceleration, and enables us to define a weight-2 conformal Carrollian scalar:

$$s = \frac{s_x}{\sqrt{a}}. \tag{2.96}$$

The latter originates from the Weyl curvature F of the pseudo-Riemannian ascendent manifold $\mathscr{M}$:

$$s = -\lim_{k\to 0} kF. \tag{2.97}$$

Notice that the ordinary scalar curvature of $\mathscr{M}$ given in (2.27) is not Weyl-covariant (see (2.28)) and can be expressed in terms of Carrollian non-Weyl-covariant scalars of $\mathbb{R}\times\mathscr{S}$:

$$R = \frac{2}{k^2}\left(\theta^2 + \frac{1}{\Omega}\partial_t\theta\right) - 2\left(\hat{\nabla}_x + \varphi_x\right)\varphi^x. \tag{2.98}$$

Besides the inverse velocity, acceleration and suracceleration, other physical data describe a Carrollian fluid.

- The energy density ε and the pressure p, related here through $\varepsilon = p$. The Carrollian energy and pressure are the zero-k limits of the corresponding relativistic quantities, and have weight 2. It is implicit that they are finite, and in order to avoid inflation of symbols, we have kept the same notation.

- The heat current $\boldsymbol{\pi} = \pi_x(t,x)\mathrm{d}x$ of conformal weight 1, inherited from the relativistic heat current (see (2.2)) as follows:[16]

$$q^x = k^2\pi^x + \mathrm{O}\left(k^4\right). \tag{2.99}$$

This translates the expected (see (2.69)) small-k behaviour of χ:

$$\chi = \chi_\pi k + \mathrm{O}\left(k^3\right), \tag{2.100}$$

[16] In arbitrary dimensions one generally admits $q^x = Q^x + k^2\pi^x + \mathrm{O}\left(k^4\right)$ (see [19]), which amounts assuming $\chi = \frac{\chi_Q}{k} + \chi_\pi k + \mathrm{O}\left(k^3\right)$. This is actually more natural because vanishing χ_Q is not a hydrodynamic-frame-invariant feature in the presence of friction. Keeping $\chi_Q \neq 0$, however, is not viable holographically in two boundary dimensions because it would create a $1/k^2$ divergence inside the derivative expansion. Since the Carrollian limit destroys anyway the hydrodynamic-frame invariance, our choice is consistent from every respect. Ultimately these behaviours should be justified within a microscopic quantum/statistical approach, missing at present.

leading to

$$\pi^x = \frac{\chi_\pi}{\sqrt{a}}. \tag{2.101}$$

- The weight-0 viscous stress tensors $\boldsymbol{\Sigma} = \Sigma_{xx}\mathrm{d}x^2$ and $\boldsymbol{\Xi} = \Xi_{xx}\mathrm{d}x^2$, obtained from the relativistic viscous stress tensor $\frac{\tau}{k^2} * \mathrm{u} * \mathrm{u}$ as

$$\tau^{xx} = -\frac{\Sigma^{xx}}{k^2} - \Xi^{xx} + \mathrm{O}\left(k^2\right). \tag{2.102}$$

For this to hold, following (2.70), we expect

$$\tau = \frac{\tau_\Sigma}{k^2} + \tau_\Xi + \mathrm{O}\left(k^2\right), \tag{2.103}$$

and find (in the Carrollian geometry, indices are lowered with $a_{xx} = a$):

$$\Sigma^x{}_x = -\tau_\Sigma, \quad \Xi^x{}_x = -\tau_\Xi - \boldsymbol{\beta}^2\tau_\Sigma. \tag{2.104}$$

As we will see later, this is in agreement with the form of τ for the relativistic systems at hand (see Eqs. (2.98) and (3.2)).

- Finally, we assume that the components of the external force density behave as follows, providing further Carrollian power and tension:

$$\begin{cases} \frac{k}{\Omega} f_0 = \frac{f}{k^2} + e + \mathrm{O}\left(k^2\right), \\ f^x = \frac{h^x}{k^2} + g^x + \mathrm{O}\left(k^2\right). \end{cases} \tag{2.105}$$

Hydrodynamic equations

The hydrodynamic equations for a Carrollian fluid are obtained as the zero-k limit of the relativistic equations (see [19]):

$$\begin{aligned}
-\left(\frac{1}{\Omega}\partial_t + 2\theta\right)\left(\varepsilon - \boldsymbol{\beta}^2\Sigma^x{}_x\right) + \left(\hat{\nabla}^x + 2\varphi^x\right)\left(\beta_x\Sigma^x{}_x\right) + \theta\left(\Xi^x{}_x - \boldsymbol{\beta}^2\Sigma^x{}_x\right) &= e, && (2.106)\\
\theta\Sigma^x{}_x &= f, && (2.107)\\
\left(\hat{\nabla}_x + \varphi_x\right)\left(\varepsilon - \Xi^x{}_x\right) + \varphi_x\left(\varepsilon - \boldsymbol{\beta}^2\Sigma^x{}_x\right) + \left(\frac{1}{\Omega}\partial_t + \theta\right)\left(\pi_x + \beta_x\left(2\varepsilon - \Xi^x{}_x\right)\right) &= g_x, && (2.108)\\
-\left(\hat{\nabla}_x + \varphi_x\right)\Sigma^x{}_x - \left(\frac{1}{\Omega}\partial_t + \theta\right)\left(\beta_x\Sigma^x{}_x\right) &= h_x. && (2.109)
\end{aligned}$$

Generically, the above equations are not invariant under Carrollian local boosts, acting as

$$\beta'_x = \beta_x + B_x \tag{2.110}$$

(vanishing-k limit of (2.72)). This should not come as a surprise. Such an invariance is exclusive to the relativistic case for obvious physical reasons, and is also known to be absent from Galilean fluid equations, which are not invariant under local Galilean boosts. Nevertheless, as we will see in Sec. 4, in specific situations a residual invariance persists.

3 Three-dimensional bulk reconstruction

3.1 Anti-de Sitter

Three-dimensional Einstein spacetimes are peculiar because the usual derivative expansion terminates at finite order. This happens also for the Fefferman–Graham expansion (see *e.g.* [21]). The reason is that most geometric and fluid tensors vanish (like the shear or the vorticity), reducing the number of available terms compatible with conformal invariance. Indeed, following the original fluid/gravity works [1–4], the ansatz for the bulk Einstein metric is a power expansion in $1/r$ such that boundary Weyl transformations (2.15) are compensated by $r \to \mathcal{B}(t,x)r$. The boundary metric has weight -2, the forms u and $*\mathrm{u}$ (velocity and dual fluid velocity) weight -1, whereas the energy and heat densities of the fluid have weight 2. The Weyl connection A has (anomalous) weight zero, as the form $\mathrm{d}r$. With these data we obtain:

$$\boxed{\mathrm{d}s^2_{\text{Einstein}} = 2\frac{\mathrm{u}}{k^2}\left(\mathrm{d}r + r\mathrm{A}\right) + r^2\mathrm{d}s^2 + \frac{8\pi G}{k^4}\mathrm{u}\left(\varepsilon\mathrm{u} + \chi * \mathrm{u}\right),} \tag{3.1}$$

where A is displayed in (2.18), ε and χ being the energy and heat densities of the fluid (as opposed to higher dimension, the heat current appears explicitly in the ansatz). These enter the fluid energy–momentum tensor (2.14) together with τ, which carries the anomaly:

$$\tau = \frac{R}{8\pi G} = \frac{1}{4\pi G k^2}\left(\Theta^2 - \Theta^{*2} + \mathrm{u}(\Theta) - *\mathrm{u}(\Theta^*)\right) \tag{3.2}$$

(we keep the conformal state equation $\varepsilon = p$). For a flat boundary this trace is absent, but Weyl transformations bring it back.

The precise coefficients of the eligible terms in the ansatz are determined by the radial-evolution subset of Einstein's equations, and this is already taken care of in expression (3.1), utterly locking the r-dependence. The remaining Einstein's equations further constrain the boundary data, *i.e.* the metric and the fluid. Summarizing, the metric (3.1) provides an *exact* Einstein, asymptotically AdS spacetime, with $R = 6\Lambda = -6k^2$, under the necessary and sufficient condition that the non-conformal fluid energy–momentum tensor (2.14) obeys

$$\nabla^\mu\left(T_{\mu\nu} + D_{\mu\nu}\right) = 0, \tag{3.3}$$

where $D_{\mu\nu}$ is a symmetric and traceless tensor which reads:

$$D_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = \frac{1}{8\pi G k^4}\left(\left(\mathrm{u}(\Theta)+*\mathrm{u}(\Theta^*)-\frac{k^2}{2}R\right)\left(\mathrm{u}^2+*\mathrm{u}^2\right)-4*\mathrm{u}(\Theta)\mathrm{u}*\mathrm{u}\right). \tag{3.4}$$

On the one hand, the holographic energy–momentum tensor is the sum $T_{\mu\nu}+D_{\mu\nu}$, and this can be shown following the Balasubramanian–Kraus method [35].[17] On the other hand, the holographic fluid is subject to an external force with density

$$f_{\nu} = -\nabla^{\mu}D_{\mu\nu}. \tag{3.5}$$

Its longitudinal and transverse components are

$$\begin{cases} u^{\mu}f_{\mu} = -\frac{1}{4\pi G}\left(*\mathrm{u}(F)+2\Theta^*F+\frac{1}{2}\Theta R\right), \\ *u^{\mu}f_{\mu} = \frac{1}{8\pi G}\left(*\mathrm{u}(R)+\Theta^* R\right). \end{cases} \tag{3.6}$$

Combining (2.30), (3.2) and (3.6) we find the following equations:

$$\begin{cases} \left(u^{\mu}+*u^{\mu}\right)\mathscr{D}_{\mu}\left(\varepsilon+\chi\right) = \frac{1}{4\pi G}*u^{\mu}\mathscr{D}_{\mu}F, \\ \left(u^{\mu}-*u^{\mu}\right)\mathscr{D}_{\mu}\left(\varepsilon-\chi\right) = \frac{1}{4\pi G}*u^{\mu}\mathscr{D}_{\mu}F. \end{cases} \tag{3.7}$$

Notice that eventually these equations are Weyl-covariant (weight-3) despite the conformal anomaly.

An important remark is in order regarding the holographic fluid. Rather than $T_{\mu\nu}$, we could have adopted $T_{\mu\nu}+D_{\mu\nu}$ as its energy–momentum tensor. The latter would have been decomposed as in (2.2), with $\tilde{\varepsilon}=\tilde{p}$ and $\tilde{\chi}$ though ($\tilde{\tau}=\tau$ since $D_{\mu\nu}$ has vanishing trace):

$$\tilde{\varepsilon} = \varepsilon+\frac{1}{8\pi G k^2}\left(\mathrm{u}(\Theta)+*\mathrm{u}(\Theta^*)\right)-\frac{R}{16\pi G}, \tag{3.8}$$

$$\tilde{\chi} = \chi-\frac{1}{4\pi G k^2}*\mathrm{u}(\Theta). \tag{3.9}$$

We did not make this choice for two reasons: (i) in the formula (3.1) we used ε and χ rather than $\tilde{\varepsilon}$ and $\tilde{\chi}$ for reconstructing the bulk; (ii) ε and χ/k are finite in the limit of vanishing k, whereas $\tilde{\varepsilon}$ and $\tilde{\chi}/k$ are not. This last fact is not an obstruction, but it would require to reconsider the Carrollian hydrodynamic equations developed in Ref. [19] and applied here.

Expression (3.1) is the most general locally AdS spacetime in Eddington–Finkelstein coordinates. The corresponding gauge includes but does not always coincide with BMS.[18] From

[17] For this computation we used the conventions of [36].

[18] There is no definition of Eddington–Finkelstein gauge. Within the three-dimensional derivative expansion, one can nevertheless refer to it as a gauge because the r-dependence is fixed. This does not exhaust all freedom, but allows comparison with BMS. Actually, fluid/gravity approach is not meant to lock completely the coordinates for describing the most general solution in terms of a minimal set of functions.

that perspective, this result is new although it may not contain any new solutions compared *e.g.* to Bañados' [18], all captured either in BMS or in Fefferman–Graham gauge (see [20]). The bonus is the hydrodynamical interpretation. Here the corresponding fluid is defined on a generally curved boundary and has an arbitrary velocity field. This should be contrasted with the treatment of three-dimensional fluid/gravity correspondence worked out in Refs. [2, 3], where the host geometry was flat, avoiding the issue of conformal anomaly. Furthermore the fluid was assumed perfect by hydrodynamic-frame choice, which permits a subclass of Bañados solutions only, as we will see in Sec. 4 by computing the conserved charges.

For practical purposes, we can work in light-cone coordinates, introduced in Eq. (2.51). Using the expression (2.54) for the congruence u, and solving the fluid equations (3.7), we obtain the fluid densities ε and χ in terms of two arbitrary chiral functions $\ell_\pm(x^\pm)$:

$$\varepsilon = \frac{e^{2\omega}}{4\pi G}\left(\frac{\ell_+}{\xi} + \xi\ell_- - \frac{3\,(\partial_+\xi)^2}{4\xi^3} + \frac{\partial_+^2\xi}{2\xi^2} + \frac{(\partial_-\xi)^2}{4\xi} - \frac{\partial_-^2\xi}{2}\right), \tag{3.10}$$

$$\chi = \frac{e^{2\omega}}{4\pi G}\left(-\frac{\ell_+}{\xi} + \xi\ell_- + \frac{3\,(\partial_+\xi)^2}{4\xi^3} - \frac{\partial_+^2\xi}{2\xi^2} + \frac{(\partial_-\xi)^2}{4\xi} - \frac{\partial_-^2\xi}{2} + \frac{\partial_+\xi\partial_-\xi}{\xi^2} - \frac{\partial_+\partial_-\xi}{\xi}\right). \tag{3.11}$$

Gathering these data together with (2.57) inside (3.1) provides, in the gauge at hand, the general class of locally AdS three-dimensional spacetime with curved conformal boundary. The conformal factor $\exp 2\omega$ can be apparently reabsorbed by setting r to $r\exp\omega$, thus bringing (3.1) to its flat-boundary form.[19] One should nevertheless be careful when making claims based on coordinate redefinitions, even in seemingly safe situations, because they can potentially alter global properties. Indeed, as discussed in Ref. [37], ω is expected to bring different asymptotics and new charges, and the corresponding solutions might generalize *Bañados'* family. In our subsequent analysis of Sec. 4.1, we will set $\omega = 0$. As we will shortly see, the arbitrary function $\xi(x^+, x^-)$ is also insidious regarding the charges, and focusing on it will be sufficient for the scope of this work.

We could proceed and display similar expressions in the Randers–Papapetrou boundary frame, describing the general locally anti-de Sitter spacetimes in terms of the three geometric data $\Omega(t,x)$, $b_x(t,x)$ and $a_{xx} = a(t,x)$, and whatever integration functions would appear in the process of solving the hydrodynamic equations (3.7). Usually, this resolution cannot be conducted explicitly as it happens in light-cone coordinates, and we end up with an implicit description of the bulk metric. We should quote here that a specific example of curved boundary[20] was investigated in Ref. [38], outside of the fluid/gravity framework, and the output agrees with our general results. We should also stress, following the discussion of

[19] This should be contrasted with the more intricate situation regarding this conformal factor inside the analogous formula in Fefferman–Graham gauge, Eq. (2.21) of Ref. [20].

[20] In that case $\Omega = \exp 2\beta$, $b_x = 0$, $a = 1$ and, in our language, the fluid velocity would have been $\mathrm{u} = -k^2 e^{2\beta} \mathrm{d}t$, *i.e.* comoving.

footnote 18, that the Randers–Papapetrou boundary frame produces in (3.1) order-r $\mathrm{d}t\mathrm{d}x$ components absent in the BMS gauge.

3.2 Ricci-flat

Our starting point is the finite derivative expansion of an asymptotically AdS$_3$ spacetime, Eq. (3.1). The fundamental question is whether the latter admits a smooth zero-k limit.

We have implicitly assumed that the Randers–Papapetrou data of the two-dimensional pseudo-Riemannian conformal boundary $\mathscr{I}$ associated with the original Einstein spacetime, a, b and Ω, remain unaltered at vanishing k, providing therefore directly the Carrollian data for the new spatial one-dimensional boundary $\mathscr{S}$ emerging at $\mathscr{I}^+$. Following again the detailed analysis performed in [19], we can match the various two-dimensional Riemannian quantities with the corresponding one-dimensional Carrollian ones:

$$\mathrm{u} = -k^2 \left(\Omega \mathrm{d}t - (b_x + \beta_x) \mathrm{d}x\right) + \mathrm{O}\left(k^4\right), \quad *\mathrm{u} = k\sqrt{a}\,\mathrm{d}x + \mathrm{O}\left(k^3\right) \tag{3.12}$$

and

$$\begin{aligned} \Theta &= \theta + \mathrm{O}\left(k^2\right), \\ \mathrm{a} &= k^2 \left(\varphi_x + \gamma_x\right) \mathrm{d}x + \mathrm{O}\left(k^4\right), \\ \mathrm{A} &= \theta\Omega \mathrm{d}t + \left(\alpha_x + \delta_x\right) \mathrm{d}x + \mathrm{O}\left(k^2\right), \end{aligned} \tag{3.13}$$

where the left-hand-side quantities are Riemannian, and the right-hand-side ones Carrollian (see (2.82), (2.83), (2.85), (2.92), (2.93)).

The closed form (3.1) is smooth at zero k. In this limit the metric reads:

$$\boxed{\begin{aligned} \mathrm{d}s^2_{\text{flat}} = &-2\left(\Omega \mathrm{d}t - \boldsymbol{b} - \boldsymbol{\beta}\right)\left(\mathrm{d}r + r\left(\boldsymbol{\varphi} + \boldsymbol{\gamma} + \theta\left(\Omega \mathrm{d}t - \boldsymbol{b} - \boldsymbol{\beta}\right)\right)\right) \\ &+ r^2 \mathrm{d}\ell^2 + 8\pi G\left(\Omega \mathrm{d}t - \boldsymbol{b} - \boldsymbol{\beta}\right)\left(\varepsilon\left(\Omega \mathrm{d}t - \boldsymbol{b} - \boldsymbol{\beta}\right) - \boldsymbol{\pi}\right), \end{aligned}} \tag{3.14}$$

Here $\mathrm{d}\ell^2$, Ω, $\boldsymbol{b} = b_x \mathrm{d}x$, $\boldsymbol{\varphi} = \varphi_x \mathrm{d}x$ and θ are the Carrollian geometric objects introduced earlier. The bulk Ricci-flat spacetime is now dual to a Carrollian fluid with kinematics captured in $\boldsymbol{\beta} = \beta_x \mathrm{d}x$ and $\boldsymbol{\gamma} = \gamma_x \mathrm{d}x$, energy density ε (zero-k limit of the corresponding relativistic function), and heat current $\boldsymbol{\pi} = \pi_x \mathrm{d}x$ (obtained in Eqs.(2.99), (2.100) and (2.101)).

For the fluid under consideration, there is also a pair of Carrollian stress tensors originating from the anomaly (3.2). Using expressions (2.98) and (2.103), we can determine τ_Σ and τ_Ξ, and Eqs. (2.104) provide in turn the Carrollian stress:

$$\Sigma^x{}_x = -\frac{1}{4\pi G}\left(\theta^2 + \frac{\partial_t \theta}{\Omega}\right), \quad \Xi^x{}_x = \frac{1}{4\pi G}\left(\left(\hat{\nabla}_x + \varphi_x\right)\varphi^x - \boldsymbol{\beta}^2\left(\theta^2 + \frac{\partial_t \theta}{\Omega}\right)\right). \tag{3.15}$$

This is the advertised Carrollian emanation of the relativistic conformal anomaly.

Expression (3.14) will grant by construction an exact Ricci-flat spacetime provided the conditions under which (3.1) was Einstein are fulfilled in the zero-k limit. These are the set of Carrollian hydrodynamic equations (2.106), (2.107), (2.108) and (2.109), with Carrollian power and force densities e, f, g_x, h_x obtained using their definition (2.105) and the expressions of f_μ displayed in (3.6) (we use for this computation the expression of the scalar curvature (2.98), and s_x as given in (2.95)). Equations (2.107) and (2.109) are automatically satisfied, whereas (2.106) and (2.108) lead to[21]

$$\begin{cases}\dfrac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon+\dfrac{1}{4\pi G}\left(\dfrac{2s_x}{\Omega}\hat{\mathscr{D}}_t\beta^x+\dfrac{\beta_x}{\Omega}\hat{\mathscr{D}}_t s^x+\hat{\mathscr{D}}^x s_x\right)=0,\\ \hat{\mathscr{D}}_x\varepsilon-\dfrac{\beta_x}{\Omega}\hat{\mathscr{D}}_t\varepsilon+\dfrac{1}{\Omega}\hat{\mathscr{D}}_t\left(\pi_x+2\varepsilon\beta_x\right)=0.\end{cases} \tag{3.16}$$

The unknown functions, which bear the fluid configuration, are $\varepsilon(t,x)$, $\pi_x(t,x)$ and $\beta_x(t,x)$. These cannot be all determined by the two equations at hand. Hence, there is some redundancy, originating from the relativistic fluid frame invariance – responsible *e.g.* for the arbitrariness of $\xi(x^+,x^-)$ in the description of AdS spacetimes using the light-cone boundary frame. More will be said about this in Sec. 4.2.

Equations (3.16) are Carroll–Weyl covariant. The Ricci-flat line element (3.14) inherits Weyl invariance from its relativistic ancestor. The set of transformations (2.81), (2.84) and (2.86), supplemented with $\varepsilon\to\mathcal{B}^2\varepsilon$ and $\pi_x\to\mathcal{B}\pi_x$, can indeed be absorbed by setting $r\to\mathcal{B}r$, resulting thus in the invariance of (3.14). In the relativistic case this invariance was due to the AdS conformal boundary. In the case at hand, this is rooted to the location of the one-dimensional spatial boundary $\mathscr{S}$ at null infinity $\mathscr{I}^+$.

We would like to close this chapter with a specific but general enough situation to encompass all Barnich–Troessaert Ricci-flat three-dimensional spacetimes. The Carrollian geometric data are $b_x=0$, $\Omega=1$ and $a=\exp 2\Phi(t,x)$, and the kinematic variable of the Carrollian dual fluid β_x is left free. Hence (3.14) reads:

$$\begin{aligned}\mathrm{d}s^2_{\text{flat}} &= -2\left(\mathrm{d}t-\beta_x\mathrm{d}x\right)\left(\mathrm{d}r+r\left(\partial_t\Phi\mathrm{d}t+\left(\partial_t-\partial_t\Phi\right)\beta_x\mathrm{d}x\right)\right)\\ &\quad +r^2\mathrm{e}^{2\Phi}\mathrm{d}x^2+8\pi G\left(\mathrm{d}t-\beta_x\mathrm{d}x\right)\left(\varepsilon\mathrm{d}t-\left(\pi_x+\varepsilon\beta_x\right)\mathrm{d}x\right),\end{aligned} \tag{3.17}$$

where $\varepsilon(t,x)$ and $\pi_x(t,x)$ obey Eqs. (3.16) in the form

$$\begin{cases}\left(\partial_t+2\partial_t\Phi\right)\varepsilon+\dfrac{1}{4\pi G}\left(2s_x\left(\partial_t+\partial_t\Phi\right)\beta^x+\beta_x\left(\partial_t+3\partial_t\Phi\right)s^x+\left(\partial_x+\partial_x\Phi\right)s^x\right)=0,\\ \partial_x\varepsilon+\left(\partial_t+\partial_t\Phi\right)\pi_x+2\varepsilon\partial_t\beta_x+\beta_x\partial_t\varepsilon=0.\end{cases} \tag{3.18}$$

[21] We remind that Weyl–Carroll covariant derivatives are defined in Eqs. (2.87), (2.88), (2.89) and (2.90). Here ε, β^x, π_x and s^x have weights 2, 1, 1 and 3. For example $\hat{\mathscr{D}}_x s^x=\hat{\nabla}_x s^x+2\varphi_x s^x=\frac{1}{\sqrt{a}}\hat{\partial}_x(\sqrt{a}s^x)+2\varphi_x s^x$.

Here, s_x takes the simple form

$$s_x = \partial_t^2 \beta_x - \partial_t \left(\beta_x \partial_t \Phi\right) - \partial_t \partial_x \Phi. \tag{3.19}$$

For vanishing β_x, the results (3.17) and (3.18) coincide precisely with those obtained in [20] by demanding Ricci-flatness in the BMS gauge. Here, they are reached from purely Carrollian-fluid considerations, and for generic $\beta_x(t,x)$, the metric (3.17) lays outside the BMS gauge.

4 Two-dimensional flat boundary and conserved charges

We will now restrict the previous analysis to Ricci-flat and Weyl-flat boundaries, both in AdS and Ricci-flat spacetimes. This enables us to compute the conserved charges following [22–24], and analyze the role of the velocity and the heat current of the boundary fluid.

4.1 Charges in AdS spacetimes

The flatness requirements are equivalent to setting $R = 0$ and $F = 0$. In the light-cone frame (2.51), this amounts to (see (2.57) and (2.58))

$$\omega = 0 \quad \text{and} \quad \xi(x^+, x^-) = -\frac{\xi^-(x^-)}{\xi^+(x^+)}, \tag{4.1}$$

where the minus sign is conventional.

Using the general solutions (3.10) and (3.11) in the bulk expression (3.1), and trading the chiral functions $\ell_\pm$ for $L_\pm$ defined as (the prime stands for the derivative with respect to the unique argument of the function)

$$\ell_\pm = \frac{1}{(\xi^\pm)^2} \left(L_\pm - \frac{(\xi^{\pm\prime})^2 - 2\xi^\pm \xi^{\pm\prime\prime}}{4} \right), \tag{4.2}$$

we obtain the following metric:

$$\begin{aligned}
\mathrm{d}s^2_{\text{Einstein}} \quad = \quad & -\frac{1}{k}\left(\sqrt{-\frac{\xi^-}{\xi^+}}\,\mathrm{d}x^+ - \sqrt{-\frac{\xi^+}{\xi^-}}\,\mathrm{d}x^-\right)\mathrm{d}r \\
& + \left(\frac{L_+}{k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\,\xi^{+\prime}\right)\left(\frac{\mathrm{d}x^+}{\xi^+}\right)^2 + \left(\frac{L_-}{k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\,\xi^{-\prime}\right)\left(\frac{\mathrm{d}x^-}{\xi^-}\right)^2 \\
& + \left(r^2 + \frac{r}{2k}\frac{1}{\sqrt{-\xi^+\xi^-}}\left(\xi^{+\prime} + \xi^{-\prime}\right) + \frac{L_+ + L_-}{k^2\xi^+\xi^-}\right)\mathrm{d}x^+\mathrm{d}x^-.
\end{aligned} \tag{4.3}$$

This metric depends on four arbitrary functions: $\xi^+(x^+)$ and $\xi^-(x^-)$ carrying information about the holographic fluid velocity (see (2.54)), and $L_+(x^+)$, $L_-(x^-)$, which together with $\xi^+(x^+)$ and $\xi^-(x^-)$ shape the energy–momentum tensor – here traceless due to the boundary flatness. Indeed we have

$$\varepsilon = -\frac{1}{4\pi G}\frac{L_+ + L_-}{\xi^+\xi^-}, \quad \chi = \frac{1}{4\pi G}\frac{L_+ - L_-}{\xi^+\xi^-}, \tag{4.4}$$

and in turn

$$T_{\pm\pm} = \frac{L_\pm}{4\pi G(\xi^\pm)^2}. \tag{4.5}$$

In three dimensions, any Einstein spacetime is locally anti-de Sitter. Hence, there exists always a coordinate transformation that can be used to bring it into a canonical AdS_3 form (say, in Poincaré coordinates). This is a large gauge transformation whenever the original Einstein spacetime has non-trivial conserved charges. The determination of the latter is therefore crucial for a faithful identification of the solution under consideration. It allows to evaluate the precise role played by the above arbitrary functions.

The charge computation requires a complete family of asymptotic Killing vectors. Those are determined according to the gauge, *i.e.* to the fall-off behaviour at large-r. The family (4.3) does not fit BMS gauge, unless $\xi^\pm$ are constant. This is equivalent to saying that the fluid has a uniform velocity, and can therefore be set at rest by an innocuous global Lorentz boost tuning $\xi^+ = 1$ and $\xi^- = -1$.[22] We will first focus on this case, where the asymptotic Killing vectors are known, and move next to the other extreme, demanding the fluid be perfect, *i.e.* in Landau–Lifshitz hydrodynamic frame. In the latter instance we will have to determine this family of vectors beforehand, as the gauge will no longer be BMS. Investigating the general situation captured by (4.3) is not relevant for our argument, which is meant to show that fluid/gravity holographic reconstruction is hydrodynamic-frame dependent.

As we will see, the charges computed following [22–24], and displayed in Eqs. (4.16) and (4.29), coincide in both cases with the modes of the energy–momentum tensor (4.5). However, they obey a different algebra due to the distinct asymptotic behaviour of the associated metric families.

Dissipative static fluid As anticipated, this class of solutions is reached by demanding $\xi^\pm = \pm 1$, while keeping $L^\pm$ arbitrary. We obtain

$$\mathrm{d}s^2_{\text{Einstein}} = -\frac{1}{k}\left(\mathrm{d}x^+ - \mathrm{d}x^-\right)\mathrm{d}r + r^2\mathrm{d}x^+\mathrm{d}x^- + \frac{1}{k^2}\left(L_+\mathrm{d}x^+ - L_-\mathrm{d}x^-\right)\left(\mathrm{d}x^+ - \mathrm{d}x^-\right), \tag{4.6}$$

which is the canonical expression of Bañados solutions in BMS gauge. Following (4.4), the boundary fluid energy and heat densities are $\varepsilon = {}^1\!/_{4\pi G}\left(L_+ + L_-\right)$ and $\chi = -{}^1\!/_{4\pi G}\left(L_+ - L_-\right)$.

[22] Observe that one may reabsorb ξ^+ and ξ^- by redefining $\mathrm{d}x^\pm \to \xi^\pm \mathrm{d}x^\pm$ and $r \to r/\sqrt{-\xi^+\xi^-}$ inside (4.3). This does not prove, however, that $\xi^\pm$ play no role, and this is why we treat them separately.

Therefore the heat current is not vanishing, and in the present hydrodynamic frame the fluid is at rest and dissipative.

The class of metrics (4.6) are form-invariant under

$$\zeta = \zeta^r \partial_r + \zeta^+ \partial_+ + \zeta^- \partial_- \tag{4.7}$$

with

$$\begin{aligned}\zeta^r &= -\frac{r}{2}\left(Y^{+\prime} + Y^{-\prime}\right) + \frac{1}{2k}\left(Y^{+\prime\prime} - Y^{-\prime\prime}\right) \\ &\quad -\frac{1}{2k^2 r}\left(L_+ - L_-\right)\left(Y^{+\prime} - Y^{-\prime}\right), &(4.8)\\ \zeta^\pm &= Y^\pm - \frac{1}{2kr}\left(Y^{+\prime} - Y^{-\prime}\right), &(4.9)\end{aligned}$$

for arbitrary chiral functions $Y^+(x^+)$ and $Y^-(x^-)$. These vector fields generate diffeomorphisms, which alter the functions appearing in (4.6) according to

$$-\mathscr{L}_\zeta g_{MN} = \delta_\zeta g_{MN} = \frac{\partial g_{MN}}{\partial L_+}\delta_\zeta L_+ + \frac{\partial g_{MN}}{\partial L_-}\delta_\zeta L_- \tag{4.10}$$

with

$$\delta_\zeta L_\pm = -Y^\pm L_\pm' - 2L_\pm Y^{\pm\prime} + \frac{1}{2}Y^{\pm\prime\prime\prime}. \tag{4.11}$$

The last term in this expression is responsible for the emergence of a central charge in the surface-charge algebra. These vectors obey an algebra for the modified Lie bracket (see e.g. [20]):

$$\zeta_3 = [\zeta_1, \zeta_2]_{\mathrm{M}} = [\zeta_1, \zeta_2] - \delta_{\zeta_2}\zeta_1 + \delta_{\zeta_1}\zeta_2 \tag{4.12}$$

with[23] $\zeta_a = \zeta\left(Y_a^+, Y_a^-\right)$ and

$$Y_3^\pm = Y_1^\pm Y_2^{\pm\prime} - Y_2^\pm Y_1^{\pm\prime}. \tag{4.13}$$

The surface charges are computed for an arbitrary metric g of the type (4.6) with global AdS_3 as reference background. The latter has metric $\bar{g}$ with $L_+ = L_- = {}^{-1}\!/_4$ *i.e.* $\varepsilon = {}^{-1}\!/_{8\pi G}$ and $\chi = 0$. The final integral is performed over the compact spatial boundary coordinate $x \in [0, 2\pi]$:

$$Q_Y[g - \bar{g}, \bar{g}] = \frac{1}{8\pi k G}\int_0^{2\pi} \mathrm{d}x \left(Y^+\left(L_+ + \frac{1}{4}\right) - Y^-\left(L_- + \frac{1}{4}\right)\right). \tag{4.14}$$

These charges are in agreement with the quoted literature,[24] and their algebra is determined

[23] Here $\delta_{\zeta_2}\zeta_1$ stands for the variation produced on ζ_1 by ζ_2, and this is not vanishing because ζ_1 depends explicitly on $L_\pm$: $\delta_{\zeta_2}\zeta_1 = \left(\frac{\partial \zeta_1^N}{\partial L_+}\delta_{\zeta_2}L_+ + \frac{\partial \zeta_1^N}{\partial L_-}\delta_{\zeta_2}L_-\right)\partial_N$.

[24] Some relative-sign differences are due to different conventions used for the light-cone coordinates, here defined as $x^\pm = x \pm kt$.

as usual:

$$\{Q_{Y_1}, Q_{Y_2}\} = \delta_{\zeta_1} Q_{Y_2} = -\delta_{\zeta_2} Q_{Y_1}. \tag{4.15}$$

Introducing the modes

$$L^{\pm}_m = \frac{1}{8\pi k G} \int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}mx^{\pm}} \left(L_{\pm} + \frac{1}{4} \right) \tag{4.16}$$

the algebra reads:

$$\mathrm{i}\left\{L^{\pm}_m, L^{\pm}_n\right\} = (m-n) L^{\pm}_{m+n} + \frac{c}{12} m \left(m^2 - 1\right) \delta_{m+n,0}, \quad \left\{L^{\pm}_m, L^{\mp}_n\right\} = 0. \tag{4.17}$$

This double realization of Virasoro algebra with Brown–Henneaux central charge $c = {}^{3}/{}_{2kG}$ was expected for Bañados solutions (4.6).

Perfect fluid with arbitrary velocity In Landau–Lifshitz frame the heat current vanishes ($\chi = 0$) and the boundary conformal fluid is perfect. Equation (4.4) requires for this

$$L_+ = L_- = \frac{M}{2}, \tag{4.18}$$

with M constant, while it gives for energy density $\varepsilon = -{}^{M}/{}_{4\pi G \xi^+ \xi^-}$. As for the general case, the reconstructed bulk family of metrics

$$\begin{aligned} \mathrm{d}s^2_{\text{Einstein}} \;=\; & -\frac{1}{k}\left(\sqrt{-\frac{\xi^-}{\xi^+}}\,\mathrm{d}x^+ - \sqrt{-\frac{\xi^+}{\xi^-}}\,\mathrm{d}x^-\right)\mathrm{d}r \\ & + \left(\frac{M}{2k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\,\xi^{+\prime}\right)\left(\frac{\mathrm{d}x^+}{\xi^+}\right)^2 + \left(\frac{M}{2k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\,\xi^{-\prime}\right)\left(\frac{\mathrm{d}x^-}{\xi^-}\right)^2 \\ & + \left(r^2 + \frac{r}{2k}\frac{1}{\sqrt{-\xi^+\xi^-}}\left(\xi^{+\prime} + \xi^{-\prime}\right) + \frac{M}{k^2\xi^+\xi^-}\right)\mathrm{d}x^+\mathrm{d}x^- \end{aligned} \tag{4.19}$$

is not in BMS gauge, unless $\xi^{\pm}$ are constant. Again this latter subset is entirely captured by $\xi^{\pm} = \pm 1$, and the resulting solution is BTZ together with all non-spinning zero-modes of Bañados family [39–41]:

$$\mathrm{d}s^2_{\text{Einstein}} = -\frac{1}{k}\left(\mathrm{d}x^+ - \mathrm{d}x^-\right)\mathrm{d}r + r^2 \mathrm{d}x^+\mathrm{d}x^- + \frac{M}{2k^2}\left(\mathrm{d}x^+ - \mathrm{d}x^-\right)^2. \tag{4.20}$$

The asymptotic structure rising in (4.19) is now respected by the following family of asymptotic Killing vectors

$$\eta = \eta^r \partial_r + \eta^+ \partial_+ + \eta^- \partial_-, \tag{4.21}$$

expressed in terms of two arbitrary chiral functions $\epsilon^{\pm}(x^{\pm})$

$$\eta^r = -\frac{r}{2}\left(\epsilon^{+\prime} + \epsilon^{-\prime}\right), \quad \eta^{\pm} = \epsilon^{\pm}. \tag{4.22}$$

These vectors, slightly different from those found for the dissipative boundary fluids (4.7), (4.8), (4.9), appear as the result of an exhaustive analysis of (4.19). They do not support subleading terms, and since they do not depend on the the functions $\xi^{\pm}$, they form an algebra for the Lie bracket:

$$[\eta_1, \eta_2] = \eta_3 \tag{4.23}$$

with $\eta_a = \eta\left(\epsilon_a^+, \epsilon_a^-\right)$ and

$$\epsilon_3^{\pm} = \epsilon_1^{\pm}\epsilon_2^{\pm\prime} - \epsilon_2^{\pm}\epsilon_1^{\pm\prime}. \tag{4.24}$$

They induce the exact transformation

$$-\mathscr{L}_\eta g_{MN} = \delta_\eta g_{MN} = \frac{\partial g_{MN}}{\partial \xi^+}\delta_\eta \xi^+ + \frac{\partial g_{MN}}{\partial \xi^{+\prime}}\delta_\eta \xi^{+\prime} + \frac{\partial g_{MN}}{\partial \xi^-}\delta_\eta \xi^- + \frac{\partial g_{MN}}{\partial \xi^{-\prime}}\delta_\eta \xi^{-\prime} \tag{4.25}$$

with

$$\delta_\eta \xi^{\pm} = \xi^{\pm}\epsilon^{\pm\prime} - \epsilon^{\pm}\xi^{\pm\prime}. \tag{4.26}$$

Following the customary pattern, we can determine the conserved charges, with global AdS_3 as reference background, now reached with $\xi^{\pm} = \pm 1$ and $M = -1/2$ (again $\varepsilon = -1/8\pi G$ and $\chi = 0$):

$$Q_\epsilon\left[g - \bar{g}, \bar{g}\right] = \frac{1}{16\pi k G}\int_0^{2\pi} \mathrm{d}x \left(\epsilon^+\left(\frac{1}{\xi^{+2}} - 1\right) - \epsilon^-\left(\frac{1}{\xi^{-2}} - 1\right)\right), \tag{4.27}$$

as well as their algebra:

$$\left\{Q_{\epsilon_1}, Q_{\epsilon_2}\right\} = \delta_{\eta_1} Q_{\epsilon_2} = -\delta_{\eta_2} Q_{\epsilon_1}. \tag{4.28}$$

Defining now

$$Z_m^{\pm} = \frac{1}{16\pi k G}\int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}mx^{\pm}}\left(\frac{1}{\xi^{\pm 2}} - 1\right) \tag{4.29}$$

we find

$$\mathrm{i}\left\{Z_m^{\pm}, Z_n^{\pm}\right\} = (m-n) Z_{m+n}^{\pm} + \frac{m}{4kG}\delta_{m+n,0}, \quad \left\{Z_m^{\pm}, Z_n^{\mp}\right\} = 0. \tag{4.30}$$

The central extension of this algebra is trivial. Indeed, it can be reabsorbed in the following redefinition of the modes $Z_m^{\pm}$

$$\tilde{Z}_m^{\pm} = Z_m^{\pm} + \frac{1}{8kG}\delta_{m,0}. \tag{4.31}$$

Therefore, (4.30) becomes

$$\mathrm{i}\left\{\tilde{Z}_m^{\pm}, \tilde{Z}_n^{\pm}\right\} = (m-n)\tilde{Z}_{m+n}^{\pm}, \quad \left\{\tilde{Z}_m^{\pm}, \tilde{Z}_n^{\mp}\right\} = 0. \tag{4.32}$$

The algebra at hand (4.32) is de Witt rather than Virasoro,[25] and this outcome demonstrates

[25] The absence of central charges occurs also in [37] for the same reason, *i.e.* a modification of the asymptotic behaviour.

the already advertised result: the family of locally anti-de Sitter spacetimes obtained holographically from two-dimensional fluids in the Landau–Lifshitz frame overlap only partially the space of Bañados solutions. This overlap encompasses the non-spinning BTZ and excess or defects geometries provided in (4.20).

4.2 Charges in Ricci-flat spacetimes

The absence of anomaly in the Carrollian framework is equivalent to setting $\Sigma^x{}_x = \Xi^x{}_x = 0$ (see (3.15)), whereas the Weyl–Carroll flatness requires $s = 0$ (see (2.96)). This amounts to taking $\Omega = a = 1$ and $b_x = 0$,[26] and with those data $s = 0$ reads

$$\partial_t^2 \beta_x = 0. \tag{4.33}$$

In the Carrollian spacetime at hand, the fluid equations of motion (3.16) are

$$\begin{cases} \partial_t \varepsilon = 0, \\ \partial_x \varepsilon + \partial_t (\pi_x + 2\varepsilon\beta_x) = 0. \end{cases} \tag{4.34}$$

Equations (4.33) and (4.34) can be integrated in terms of four arbitrary functions of x: $\varepsilon(x)$, $\varpi(x)$, $\lambda(x)$ and $\mu(x)$. We find

$$\beta_x(t,x) = \frac{\lambda(x)}{2\varepsilon(x)} - \frac{t}{2}\partial_x \ln \mu(x), \tag{4.35}$$

$$\pi_x(t,x) = -2\varepsilon(x)\beta_x(t,x) + \varpi(x) - t\partial_x\varepsilon(x) \tag{4.36}$$

(this parameterization of β_x will be appreciated later). The Ricci-flat (even locally flat) holographically reconstructed spacetime from these Carrollian fluid data is obtained from the general expression (3.14):

$$\begin{aligned} \mathrm{d}s^2_{\text{flat}} = &-2\left(\mathrm{d}t - \beta_x \mathrm{d}x\right)\left(\mathrm{d}r + r\partial_t\beta_x \mathrm{d}x\right) + r^2 \mathrm{d}x^2 \\ &+ 8\pi G\left(\varepsilon(\mathrm{d}t - \beta_x\mathrm{d}x)^2 - \pi_x \mathrm{d}x(\mathrm{d}t - \beta_x\mathrm{d}x)\right), \end{aligned} \tag{4.37}$$

where β_x and π_x are meant to be as in (4.35) and (4.36).

On the one hand, the arbitrary functions $\varepsilon(x)$ and $\varpi(x)$ are reminiscent of the functions $L_\pm(x^\pm)$ (or $\varepsilon(t,x)$ and $\chi(t,x)$) present in the AdS solutions. A vanishing-k limit was indeed used in Ref. [25] to obtain $\varepsilon(x)$ and $\varpi(x)$ from $L_\pm(x^\pm)$. On the other hand, $\lambda(x)$ and $\mu(x)$ remind $\xi^\pm(x^\pm)$, and are indeed a manifestation of a residual hydrodynamic frame invariance, which survives the Carrollian limit. Considering indeed the Carrollian hydrodynamic-frame

[26] Actually the absence of anomaly requires rather $\Omega = \Omega(t)$, $a = a(x)$ and $b_x = b_x(x)$, which can be reabsorbed trivially with Carrollian diffeomorphisms.

transformations (2.110)

$$\beta'_x = \beta_x + B_x, \tag{4.38}$$

in the present framework ($\Sigma^x{}_x = \Xi^x{}_x = 0$), and using Eqs. (2.73), (2.74), (2.75), (2.76), (2.99), (2.100), (2.101), we obtain the transformations:

$$\varepsilon' = \varepsilon, \quad \pi'_x = \pi_x - 2\varepsilon B_x, \tag{4.39}$$

which leave the Carrollian fluid equations (4.34) invariant. The new velocity field β'_x is compatible with the Weyl–Carroll flatness (4.33) provided the transformation function B_x is linear in time, hence parameterized in terms of two arbitrary functions of x. This is how $\lambda(x)$ and $\mu(x)$ emerge.

Observe also that the residual Carrollian hydrodynamic frame invariance enables us to define here a Carrollian Landau–Lifshitz hydrodynamic frame. Indeed, combining (4.36) and (4.35) we obtain

$$\pi_x(t,x) = -\lambda(x) + \varpi(x) + t\varepsilon(x)\partial_x \ln\frac{\mu(x)}{\varepsilon(x)}. \tag{4.40}$$

Adjusting the velocity field β_x such that

$$\varpi(x) = \lambda(x) \quad \text{and} \quad \frac{\varepsilon(x)}{\mu(x)} = \varepsilon_0 \tag{4.41}$$

with ε_0 a constant, makes the Carrollian fluid perfect: $\pi_x = 0$.

In complete analogy with the AdS analysis, we will first compute the charges for vanishing velocity $\beta_x = 0$ (which is given by $\lambda(x) = 0$ and $\mu(x) = 1$) in terms of $\varepsilon(x)$ and $\varpi(x)$, and next perform the similar computation for perfect fluids with velocity β_x parameterized with two arbitrary functions $\lambda(x)$ and $\mu(x)$. Here empty Minkowski bulk is realized with $\mu = 1$, $\lambda = 0$, $\varpi = 0$ and $\varepsilon_0 = -1/8\pi G$.

As for the AdS instance discussed in Sec. 4.1, the class (4.37) is not in the BMS gauge, unless β_x is constant, which can then be reabsorbed by a global Carrollian boost (constant B_x).[27] We will first discuss this situation, where the asymptotic Killings are the canonical generators of $\mathfrak{bms}_3$. Outside the BMS, we will determine the asymptotic isometry for metrics reconstructed from perfect fluids, and proceed with the surface charges and their algebra. Our conclusion is here that asymptotically flat fluid/gravity correspondence is sensitive to the residual hydrodynamic-frame invariance.

[27]The functions $\lambda(x)$ and $\mu(x)$ entering (4.37) via (4.36) and (4.35) can be reabsorbed in any case by performing the coordinate transformation $\mathrm{d}x \to \frac{\mathrm{d}x}{\sqrt{\mu(x)}}$, $\mathrm{d}t \to \frac{1}{\sqrt{\mu(x)}}(\mathrm{d}t + \beta_x \mathrm{d}x)$ and $r \to r\sqrt{\mu(x)}$. This leads to the same form as the one reached by setting $\mu = 1$ and $\lambda = 0$, *i.e* (4.42).

Dissipative static fluid The metric (4.37) for vanishing β_x takes the simple form (again the prime signals a derivative)

$$\mathrm{d}s^2_{\text{flat}} = -2\mathrm{d}t\mathrm{d}r + r^2\mathrm{d}x^2 + 8\pi G\left(\varepsilon \mathrm{d}t - \left(\varpi - t\varepsilon'\right)\mathrm{d}x\right)\mathrm{d}t, \tag{4.42}$$

compatible with BMS gauge with asymptotic Killing vectors

$$\zeta = \zeta^r\partial_r + \zeta^t\partial_t + \zeta^x\partial_x, \tag{4.43}$$

where

$$\zeta^r = -rY' + H'' + tY''' + \frac{4\pi G}{r}\left(\varpi - t\varepsilon'\right)\left(H' + tY''\right), \tag{4.44}$$
$$\zeta^t = H + tY', \tag{4.45}$$
$$\zeta^x = Y - \frac{1}{r}\left(H' + tY''\right). \tag{4.46}$$

Here H and Y are functions of x only. Vectors (4.44), (4.45), (4.46) are the vanishing-k limit of (4.7), (4.8), (4.9), reached by trading light-cone frame as $x^\pm = x \pm kt$, and setting $Y^\pm(x^\pm) = Y(x) \pm k(H(x) + tY'(x))$.

This family of vectors produces the following variation on the metric fields:

$$-\mathscr{L}_\zeta g_{MN} = \delta_\zeta g_{MN} = \frac{\partial g_{MN}}{\partial\varepsilon}\delta_\zeta\varepsilon + \frac{\partial g_{MN}}{\partial\varepsilon'}\delta_\zeta\varepsilon' + \frac{\partial g_{MN}}{\partial\varpi}\delta_\zeta\varpi, \tag{4.47}$$

with

$$\delta_\zeta\varepsilon = -2\varepsilon Y' - Y\varepsilon' + \frac{Y'''}{4\pi G}, \tag{4.48}$$
$$\delta_\zeta\varpi = -\frac{H'''}{4\pi G} + \frac{1}{H}\left(\varepsilon H^2\right)' - \frac{1}{Y}\left(\varpi Y^2\right)'. \tag{4.49}$$

Their algebra closes for the same modified Lie bracket (4.12) with $\zeta_a = \zeta\left(H_a, Y_a\right)$ and

$$Y_3 = Y_1Y_2' - Y_2Y_1' \quad H_3 = Y_1H_2' + H_1Y_2' - Y_2H_1' - H_2Y_1'. \tag{4.50}$$

We can compute the charges of g in (4.42), using Minkowski as reference background $\bar{g}$. They read:

$$Q_{H,Y}[g - \bar{g}, \bar{g}] = \frac{1}{2}\int_0^{2\pi}\mathrm{d}x\left[H\left(\varepsilon + \frac{1}{8\pi G}\right) - Y\varpi\right]. \tag{4.51}$$

With a basis of functions $\exp imx$ for H and Y, we find the standard collection of charges

$$P_m = \frac{1}{2}\int_0^{2\pi}\mathrm{d}x\,\mathrm{e}^{imx}\left(\varepsilon + \frac{1}{8\pi G}\right), \quad J_m = -\frac{1}{2}\int_0^{2\pi}\mathrm{d}x\,\mathrm{e}^{imx}\varpi, \tag{4.52}$$

which coincide with the computation performed *e.g.* in [25]. Using

$$\{Q_{H_1,Y_1}, Q_{H_2,Y_2}\} = \delta_{\zeta_1} Q_{H_2,Y_2} = -\delta_{\zeta_2} Q_{H_1,Y_1}, \tag{4.53}$$

we obtain the following surface-charge algebra:

$$\mathrm{i}\{J_m, P_n\} = (m-n)P_{m+n} + \frac{c}{12} m\left(m^2-1\right)\delta_{m+n,0}, \quad \mathrm{i}\{J_m, J_n\} = (m-n)J_{m+n}, \quad \{P_m, P_n\} = 0 \tag{4.54}$$

with $c = {}^{3}/_{G}$. This is the $\mathfrak{bms}_3$ algebra, and this analysis demonstrates that a non-perfect Carrollian fluid, even with $\beta_x = 0$, is sufficient for generating holographically all Barnich–Troessaert flat three-dimensional spacetimes. This goes along with the analogue conclusion reached in AdS for Bañados spacetimes.

Perfect fluid with velocity Consider now the resummed metric (4.37) assuming (4.41). We obtain

$$\mathrm{d}s^2_{\text{flat}} = -2\left(\mathrm{d}t - \beta_x \mathrm{d}x\right)\left(\mathrm{d}r - \frac{r\mu'}{2\mu}\mathrm{d}x\right) + r^2\mathrm{d}x^2 + 8\pi G\varepsilon_0\mu\left(\mathrm{d}t - \beta_x\mathrm{d}x\right)^2 \tag{4.55}$$

with β_x given by

$$\beta_x = \frac{1}{2\mu}\left(\frac{\lambda}{\varepsilon_0} - t\mu'\right). \tag{4.56}$$

Unless β_x is constant, the metrics (4.55) are not in BMS gauge. The BMS subset is entirely captured by $\mu = 1$, $\lambda = 0$ with resulting solutions plain Minkowski ($\varepsilon_0 = {}^{-1}/_{8\pi G}$) and the non-spinning zero-modes of Barnich–Troessaert family:

$$\mathrm{d}s^2_{\text{flat}} = -2\mathrm{d}t\mathrm{d}r + r^2\mathrm{d}x^2 + 8\pi G\varepsilon_0\mathrm{d}t^2. \tag{4.57}$$

The asymptotic isometries of (4.55) are now generated by[28]

$$\eta = \eta^r\partial_r + \eta^t\partial_t + \eta^x\partial_x, \tag{4.58}$$

expressed in terms of two arbitrary functions $h(x)$ and $\rho(x)$

$$\eta^r = -r\rho', \quad \eta^t = h + t\rho', \quad \eta^x = \rho. \tag{4.59}$$

The algebra of asymptotic Killing vectors closes for the ordinary Lie bracket

$$[\eta_1, \eta_2] = \eta_3 \tag{4.60}$$

[28] Again the fields (4.58), (4.59) are alternatively obtained by an appropriate zero-k limit of (4.21) and (4.22).

with $\eta_a = \eta\,(h_a, \rho_a)$ and

$$\rho_3 = \rho_1 \rho_2' - \rho_2 \rho_1', \quad h_3 = \rho_1 h_2' + h_1 \rho_2' - \rho_2 h_1' - h_2 \rho_1'. \tag{4.61}$$

It respects the form of the metric

$$-\mathscr{L}_\eta g_{MN} = \delta_\eta g_{MN} = \frac{\partial g_{MN}}{\partial \mu} \delta_\eta \mu + \frac{\partial g_{MN}}{\partial \mu'} \delta_\eta \mu' + \frac{\partial g_{MN}}{\partial \lambda} \delta_\eta \lambda \tag{4.62}$$

with

$$\delta_\eta \lambda = -2\lambda\rho' - \rho\lambda' + \varepsilon_0 \left(2\mu h' + h\mu'\right), \tag{4.63}$$

$$\delta_\eta \mu = -2\mu\rho' - \rho\mu'. \tag{4.64}$$

The charges of g in (4.55) are computed as usual with Minkowski as reference background $\bar{g}$. They read:

$$Q_{h,\rho}[g - \bar{g}, \bar{g}] = \frac{1}{2} \int_0^{2\pi} \mathrm{d}x \left[h \left(\varepsilon_0 \mu + \frac{1}{8\pi G} \right) - \rho\lambda \right]. \tag{4.65}$$

With a basis of unimodular exponentials for h and ρ, we find now

$$M_m = \frac{1}{2} \int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}mx} \left(\varepsilon_0 \mu + \frac{1}{8\pi G} \right), \quad I_m = -\frac{1}{2} \int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}mx} \lambda, \tag{4.66}$$

and

$$\left\{ Q_{h_1,\rho_1}, Q_{h_2,\rho_2} \right\} = \delta_{\eta_1} Q_{h_2,\rho_2} = -\delta_{\eta_2} Q_{h_1,\rho_1} \tag{4.67}$$

provide the surface-charge algebra:

$$\mathrm{i}\left\{ I_m, M_n \right\} = (m-n) M_{m+n} - \frac{m}{4G} \delta_{m+n,0}, \quad \mathrm{i}\left\{ I_m, I_n \right\} = (m-n) I_{m+n}, \quad \left\{ M_m, M_n \right\} = 0. \tag{4.68}$$

As for the anti-de Sitter case, the central extension of this algebra is trivial. By translating the modes

$$\tilde{M}_m = M_m - \frac{1}{8G} \delta_{m,0}, \tag{4.69}$$

we obtain

$$\mathrm{i}\left\{ I_m, \tilde{M}_n \right\} = (m-n) \tilde{M}_{m+n}, \quad \mathrm{i}\left\{ I_m, I_n \right\} = (m-n) I_{m+n}, \quad \left\{ \tilde{M}_m, \tilde{M}_n \right\} = 0. \tag{4.70}$$

This algebra, which could have been obtained from (4.32) in the zero-k limit, has no central charge. Therefore, our computation shows unquestionably that holographic locally flat spacetimes based on perfect Carrollian fluids – fluids in Carrollian Landau–Lifshitz frame – cover only in some measure the family on Barnich–Troessaert solutions. Among those one finds (4.57).

5 Conclusion

We can now summarize our achievements. The motivations of the present work have been twofold: (i) reconstruct asymptotically anti-de Sitter and flat three-dimensional spacetimes using fluid/gravity holographic correspondence in a unified framework; (ii) investigate the emergence of hydrodynamic-frame invariance and its potential holographic breakdown.

Solutions to three-dimensional vacuum Einstein's equations have been searched systematically since the seminal work of BTZ, and their asymptotic symmetries as well as the corresponding conserved charges are thoroughly understood. In parallel, many aspects of their boundary properties in the anti-de Sitter case were discussed before the advent of the holographic correspondence, and lately for the flat case in relation with the BMS asymptotic symmetries. However, setting up a precise correspondence between a general two-dimensional relativistic fluid defined on an arbitrary background and a three-dimensional anti-de Sitter spacetime was only superficially analyzed, whereas the possible relationship among flat spacetimes and Carrollian fluid dynamics had never been considered. This has been the core of our inquiry.

Because relativistic fluid dynamics in two spacetime dimensions is rather simple, it allows to perform an exhaustive and exact study of the equations of motion, and of their form invariance under hydrodynamic-frame transformations – local Lorentz boosts. We have assumed for commodity a conformal equation of state, keeping the fluid non-conformal though (*i.e.* with non-zero viscous bulk pressure). Hence, the relativistic fluid is described by an arbitrary velocity field, the energy and heat densities, and the viscous pressure, all transforming appropriately under local Lorentz boosts so as to keep the energy–momentum tensor invariant. The extreme situation corresponds to the Landau–Lifshitz frame, where the heat current vanishes and the energy–momentum tensor is diagonal.

Three-dimensional Einstein spacetime reconstruction is then achieved with the derivative expansion, following the usual pattern of higher dimensions. Here it is not an expansion but a finite sum, involving all boundary data. Holographic fluids have an anomalous viscous pressure proportional to the curvature of the host geometry. Owing to this fact, the holographic fluid does not move freely, but is subject to a force, entirely determined by its kinematical configuration and by the geometry. Using light-cone coordinates and conformally flat boundary makes it easy to obtain the general fluid configuration, and a general and closed expression for locally anti-de Sitter spacetimes, in a gauge which is less stringent than BMS.

With this general result, it is possible to address the question of whether a boundary fluid configuration observed from different hydrodynamic frames gives rise to distinct bulk geometries. This is discussed in the simpler (but sufficient for the argument) case of flat boundaries with vanishing Weyl curvature, for which the fluid is conformal (no trace). The reconstructed bulk geometries are then described in terms of two pairs of chiral functions, $\xi^{\pm}$ and $L_{\pm}$. The former parameterize the velocity of the fluid, while the latter its energy and heat densities. With these data two extreme configurations emerge: (i) a fluid at rest with heat current; (ii) a fluid with arbitrary velocity and vanishing heat current (hence perfect since the viscous pressure is also zero) *i.e.* in the Landau–Lifshitz frame. For both cases one determines the bulk asymptotic Killing vectors together with the algebra of conserved surface charges. In the first instance, the left and right Virasoro algebras appear with their canonical central charges. In the second, the central charges can be reabsorbed by a redefinition of the elementary modes, demonstrating thereby that the bulk-metric derivative expansion is sensitive to the boundary-fluid hydrodynamic frame. In particular, the Landau–Lifshitz frame fails to reproduce faithfully all Bañados' solutions, contrary to the common expectation.

The above pattern has been resumed for the Ricci-flat spacetimes. The conformal boundary is now at null infinity, and is endowed with a Carrollian $1+1$-dimensional structure. Boundary dynamics is carried by a Carrollian fluid, obeying a set of hydrodynamic equations for energy and heat densities, two viscous stress scalars as well as a kinematic variable referred to as "inverse-velocity". Generically, these equations do not exhibit any sort of hydrodynamic-frame invariance.

The reconstruction of three-dimensional Ricci-flat spacetimes is achieved by considering the vanishing-k limit of the anti-de Sitter derivative expansion, which is finite. Information is supplied in this Ricci-flat derivative expansion by the Carrollian fluid defined at null infinity. In particular, the original conformal anomaly is carefully identified as a source of Carrollian stress.

As for Einstein spacetimes, we do not consider the most general situation, but impose equivalent restrictions: absence of anomaly and zero Weyl–Carroll curvature. The derivative-expansion gauge is slightly less restrained than BMS, and a residual hydrodynamic-frame-like invariance emerges, which allows to treat the same Carrollian dynamics from two equivalent perspectives: (i) a Carrollian fluid with vanishing inverse velocity and non-zero heat current; (ii) a Carrollian fluid with inverse velocity and vanishing heat current (*i.e.* a sort of Carrollian Landau–Lifshitz frame). Although equivalent from the Carrollian-fluid perspective, these two patterns lead to Ricci-flat spacetimes with different surface charge algebras. The former family fits in BMS gauge and reproduces all Barnich–Troessaert spacetimes with the appropriate charges. The algebra is $\mathfrak{bms}_3$ with central charge. The set of Ricci-flat metrics obtained with a Carrollian perfect fluids exhibit an algebra whose central charge can be ultimately reabsorbed.

The above is the bottom line of our work. Our findings raise several questions that we briefly sort in the following as possible physical applications, in three dimensions or beyond, and on either side of the fluid/gravity holographic correspondence.

At the first place, it is legitimate to ask where the origin of the hydrodynamic-frame invariance breaking stands. We have implicitly or explicitly stated in our presentation that the responsible agent was fluid/gravity duality. This view is supported by the explicit expressions of surface charges (Eqs. (4.16), (4.29), (4.52) and (4.66)), which appear as modes of the energy–momentum tensor for the relativistic fluid (or its Carrollian descendants), irrespective of the chosen velocity field. The breaking then occurs in the structure of the algebra, which is sensitive to the bulk-metric asymptotic behaviour, itself depending on the boundary-fluid velocity congruence. This reasoning is not a proof, and does not exclude that relativistic fluids might be, in their own right, globally sensitive to the locally arbitrary velocity field.[29] Furthermore, our discussion has been confined to three bulk dimensions, where the observed breaking is necessarily global, as opposed to local (in three dimensions asymptotically AdS or flat translates into locally AdS and Minkowskian). Nothing excludes *a priori* that in higher dimension, other obstructions of purely local nature emerge against the free choice of a relativistic congruence. The possible breakdown of the Landau–Lifshitz-frame paradigm has been quoted indeed for three-dimensional fluids in [42], in relation with the entropy current. No general concrete results are available at present though, and these questions remain relevant both for fluid dynamics and for the subject of fluid/gravity correspondence.

The second important issue concerns the systematic analysis of asymptotic Killing vectors and conserved charges for the fall-offs suggested by the derivative expansion. This question is valid in both anti-de Sitter (Eq. (3.1), or the further restricted versions presented in Sec. 3.1) and flat spacetime (Eq. (3.14) and other realizations in Sec. 3.2). In this respect, one should remind that the investigation of fall-off conditions generalizing Brown–Henneaux's was carried in Refs. [37, 43–45]. Finding solutions to Einstein's equations obeying these more general asymptotic behaviours, *i.e.* standing beyond Bañados or Barnich–Troessaert, persists, and is worth pursuing in our framework (see the comment after Eq. (3.11) and Ref. [46]). In parallel, the Ricci-flat case calls for a deeper Hamiltonian understanding of the charges within the appropriate intrinsic Carrollian setup recently developed in [47].

This latter comment opens Pandora's box for Carrollian physics, *i.e.* physics in the ultra-relativistic regime, which is generally unexplored in a systematic fashion. Our study of Sec. 2.2, and Eqs. (2.106)–(2.109) in particular, exhibit the dynamics of two-dimensional ultra-relativistic fluids. It is remarkable that these physical systems are dual to Ricci-flat spacetimes. Equation (3.1) is instrumental in setting this duality: it starts from the ordinary relativistic regime and reaches the Carrollian limit, from the gravitational side, as a Ricci-flat limit. This formalism is expected to have genuine physical applications in many-body one-dimensional systems – and beyond one space dimension, as discussed in [11].

[29] Changing hydrodynamic frame is a gauge transformation. As such, it can affect global properties.

Last and aside from the interplay between gravity and fluids, a purely hydrodynamic issue was also discussed, which remains puzzling: the entropy current. No microscopic definition or closed expression exist and this object is usually constructed order-by-order in the derivative expansion, physically restricted to comply with fundamental laws. In relativistic systems, this current is expected to be hydrodynamic-frame invariant, by essence of this invariance. Hence, any obstruction to the existence of such a frame-invariant current might dispute or hamper the freedom of choosing at wish the fluid velocity field. In two dimensions, we have the possibility to implement frame invariance exactly and we proposed a closed expression, which however is not unique and deserves further investigation. One should understand whether and why this is the proper choice, and possibly wonder if it provides a helpful guideline for handling the entropy current in systems of dimension higher than two. Ultimately, in the spirit of considering its Carrollian limit, one should try to give a meaning to entropy in ultra-relativistic systems.

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Carrollian Physics at the Black Hole Horizon

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Abstract

We show that the geometry of a black hole horizon can be described as a Carrollian geometry emerging from an ultra-relativistic limit where the near-horizon radial coordinate plays the role of a virtual velocity of light tending to zero. We prove that the laws governing the dynamics of a black hole horizon, the null Raychaudhuri and Damour equations, are Carrollian conservation laws obtained by taking the ultra-relativistic limit of the conservation of an energy–momentum tensor; we also discuss their physical interpretation. We show that the vector fields preserving the Carrollian geometry of the horizon, dubbed Carrollian Killing vectors, include BMS-like supertranslations and superrotations and that they have non-trivial associated conserved charges on the horizon. In particular, we build a generalization of the angular momentum to the case of non-stationary black holes. Finally, we discuss the relation of these conserved quantities to the infinite tower of charges of the covariant phase space formalism.

Contents

1 Introduction

In the membrane paradigm formalism [1–3], the black hole event horizon is seen as a two-dimensional membrane that lives and evolves in three-dimensional spacetime. This viewpoint was originally motivated by Damour's seminal observation that a generic black hole horizon is similar to a fluid bubble with finite values of electrical conductivity, shear and bulk viscosity [4–6]. It was moreover shown that the equations governing the evolution of the horizon take the familiar form of an Ohm's law, Joule heating law, and Navier-Stokes equation. The membrane paradigm developed by Thorne and Macdonald for the electromagnetic aspects, and by Price and Thorne for gravitational and mechanical aspects, combines Damour's results with the $3+1$ formulation of general relativity, where one trades the true horizon for a $2+1$-dimensional timelike surface located slightly outside it, called "stretched horizon" or "membrane". The laws of evolution of the stretched horizon then become boundary conditions on the physics of the external universe, hence making the membrane picture a convenient tool for astrophysical purposes. In order to derive the evolution equations of the membrane, a crucial step in [3] was to renormalize all physical quantities (energy density, pressure, etc) on the membrane, as they turned out to be divergently large as one approaches the real horizon. We will show that a better approach to this issue is to interpret the near-horizon limit as an ultra-relativistic limit for the stretched horizon, where the radial coordinate plays the role of a virtual speed of light. This ultra-relativistic limit results in the emergence of Carrollian physics at the horizon.

The Carroll group was originally introduced in [7] as an ultra-relativistic limit of the Poincaré group where the speed of light is tending to zero (as opposed to the more familiar non-relativistic limit leading to the Galilean group). Recently, there has been a renewed interest in Carrollian physics due to its relation to asymptotically flat gravity. The symmetry group of asymptotically flat spacetimes is the Bondi-Metzner-Sachs (BMS) group; it is an infinite-dimensional extension of the Poincaré group, and its connection with soft theorems has shead a new light on the infrared structure of gravitational theories [8]. Interestingly, the BMS group was also shown to be isomorphic to a conformal extension of the Carroll group in [9], while the dynamics of asymptotically flat spacetimes has been rephrased in terms of ultra-relativistic conservations laws on null infinity [10]. This leads to think that theories holographically dual to asymptotically flat gravity should be ultra-relativistic and enjoy a Carrollian symmetry [11]. Actually, it is now understood that *any* null hypersurface is endowed with a Carrollian geometry[1] [10,12–16] and that the associated constraint equations are ultra-relativistic conservation laws [17]. The aim of this paper is to give a complete analysis of this statement at the level of another physically interesting null hypersurface, the horizon of a black hole. The Carrollian symmetry emerging at the horizon was also used in [18] to explain the vanishing of Love numbers for the Schwarzschild black hole.

The recent focus on the symmetries of near-horizon geometries has been motivated by the fact that they exhibit, in some instances, a BMS-like algebra composed of supertranslations and superrotations [19–32]. Moreover, one can associate non-trivial charges to these large diffeomorphisms: they generate the so-called *soft hair* on black holes [22–25], which were pointed out to have implications for the information paradox. We will show that this rich symmetry structure is in fact naturally encoded in the Carrollian geometry of the horizon. To do so, we will interpret the near-horizon limit as an ultra-relativistic limit, where the radial coordinate ρ plays the role of a virtual speed of light for constant ρ hypersurfaces. This will allow to define proper, rather than ad hoc, finite quantities on the horizon. Moreover, we will prove that the laws governing the dynamics of the black hole horizon are Carrollian conservation laws. These are the ultra-relativistic equivalent of the conservation of an energy–momentum tensor. Through the near-horizon analysis, we will derive the isometries of the induced Carrollian geometry on the horizon and show that they include supertranslations and superrotations. We will also construct associated conserved charges; in particular, the one associated with superrotations will provide a generalization of the angular momentum for very generic non-stationary black holes. Finally, the relation of these conserved quantities to the charges of the covariant phase space formalism will also be discussed.

The paper is organized as follows: in Sec. 2, we introduce a suitable coordinate system for the study of near-horizon geometries. We define the intrinsic and extrinsic objects of the horizon and write the constraint equations governing the dynamics, *i.e.* the null Raychaudhuri and Damour equations. We then review the set of vector fields preserving the near-horizon metric and the derivation of their associated surface charges defined in the covariant phase space formalism. In Sec. 3, we present the Carrollian geometry associated with the black hole horizon. By identifying the radial coordinate ρ as the square of a virtual speed of light for constant ρ hypersurfaces, we interpret the near-horizon limit ($\rho \to 0$) as an ultra-relativistic limit and compute the horizon Carrollian geometric fields. We then define

[1] Carrollian geometry is the degenerate geometry that one obtains when taking the ultra-relativistic limit of a Lorentzian metric. It is composed of a non-degenerate metric on spatial sections and a transverse vector field that corresponds to the time direction.

the energy–momentum tensor associated with a constant ρ hypersurface in terms of its extrinsic curvature. The analysis of its scaling w.r.t. the radial coordinate allows us to define the Carrollian momenta which are the ultra-relativistic equivalent of the energy–momentum tensor. We give a physical interpretation of those quantities in terms of energy density, pressure, heat current and dissipative tensor. Ultra-relativistic conservation laws are written in terms of the Carrollian momenta and are shown to match perfectly the null Raychaudhuri and Damour equations. Finally, we consider the Killing fields which preserve the Carrollian geometry induced on the horizon and construct associated conserved charges. The latter provides a generalization of the angular momentum for non-stationary black holes. We extend this analysis to conformal Killing vectors of the Carrollian geometry and show that the charges are now conserved provided a conformal state equation involving the energy density and the pressure is satisfied. We also write an interesting relation between these conserved charges and the one obtained in the covariant phase space formalism. We conclude in Sec. 4 with a discussion of open questions.

2 Near-horizon geometry and dynamics

In this section, we describe the near-horizon geometry of a black hole and its dynamics. To do so, we introduce a coordinate system adapted to the study of the spacetime geometry near a null hypersurface. This will allow us to define the intrinsic and extrinsic geometry of the horizon. The projection of Einstein equations on the horizon gives rise to two constraint equations on the extrinsic geometry, the *null Raychaudhuri* equation and the *Damour* equation. These are the constraints that we ultimately want to interpret as ultra-relativistic conservation laws. Finally, we turn to the asymptotic symmetries preserving the form of the near-horizon geometry we have introduced, and present the associated charges computed through the covariant phase space formalism. They have the particularity of being generically non-integrable.

2.1 Intrinsic and extrinsic geometry of the horizon

We consider a D-dimensional spacetime whose coordinates are $x^a = (x^\alpha, x^A)$, with $x^\alpha = (v, \rho)$ where v is the advanced time and ρ the radial coordinate. The surfaces of constant v and ρ are $(D-2)$-dimensional spheres $S_{v,\rho}$ and parametrized by x^A $(A = 3, \cdots, D)$, the set of all these angular coordinates will be denoted $\mathbf{x}$. Throughout the paper, when we refer to spatial objects, it will be with respect to the angular coordinates. The constant v surfaces are null, and constant ρ are timelike. Finally, we assume the existence of a horizon $\mathcal{H}$ sitting at $\rho = 0$.

It is alway possible to find a coordinates system, usually called *null Gaussian coordinates*, such that the near-horizon geometry is given by [33][2]

$$ds^2 = -2\kappa\rho dv^2 + 2d\rho dv + 2\theta_A \rho dv dx^A + (\Omega_{AB} + \lambda_{AB}\rho)dx^A dx^B + \mathcal{O}(\rho^2), \tag{2.1}$$

[2]See also [34], p. 48 for a review.

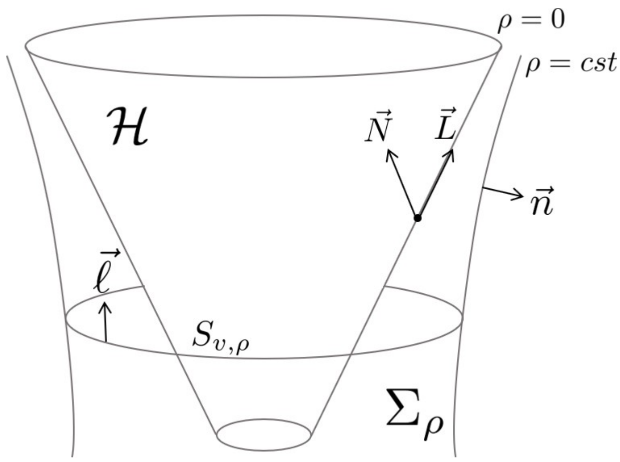

Figure 1: The horizon is a null hypersurface situated at $\rho = 0$ and Σ_ρ is a timelike constant ρ hypersurface near the horizon. We define also four vectors that are useful for our analysis, the null vector $\vec{L}$ is the normal to the horizon while $\vec{N}$ is transverse but also null. The spacelike vector $\vec{n}$ is the normal to Σ_ρ and the timelike vector $\vec{\ell}$ is the normal to a constant v section of Σ_ρ.

where κ, Ω_{AB}, λ_{AB}, θ_A in principle depend on the coordinates $\mathbf{x}$ and v. The spatial metric Ω_{AB} will be used to raise and lower spatial indexes. We will sometime refer to the D-dimensional spacetime as the *bulk*.

There are now two types of geometrical objects we can define on $\mathcal{H}$: the first ones are intrinsic and the others extrinsic. In a Hamiltonian perspective, they are canonical conjugate of each other. Moreover, the canonical momenta satisfy constraint equations that are imposed by the gravitational dynamics [35, 36]. The induced geometry on $\mathcal{H}$ is degenerate and reads

$$ds^2_{\mathcal{H}} = 0 \cdot dv^2 + 0 \cdot dv dx^A + \Omega_{AB} dx^A dx^B, \tag{2.2}$$

the intrinsic geometry being then entirely specified by the spatial metric in this gauge. We now perform a decomposition of the bulk metric adapted to the study of null hypersurfaces:

$$g_{ab} = q_{ab} + L_a N_b + N_b L_a, \tag{2.3}$$

where

$$\vec{L} = L^a \partial_a = \partial_v - \rho\theta^A \partial_A + \kappa\rho\partial_\rho \quad \text{and} \quad N = N_a dx^a = dv, \tag{2.4}$$

are respectively a null vector and a null form. They satisfy $N(\vec{L}) = 1$ and will allow us to define all the extrinsic curvature elements of $\mathcal{H}$. The vector $\vec{L}$ coincides with the normal to the horizon on $\mathcal{H}$, and has the particularity of being also tangent to the horizon. On the other hand the vector $\vec{N} \equiv g^{-1}(N)$ is transverse to the horizon and together with $\vec{L}$ they define q_{ab}, the projector perpendicular to $\vec{L}$ and $\vec{N}$. In his work [5, 6], T. Damour maps the black hole dynamics to the hydrodynamics of a fluid living on the horizon, and the vector $\vec{L}$ defines the fluid's velocity through $\vec{L}_{\mathcal{H}} = \partial_v + v^A \partial_A$. We have $v^A = 0$, as we have chosen comoving coordinates, *i.e.*, in Damour's interpretation the fluid would be at rest but on a dynamical surface[3].

The extrinsic geometry of the horizon is captured by a triple $(\Sigma^{AB}, \omega_A, \tilde{\kappa})$ where Σ^{AB} is the deformation tensor (or second fundamental form), ω_A is the twist field (Hajicek one-form) and $\tilde{\kappa}$ the surface gravity, defined as follows:

$$\Sigma_{AB} = \frac{1}{2} q_A^a q_B^b \mathcal{L}_{\vec{L}} q_{ab}, \quad \omega_A = q_A^a (N_b \mathcal{D}_a L^b) \quad \text{and} \quad L^b \mathcal{D}_b L^a = \tilde{\kappa} L^a, \tag{2.5}$$

where $\mathcal{L}$ denotes the Lie derivative, and $\mathcal{D}_a$ is the Levi-Civita associated with g_{ab}. Using the bulk metric (2.1), these quantities become on $\mathcal{H}$

$$\Sigma_{AB} = \frac{1}{2} \partial_v \Omega_{AB}, \quad \omega_A = -\frac{1}{2} \theta_A \quad \text{and} \quad \tilde{\kappa} = \kappa. \tag{2.6}$$

We see that κ really plays the role of the surface gravity and that θ_A is proportional to the twist. The deformation tensor gives rise to two new extrinsic objects: its trace and its traceless part, which are respectively the horizon expansion and the shear tensor:

$$\begin{aligned} \Theta &= \Omega^{AB} \Sigma_{AB} = \partial_v \ln \sqrt{\Omega}, \\ \sigma_{AB} &= \frac{1}{2} \partial_v \Omega_{AB} - \frac{\Theta}{D-2} \Omega_{AB}, \end{aligned} \tag{2.7}$$

where $\sqrt{\Omega}$ is the volume form of the spatial metric. The scalar expansion Θ measures the rate of variation of the surface element of the spatial section of $\mathcal{H}$.[4] It is possible to show, under the assumption that matter fields satisfy the null energy condition and that the null Raychaudhuri equation (see next section) is satisfied, that Θ is positive everywhere on $\mathcal{H}$, which implies that the surface area of the horizon can only increase with time (see e.g. [37]).

2.2 Raychaudhuri and Damour equations

Those quantities being defined, we can deduce from Einstein equations two conservation laws (or constraint equations) that belong to $\mathcal{H}$: the null Raychaudhuri equation [38] and Damour equation [5, 6], which are respectively

$$L^a L^b R_{ab} = 0 \quad \text{and} \quad q_A^a L^b R_{ab} = 0; \tag{2.8}$$

they are thus given by projections of vacuum Einstein equations on the horizon. The first one is scalar and the second one is a vector equation w.r.t. the spatial section of $\mathcal{H}$. Using the near-horizon geometry (2.1), the null Raychaudhuri equation becomes

$$\partial_v \Theta - \kappa \Theta + \frac{\Theta^2}{D-2} + \sigma_{AB} \sigma^{AB} = 0, \tag{2.9}$$

[3]As pointed out in [3], one can always set $v^A = 0$, namely the spatial coordinates x^A can always be taken to be comoving, except at caustics.

[4]By definition, a non-expanding horizon has $\Theta = 0$.

where $\sigma^{AB} = \Omega^{AC}\Omega^{BD}\sigma_{CD}$. This equation describes how the expansion evolves along the null geodesic congruence $\vec{L}$ and is a key ingredient in the proofs of singularity theorems. Damour equation[5] becomes

$$(\partial_v + \Theta)\,\theta_A + 2\nabla_A\left(\kappa + \frac{D-3}{D-2}\Theta\right) - 2\nabla_B\sigma^B_A = 0, \tag{2.10}$$

where ∇_A is the Levi-Civita connection associated with Ω_{AB}. Damour has interpreted this last equation as a $(D-2)$-dimensional Navier-Stokes equation for a viscous fluid; notice that the fluid velocity is not appearing here because we have chosen a comoving coordinate system as explained earlier.

It is useful to know what these equations become when considering the conformal gauge, *i.e.* when the spatial metric can be written as a conformal factor times a purely spatial metric:

$$\Omega_{AB} = \gamma(v,\mathbf{x})\bar{\Omega}_{AB}(\mathbf{x}). \tag{2.11}$$

One can check that this is equivalent to asking the shear to be zero. If we make this choice, $\bar{\Omega}_{AB}$ disappears and the two conservation equations read

$$\begin{aligned} &\partial_v^2\gamma - \frac{1}{2}\gamma^{-1}(\partial_v\gamma)^2 - \kappa\partial_v\gamma = 0, \\ &\partial_v\theta_A + 2\partial_A\kappa + (D-3)\gamma^{-1}\partial_A\partial_v\gamma - (D-3)\gamma^{-2}\partial_A\gamma\partial_v\gamma + \frac{(D-2)}{2}\gamma^{-1}\theta_A\partial_v\gamma = 0. \end{aligned} \tag{2.12}$$

In particular, one can verify that these equations reproduce the field equations studied in [20] in the $D = 3$ and $D = 4$ cases.

2.3 Bulk symmetries and associated charges

We now turn our attention to the bulk symmetries of the near-horizon gauge. The vector fields $\chi = \chi^a\partial_a$ that preserve the shape of the metric (2.1) were shown in [20] to involve of a smooth arbitrary function $f(v,\mathbf{x})$, which depends on the advanced time and the sphere coordinates, and a vector field of the sphere $Y^A(\mathbf{x})$; they are given by

$$\begin{aligned} \chi^v &= f(v,\mathbf{x}), \\ \chi^\rho &= -\partial_v f\rho + \frac{1}{2}\theta^A\partial_A f\rho^2 + \mathcal{O}(\rho^3), \\ \chi^A &= Y^A(\mathbf{x}) + \Omega^{AC}\partial_C f\rho + \frac{1}{2}\lambda^{AC}\partial_C f\rho^2 + \mathcal{O}(\rho^3), \end{aligned} \tag{2.13}$$

and in any dimension D. We will call them asymptotic Killing vectors even though the gauge introduced does not involve a notion of infinity. We notice an important feature, which is that these vector fields projected on the horizon become

$$\chi = f(v,\mathbf{x})\partial_v + Y^A(\mathbf{x})\partial_A \quad \text{projected on } \mathcal{H}, \tag{2.14}$$

[5] Raychaudhuri and Damour equations are called respectively the "focusing equation" and the "Hajicek equation" in Price-Thorne [3]. The tidal-force equation expresses components of the Weyl tensor in terms of the evolution of the shear, and will not play a role here.

and as f and Y^A are totally generic for the moment, this is exactly the infinitesimal version of a particular type of diffeomorphisms on the horizon that we will define in Sec. 3: the Carrollian diffeomorphisms. Following [19, 20], we will call f a supertranslation and Y^A a superrotation. They act on the horizon fields in the following way:

$$\begin{aligned}
&\delta_\chi \kappa = Y^A \partial_A \kappa + \partial_v(\kappa f) + \partial_v^2 f, \\
&\delta_\chi \Omega_{AB} = f\partial_v \Omega_{AB} + \mathcal{L}_Y \Omega_{AB}, \\
&\delta_\chi \theta_A = \mathcal{L}_Y \theta_A + f\partial_v \theta_A - 2\kappa \partial_A f - 2\partial_v \partial_A f + \partial_v \Omega_{AB} \partial^B f, \\
&\delta_\chi \lambda_{AB} = f\partial_v \lambda_{AB} - \lambda_{AB}\partial_v f + \mathcal{L}_Y \lambda_{AB} + \theta_A \partial_B f + \theta_B \partial_A f - 2\nabla_A \nabla_B f.
\end{aligned} \tag{2.15}$$

To each of these vector fields preserving the near-horizon metric, one can associate a surface charge through the covariant phase space formalism [39].[6] More precisely, the quantity which is constructed at first is not a charge, but rather the field-variation of a charge (namely a one-form in the configuration space). For an on shell metric g and variation $h \equiv \delta g$, it is given by:

$$\not\delta Q_\chi[g,h] = \oint_{S_{v,\rho}} \mathbf{k}_\chi[g,h], \tag{2.16}$$

where χ is an asymptotic Killing vector and $\mathbf{k}_\chi[g,h]$ is a one-form w.r.t. the field configuration space but a $(D-2)$-form w.r.t. the spacetime. It is defined as follows:[7]

$$\mathbf{k}_\chi[g,h] = \frac{\sqrt{-g}}{8\pi G}(d^{D-2}x)_{ab}\left(\chi^a \nabla_c h^{bc} - \chi^a \nabla^b h + \chi_c \nabla^b h^{ac} + \frac{1}{2} h \nabla^b \chi^a - h^{cb}\nabla_c \chi^a\right), \tag{2.17}$$

where $h = g^{ab}h_{ab}$ and $(d^{D-2}x)_{ab} = \frac{1}{2(D-2)!}\epsilon_{abc_1\ldots c_{D-2}} dx^{c_1} \wedge \ldots \wedge dx^{c_{D-2}}$. The $\not\delta$ is a notation that emphasizes the fact that the charges (2.16) are a priori non-integrable (namely not δ-exact). In the integrable case, Q_χ represents the generator of the associated infinitesimal transformation χ. Computing $\not\delta Q[g,h]$ for the metric written in the horizon gauge (2.1), the associated preserving vector fields (2.13) and integrated on a spatial section of $\mathcal{H}$, one obtains [20]:

$$\begin{aligned}
\not\delta Q_{(f,Y^A)}[g,\delta g] = \frac{1}{16\pi G}\oint_{S^{D-2}} d^{D-2}x \Big(& 2f\kappa\delta\sqrt{\Omega} + 2\partial_v f \delta\sqrt{\Omega} - 2f\sqrt{\Omega}\delta\Theta + \frac{1}{2} f\sqrt{\Omega}\partial_v \Omega_{AB}\delta\Omega^{AB} \\
& - Y^A \delta(\theta_A \sqrt{\Omega})\Big).
\end{aligned} \tag{2.18}$$

We can see that these charges are not integrable in full generality, due to the presence of the three following terms: $2f\sqrt{\Omega}\delta\Theta$, $2f\kappa\delta\sqrt{\Omega}$ and $\frac{1}{2}f\sqrt{\Omega}\partial_v\Omega_{AB}\delta\Omega^{AB}$. The authors of [20] circumvent this issue by restricting the phase space to the configurations where κ is a constant. They also use the fact that they work in four dimensions to choose a spatial metric related to the usual metric on the 2-sphere by a Weyl transformation. We would like instead for the moment to keep all possible dependencies of the fields.

[6] See also [40] for a pedagogical introduction to this formalism.

[7] There is actually an ambiguity in this definition (see [40]), as one can add to the definition of $\mathbf{k}_\chi[g,h]$ the term $\alpha\frac{\sqrt{-g}}{16\pi G}(d^{D-2}x)_{ab}\left(h^{cb}\nabla^a\chi_c + h^{cb}\nabla_c\chi^a\right)$, where α is any constant. But one can show that for the metric and vector fields at hand, this term vanishes when evaluated on the horizon.

When surface charges are non-integrable, there is still a way to obtain a representation of the asymptotic Killing algebra through the definition of a modified bracket [41]. To do so, we split $\not{\delta} Q_\chi$ into an integrable part Q_χ^{int} and a non-integrable part Ξ_χ:

$$\not{\delta} Q_\chi[g, \delta g] = \delta(Q_\chi^{\text{int}}[g]) + \Xi_\chi[g, \delta g], \tag{2.19}$$

where

$$Q_\chi^{\text{int}}[g] = \frac{1}{16\pi G} \oint_{S^{D-2}} d^{D-2}x \sqrt{\Omega} \left(2f\kappa + 2\partial_v f - \frac{2}{D-2} f\Theta - Y^A \theta_A \right), \tag{2.20}$$

and

$$\Xi_\chi[g, \delta g] = -\frac{1}{8\pi G} \oint_{S^{D-2}} d^{D-2}x \sqrt{\Omega}\, f \left(\delta\kappa + \frac{D-3}{D-2} \delta\Theta - \frac{1}{2} \sigma_{AB} \delta\Omega^{AB} \right). \tag{2.21}$$

From this splitting[8] we can see directly why, for three-dimensional bulk spacetimes, the condition $\delta\kappa = 0$ considered in [20] was sufficient to insure integrability of the charges (the shear vanishes by definition and the factor $(D-3)$ cancels the contribution of the expansion in (2.21)). We now define the following modified Dirac bracket

$$\{Q_\chi^{\text{int}}[g], Q_\eta^{\text{int}}[g]\}^* \equiv \delta_\eta Q_\chi^{\text{int}}[g] + \Xi_\eta[g, \mathcal{L}_\chi g]. \tag{2.22}$$

It was first introduced in [41] for the study of the BMS charges in four dimensions, which are also generically non-integrable. They also noticed that the splitting is not unique in the sense that for some $N_\chi[g]$ we can always choose

$$\tilde{Q}_\chi^{\text{int}} = Q_\chi^{\text{int}} - N_\chi \quad \text{with} \quad \tilde{\Theta}_\chi + \delta N_\chi. \tag{2.23}$$

However, we will see that the separation (2.20), (2.21) we have chosen happens to be relevant in the Carrollian anaysis that we perform in Sec. 3. This modified bracket defines a representation of the asymptotic Killing algebra: indeed, letting (f_1, Y_1^A) and (f_2, Y_2^A) to be two asymptotic Killing fields, one can show that

$$\{Q^{\text{int}}_{(f_1, Y_1^A)}, Q^{\text{int}}_{(f_2, Y_2^A)}\}^* = Q^{\text{int}}_{(f_{12}, Y_{12}^A)}, \tag{2.24}$$

where $f_{12} = f_1 \partial_v f_2 - f_2 \partial_v f_1 + Y_1^A \partial_A f_2 - Y_2^A \partial_A f_1$ and $Y_{12}^A = Y_1^B \partial_B Y_2^A - Y_2^B \partial_B Y_1^A$. We notice that this algebra does not involve any central extension. A direct consequence of (2.24) is that the non-integrable part of the charges plays the role of a source for the non-conservation of Q^{int}. Indeed choosing (f_2, Y_2^A) to be $(1, 0)$ we obtain

$$\delta_{(1,0)} Q_\chi^{\text{int}}[g] + Q^{\text{int}}_{(\partial_v f, 0)}[g] = -\Xi_{(1,0)}[g, \mathcal{L}_\chi g], \tag{2.25}$$

moreover $\delta_{(1,0)}$ acts like a time derivative on the fields (2.15), so we finally obtain

$$\frac{d}{dv} Q_\chi^{\text{int}}[g] = -\Xi_{(1,0)}[g, \mathcal{L}_\chi g]. \tag{2.26}$$

[8] This splitting coincides with expression obtained in [36] in the Hamiltonian framework, while another splitting was considered in [20] for the case $\kappa = cst$.

3 Near-horizon or ultra-relativistic limit

One of the particularity of null hypersurfaces is that they are equipped with a degenerate induced metric Ω in the sense that there exists a vector field $\vec{u}$ that belongs to its kernel:

$$\Omega(.,\vec{u}) = 0. \tag{3.1}$$

In the case of the horizon described above, $\Omega = \Omega_{AB}(v,\mathbf{x})dx^A dx^B$ and $\vec{u} = f(v,\mathbf{x})\partial_v$, for any function f on $\mathcal{H}$. It was understood, for example in [13, 14, 18], that this defines a *Carrollian geometry*, the natural non-Riemannian geometry that ultra-relativistic theories couple to. This means that any null hypersurface can be thought of as an ultra-relativistic spacetime. In particular, for the near-horizon geometry presented above, we are going to show that the limit $\rho \to 0$, can be understood as an ultra-relativistic limit where $\sqrt{\rho}$ plays the role of a virtual velocity of light c. Notice that this parameter should not be confused with the physical velocity of light of the bulk spacetime that is set to 1 in (2.1).

This feature has strong consequences on the dynamics of the horizon, *i.e.* the null Raychaudhuri and Damour equations: indeed, we will show that they match ultra-relativistic conservation laws written in terms of the Carrollian geometry and the *Carrollian momenta*, sort of ultra-relativistic equivalent of the energy–momentum tensor.

Finally, we will study the symmetries and charges associated with the horizon that we interpret as *Carrollian Killing*, defined as the vector fields on $\mathcal{H}$ that preserve the Carrollian geometry. In some instances, the symmetry algebra will be shown to have a BMS-like structure in the sense that it includes superrotations and supertranslations on the horizon [19, 20, 32].

3.1 Carrollian geometry: Through the Looking-Glass

Carrollian geometry emerges from an ultra-relativistic ($c \to 0$) limit of the relativistic metric and was shown to have a rich mathematical structure and interesting dynamics [7, 9, 10, 12, 13, 16, 42]. It was shown in [10, 16] that the $c \to 0$ limit of relativistic general-covariant theories is covariant under a subset of the diffeomorphisms dubbed *Carrollian diffeomorphisms*

$$v' = v'(v,\mathbf{x}) \,, \qquad \mathbf{x}' = \mathbf{x}'(\mathbf{x}), \tag{3.2}$$

whose infinitesimal version is given by the vector fields

$$\xi = f(v,\mathbf{x})\partial_v + Y^A(\mathbf{x})\partial_A, \tag{3.3}$$

for any f and Y^A. This suggests that space and time decouple and an adequate parametrization to study the ultra-relativistic limit is the so-called Randers–Papapetrou parametrization, where the metric is decomposed as[9]

$$a = \begin{pmatrix} -c^2\alpha^2 & c^2\alpha b_A \\ c^2\alpha b_B & \Omega_{AB} - c^2 b_A b_B \end{pmatrix}_{\{dv,dx^A\}} \underset{c\to 0}{\longrightarrow} \quad \Omega_{AB}dx^A dx^B. \tag{3.4}$$

[9] Any spacetime metric can be parametrized in that way.

After the limit is performed, one thus trade the metric a for $\alpha(v, \mathbf{x})$ the time lapse, $b_A(v, \mathbf{x})$ the temporal connection, and $\Omega_{AB}(v, \mathbf{x})$ the spatial metric. These functions define the Carrollian geometry and one can check that they transform covariantly under Carrollian diffeomorphisms (see Sec. 2 of [10] for a complete presentation). Out of the Carrollian geometry, one can build the following first-derivative quantities:

$$\begin{aligned}
\varphi_A &= \alpha^{-1}(\partial_v b_A + \partial_A \alpha),\\
\beta &= \alpha^{-1}\partial_v \ln\sqrt{\Omega},\\
\xi_{AB} &= \alpha^{-1}\left(\frac{1}{2}\partial_v \Omega_{AB} - \frac{\Omega_{AB}}{D-2}\partial_v \ln\sqrt{\Omega}\right),\\
\omega_{AB} &= \partial_{[A} b_{B]} + \alpha^{-1}(b_{[A}\partial_{B]}\alpha + b_{[A}\partial_v b_{B]});
\end{aligned} \tag{3.5}$$

they are respectively, the Carrollian acceleration, expansion, shear and vorticity. They also transform covariantly under Carrollian diffeomorphisms, and will play an important role in the Carrollian conservation laws we will discuss in the next section.

Let us come back to the black hole near-horizon metric (2.1). On each constant ρ hypersurface, called Σ_ρ in Fig. 1, it induces a Lorentzian signature metric that becomes degenerate when taking the near-horizon limit:

$$a = ds^2_{\rho=cst} = \begin{pmatrix} -2\rho\kappa & \rho\theta_A \\ \rho\theta_B & \Omega_{AB} + \rho\lambda_{AB} \end{pmatrix}_{\{dv, dx^A\}} \underset{\rho\to 0}{\longrightarrow} \quad \Omega_{AB} dx^A dx^B. \tag{3.6}$$

If we now compare this induced metric with the Randers–Papapetrou one, we are tempted to make the following identifications:[10]

$$c^2 = \rho, \quad \alpha = \sqrt{2\kappa}, \quad \text{and} \quad b_A = \frac{\theta_A}{\sqrt{2\kappa}}. \tag{3.7}$$

We thus identify the radial coordinate with the square of a virtual speed of light for the Lorentzian spacetime Σ_ρ. As the horizon is located at $\rho = 0$, it is an ultra-relativistic spacetime endowed with a Carrollian geometry given in terms of the surface gravity, the twist and the induced spatial metric Ω_{AB}. After this identification, we can re-express the first-derivative Carrollian tensors (3.5) in terms of the extrinsic geometry of the horizon (2.6):

$$\begin{aligned}
\varphi_A &= \frac{1}{2\kappa}\left(\partial_A\kappa + \partial_v\theta_A - \frac{\theta_A}{2\kappa}\partial_v\kappa\right),\\
\beta &= \frac{\Theta}{\sqrt{2\kappa}},\\
\xi_{AB} &= \frac{1}{\sqrt{2\kappa}}\sigma_{AB},\\
\omega_{AB} &= \frac{1}{2}\left(\frac{\partial_A\theta_B}{\sqrt{2\kappa}} + \frac{2\theta_A\partial_B\kappa + \theta_A\partial_v\theta_B}{(2\kappa)^{3/2}}\right) - (A \leftrightarrow B).
\end{aligned} \tag{3.8}$$

We notice that the Carrollian expansion and the Carrollian shear are proportional respectively to the expansion and the shear of the horizon defined extrinsically in Sec. 2.1.

[10] One notices that, following this identification, we should also have $\lambda_{AB} = -b_A b_B$, which becomes $\lambda_{AB} = -\frac{\theta_A\theta_B}{2\kappa}$. This would then impose a constraint on the near-horizon geometry, but we will actually not have to do that as λ_{AB} will always appear at subleading order in the equations we are going to consider.

3.2 Horizon dynamics as ultra-relativistic conservation laws

We now turn our attention to the gravitational dynamics of the horizon. Consider again the hypersurface Σ_ρ near $\rho = 0$. Its unit normal is given by

$$n = \frac{d\rho}{\sqrt{2\kappa\rho}}, \tag{3.9}$$

and allows us to define the extrinsic curvature and the momentum conjugate to the induced metric:

$$T_{ab} = \frac{1}{8\pi G}(K a_{ab} - K_{ab}), \tag{3.10}$$

where $K^a_b = a^c_b \mathcal{D}_c n^a$ is the extrinsic curvature of Σ_ρ, $K = K^a_a$ its trace and $a_{ab} = g_{ab} - n_a n_b$ is the projector on the hypersurface perpendicular to n.[11] This hypersurface is sometimes referred to as the stretched horizon or membrane, while T_{ab} is called the "membrane energy–momentum tensor" [2, 3, 28].[12] Einstein equations ensure that it is conserved:

$$\bar{\nabla}_j T^{ji} = 0, \tag{3.11}$$

where the index i refers to $\{v, \mathbf{x}\}$, and $\bar{\nabla}_i$ is the Levi-Civita connection associated with the induced metric (3.6). The membrane is then interpreted as a fluid whose equations of motion are given by this conservation law. One notices that (3.11) describes the dynamics of a *relativistic* fluid that lies in the $(D-1)$-dimensional spacetime given by the constant ρ hypersurface and equipped with the metric a. We are going to show that, to obtain the null Raychaudhuri (2.9) and Damour equations (2.10), one has to take the near-horizon limit of this conservation law which, at the level of the fluid, is interpreted as an ultra-relativistic limit through the identification $\rho = c^2$.

Using (2.1), we compute the membrane energy–momentum tensor near the horizon,

$$\begin{aligned}
8\pi G T^{vv} &= \frac{\Theta}{2\sqrt{2}(\rho\kappa)^{\frac{3}{2}}} + \mathcal{O}(1/\sqrt{\rho}),\\
8\pi G T^{vA} &= -\frac{1}{2\sqrt{2\rho}\kappa^{3/2}}\left(\partial^A\kappa + \theta^A(\kappa+\Theta)\right) + \mathcal{O}(\sqrt{\rho}),\\
8\pi G T^{AB} &= -\frac{1}{\sqrt{2\rho\kappa}}\left(\Omega^{AB}(\kappa + \Theta - \frac{\partial_v\kappa}{2\kappa}) + \frac{1}{2}\partial_v\Omega^{AB}\right) + \mathcal{O}(\sqrt{\rho}).
\end{aligned} \tag{3.12}$$

We now decompose T^{ij} into the *Carrollian momenta*, which are defined such that they are independent of the speed of light and covariant under Carrollian diffeomorphisms [10],

$$\begin{aligned}
8\pi G T^{vv} &= c^{-3}\alpha^{-2}\mathcal{E} + \mathcal{O}(c^{-1}),\\
8\pi G T^{vA} &= c^{-1}\alpha^{-1}(\pi^A - 2b_B\mathcal{A}^{AB}) + \mathcal{O}(c),\\
8\pi G T^{AB} &= -2c^{-1}\mathcal{A}^{AB} + \mathcal{O}(c),
\end{aligned} \tag{3.13}$$

[11] The projector a_{ab} coincides with (3.6) when one consider its $\{v, A\}$ components only.

[12] In those papers, the approach is to study this membrane energy–momentum tensor for a small ρ and use it to define the fluid quantities like the energy density, the pressure, etc. The problem is that those quantities diverge when ρ is sent to zero. Their solution is to rescale them by hand to obtain finite quantities. We propose another approach and define the Carrollian momenta that are finite on the horizon and well suited for the ultra-relativistic interpretation.

with $\mathcal{E}$ a scalar, π^A a spatial vector and $\mathcal{A}^{AB}$ a spatial symmetric 2-tensor. They are the ultra-relativistic equivalent of an energy–momentum tensor. They can be thought of respectively as the energy density, the heat current and the total stress tensor. The latter can be decomposed into its trace and traceless part

$$\mathcal{A}^{AB} = -\frac{1}{2}\left(\mathcal{P}\Omega^{AB} - \Xi^{AB}\right), \tag{3.14}$$

which are interpreted respectively as the pressure and the dissipative tensor.

Comparing (3.12) with (3.13), we read the following Carrollian momenta:

$$\begin{aligned}
\mathcal{E} &= \frac{1}{\sqrt{2\kappa}}\Theta, \\
\mathcal{P} &= -\frac{1}{\sqrt{2\kappa}}\left(\kappa + \frac{D-3}{D-2}\Theta - \frac{\partial_v\kappa}{2\kappa}\right), \\
\Xi_{AB} &= -\frac{1}{\sqrt{2\kappa}}\sigma_{AB}, \\
\pi_A &= -\frac{1}{2}\left(\frac{\partial_A\kappa}{\kappa} + \frac{\theta^B}{2\kappa}\partial_v\Omega_{BA} + \frac{\theta_A}{2\kappa^2}\partial_v\kappa\right).
\end{aligned} \tag{3.15}$$

We have obtained that the energy density is proportional to the expansion of the horizon. The pressure is related to the combination

$$\mu = \kappa + \frac{D-3}{D-2}\Theta, \tag{3.16}$$

which is referred to in [36] as the "gravitational pressure" and receives corrections from the time evolution of the surface gravity. The dissipative tensor is proportional to the shear of the horizon (2.7). The heat current π_A is harder to interpret but we notice that it receives a contribution from the gradient of κ, which can be thought of as a local temperature on the black hole horizon (see the discussion at the end of [31]).

These Carrollian momenta satisfy conservation equations that are given by the ultra-relativistic (*i.e.* near-horizon) limit of the energy–momentum conservation (3.11).[13] Using the decompositions for the metric (3.4) and the energy–momentum tensor (3.13), we obtain:

$$\begin{aligned}
&\left(\alpha^{-1}\partial_v + \beta\right)\mathcal{E} - \mathcal{A}^{AB}\alpha^{-1}\partial_v\Omega_{AB} = 0, \\
&2\left(\hat{\nabla}_A + \varphi_A\right)\mathcal{A}^A_B - \mathcal{E}\varphi_B - \left(\alpha^{-1}\partial_v + \beta\right)\pi_B = 0.
\end{aligned} \tag{3.17}$$

These equations[14] are covariant w.r.t. Carrollian diffeomorphisms, in the sense that the first one transforms like a scalar and the second one like a spatial vector and they are independant of c (or ρ, the radial coordinate). We have introduced a new object $\hat{\nabla}_A$, which is a Carroll-covariant derivative:

$$\hat{\nabla}_A v^B = \hat{\partial}_A v^B + \hat{\gamma}^A_{BC} v^C, \tag{3.18}$$

[13]This limit was considered for the first time for a relativistic fluid in [16].

[14]These conservation equations were also shown to reproduce the constraint equations on the null infinity for asymptotically flat spacetimes in the Bondi gauge, see [10].

where

$$\hat{\partial}_A = \partial_A + \frac{b_A}{\alpha}\partial_v \quad \text{and} \quad \hat{\gamma}^A_{BC} = \frac{1}{2}\Omega^{AD}\left(\hat{\partial}_B\Omega_{DC} + \hat{\partial}_C\Omega_{DB} + \hat{\partial}_D\Omega_{BC}\right). \tag{3.19}$$

If v^A transforms like a spatial vector, *i.e.* $v'^A = \frac{\partial x'^A}{\partial x^B}v^B$ under a Carrollian diffeomorphism (3.2), then $\hat{\nabla}_A v^B$ will transform like a spatial 2-tensor. One can check that this would not be the case for the usual Levi-Civita connection associated with Ω_{AB}. The first equation of (3.17) can be interpreted as a conservation of energy on a curved background, but an exotic one: indeed, one would expect the gradient of the heat current to appear while here it is absent even when the heat current is non zero. This feature is a signature of the ultra-relativistic limit [16].

The main result of this section is that, considering the Carrollian geometry (3.7) and the Carrollian momenta (3.15) and after a lenghty computation, one can show that the scalar equation is exactly the null Raychaudhuri equation (2.9) while the spatial one gives the Damour equation (2.10). This confirms that the dynamics of a black hole is mapped to ultra-relativistic conservation laws when the near-horizon radial coordinate is identified with a virtual speed of light.

3.3 Conserved charges on the horizon

Using the results of the previous section we would like now to build conserved charges associated with the horizon. The idea is to use the techniques we know from relativistic physics to build charges on a constant ρ hypersurface and then send the radial coordinate to zero to obtain conserved charges on the horizon. The latter will be conserved on shell and associated to the symmetries of the induced Carrollian geometry on the horizon. At the end of this section, we discuss their relationship with the one obtained through the covariant phase space formalism in Sec. 2.3.

Charges associated to Carrollian Killing fields on the horizon

Consider again the energy–momentum tensor of the membrane (3.10): vacuum Einstein equations imply that it is conserved:

$$\bar{\nabla}_j T^{ji} = 0. \tag{3.20}$$

It is thus possible to build a conserved current associated with any vector field of Σ_ρ that satisfies the Killing equation for the induced metric a_{ij}:

$$\bar{\nabla}_i \xi_j + \bar{\nabla}_j \xi_i = 0, \tag{3.21}$$

where we recall that $\bar{\nabla}_i$ is the Levi-Civita associated with a. This current is given by $J^i = \xi_j T^{ji}$; it is conserved

$$\bar{\nabla}_i J^i = 0, \tag{3.22}$$

and allows to build, for any small ρ, a conserved charge w.r.t. the v coordinate:

$$\mathcal{Q}^\rho_\xi = \oint_{S_{v,\rho}} d^{D-2}x\sqrt{q}\ \ell_i J^i, \tag{3.23}$$

where

$$q_{AB} = \Omega_{AB} + \rho\lambda_{AB} + \mathcal{O}(\rho^2) \quad \text{and} \quad \ell = \sqrt{2\kappa\rho}\, dv + \mathcal{O}(\rho^{\frac{3}{2}}), \tag{3.24}$$

are respectively the induced metric on a spatial section of the constant ρ hypersurface, *i.e.* $S_{v,\rho}$, and the unit timelike normal to the spatial section in the constant ρ hypersurface, see Fig. 1.

We are now ready to perform the near-horizon limit of this construction. We consider first the Killing equation for the vector ξ that we decompose as $\xi = f(v,\mathbf{x})\partial_v + Y^A(v,\mathbf{x})\partial_A$. The zero-$\rho$ limit of (3.21) becomes

$$\begin{aligned} &\partial_v Y^A = 0, \\ &f\partial_v\kappa + Y^A\partial_A\kappa + 2\kappa\partial_v f = 0, \\ &f\partial_v\Omega_{AB} + \nabla_A Y_B + \nabla_B Y_A = 0. \end{aligned} \tag{3.25}$$

The first thing to notice is that the near-horizon limit of the Killing equation imposes the vector field ξ to be Carrollian! Moreover, these three equations have an interesting geometrical interpretation: indeed, consider the degenerate metric induced on the horizon $\Omega = \Omega_{AB}(v,\mathbf{x})dx^A dx^B$ and the vector field $\vec{v} = \alpha^{-1}\partial_v$ (where α is given by the identification (3.7)), they are equivalent to asking

$$\mathcal{L}_\xi \vec{v} = 0 \quad \text{and} \quad \mathcal{L}_\xi \Omega = 0. \tag{3.26}$$

Following [13], the triple $(\mathcal{H},\, \Omega,\, \vec{v})$ defines a non-Riemannian geometry called *weak Carroll manifold*.[15] The latter is the natural structure that appears when one wants to study ultra-relativistic symmetries. Things appear to be consistent: we have considered the symmetries of the relativistic metric a, *i.e.* its Killing vector fields, then we have taken the near-horizon limit, interpreted as an ultra-relativistic limit for $\rho = c^2$, and we obtain the symmetries of the corresponding Carrollian geometry. These symmetries given by Eq. (3.26) will be called *Carrollian Killing* symmetries.

We can also perform the near-horizon limit of the charge (3.23) using the value of the membrane energy–momentum tensor derived in Sec. 3.2; we obtain

$$\mathcal{Q}^\rho_\xi \underset{\rho\to 0}{\longrightarrow} \mathcal{C}_\xi = \frac{1}{16\pi G}\oint_{S^{D-2}} \mathrm{d}^{D-2}x\sqrt{\Omega}\left(-2f\Theta - Y^A\left(\theta_A + \frac{\partial_A\kappa}{\kappa}\right)\right). \tag{3.27}$$

This charge is conserved provided that the null Raychaudhuri and the Damour equations are satisfied and the couple (f, Y^A) satisfies the Carrollian Killing equations (3.25). Taking the trace of the last equation of (3.25) we obtain $f\Theta = -\nabla_A Y^A$, therefore the integration on the sphere of this term vanishes. The charge becomes

$$\mathcal{C}_\xi = \frac{-1}{16\pi G}\oint_{S^{D-2}} \mathrm{d}^{D-2}x\sqrt{\Omega}Y^A\left(\theta_A + \frac{\partial_A\kappa}{\kappa}\right). \tag{3.28}$$

This is a sort of generalization of the angular momentum to the case of non-stationary black holes. We would like indeed to stress that in this formula, Ω_{AB}, κ and θ_A depend generically

[15]The Carrollian geometry also involves the temporal connection b_A but is does not appear in the definition of Carrollian Killings.

on both v and x^A, so the conservation of this charge is really non-trivial. Therefore, to any isometry of the induced Carrollian geometry on the horizon, we have associated a charge that is conserved on-shell.

When we consider the case $\kappa = cst$ and $\Omega_{AB} = \bar{\Omega}_{AB}(\mathbf{x})$, the solutions to the Carrollian Killing equations are a supertranslation $f = \mathcal{T}(\mathbf{x})$ together with a real Killing of the metric $\bar{\Omega}_{AB}$ and if one considers the near-horizon geometry of a Kerr black hole and the spatial Killing $Y = \partial_\varphi$, this charge reproduces the constant angular momentum J [20].

The conformal case

The same analysis can be carried out for a conformal Killing on the constant ρ hypersurface Σ_ρ, *i.e.* a vector ξ that satisfies

$$\bar{\nabla}_i \xi_j + \bar{\nabla}_j \xi_i = 2\lambda a_{ij}, \tag{3.29}$$

where $\lambda(v, \mathbf{x})$ is any function. We can build the same current by projecting ξ on the energy–momentum tensor. However, if $\lambda \neq 0$, the associated charge will be conserved on-shell only if T^{ij} satisfies the tracelessness condition

$$T_i^i = 0. \tag{3.30}$$

The near-horizon limit of the conformal Killing equation is

$$\begin{aligned}
&\partial_v Y^A = 0, \\
&f \partial_v \kappa + Y^A \partial_A \kappa + 2\kappa \partial_v f = 2\kappa\lambda, \\
&f \partial_v \Omega_{AB} + \nabla_A Y_B + \nabla_B Y_A = 2\lambda \Omega_{AB}.
\end{aligned} \tag{3.31}$$

Again, it admits a nice interpretation as the conformal isometries of the weak Carroll manifold induced on the horizon. Indeed, (3.31) is equivalent to

$$\mathcal{L}_\xi \vec{v} = -\lambda \vec{v} \quad \text{and} \quad \mathcal{L}_\xi \Omega = 2\lambda \Omega, \tag{3.32}$$

and, according to [13], this is the definition of the level-2 conformal isometries of $(\mathcal{H}, g, \vec{v})$; we will call them *conformal Carrollian Killing* vectors. To any conformal Carrollian Killing ξ we can associate the following charge:

$$\mathcal{C}_\xi = \frac{1}{16\pi G} \oint_{S^{D-2}} \mathrm{d}^{D-2}x \sqrt{\Omega} \left(-2f\Theta - Y^A \left(\theta_A + \frac{\partial_A \kappa}{\kappa} \right) \right), \tag{3.33}$$

which is the same as in the previous section, obtained through the near-horizon limit of $\mathcal{Q}_\xi$. The only difference is that, if $\lambda \neq 0$, this charge will not be generically conserved on-shell. It is generically conserved only if the near-horizon limit of the tracelessness condition (3.30) is satisfied, *i.e.*

$$\mathcal{S} \equiv \Theta + \kappa - \frac{\partial_v \kappa}{2\kappa} = 0, \tag{3.34}$$

where the function $\mathcal{S}$ has been defined through

$$T^i_i \underset{\rho\to 0}{\longrightarrow} \frac{-1}{8\pi G\sqrt{2\kappa}\sqrt{\rho}}\mathcal{S}. \tag{3.35}$$

Asking $\mathcal{S}$ to be zero is a non-trivial additional constraint on the surface gravity and the expansion, that we will call the *conformal state equation*. Indeed, if we reintroduce the Carrollian momenta (3.15) we obtain that

$$\mathcal{S} = 0 \quad \Leftrightarrow \quad \mathcal{E} = (D-2)\mathcal{P}. \tag{3.36}$$

We recognize the usual state equation satisfied by the energy and the pressure of a conformal fluid (see [43] or [16]).

We consider now the case $\kappa = cst$ and $\Omega_{AB} = \bar{\Omega}_{AB}(\mathbf{x})$, the corresponding Carrollian Killings are given by

$$\xi = \left(\frac{v}{D-2}\nabla_A Y^A + \mathcal{T}(\mathbf{x})\right) + Y^A(\mathbf{x})\partial_A, \tag{3.37}$$

where $\mathcal{T}$ is a supertranslation and Y^A is a conformal Killing of $\bar{\Omega}_{AB}$. When the spatial metric is chosen to be the round metric on S^{D-2} we obtain the bms_D algebra. The conformal state equation becomes $\kappa = 0$. This constraint is obviously very restricting but actually, in this particular case, we will not have to impose it to obtain conserved charges. Indeed the charge $\mathcal{C}_\xi$ becomes

$$\mathcal{C}_\xi = \frac{-1}{16\pi G}\oint_{S^{D-2}} \mathrm{d}^{D-2}x\sqrt{\bar{\Omega}}Y^A\theta_A, \tag{3.38}$$

and the Damour equation becomes

$$\partial_v\theta_A = 0. \tag{3.39}$$

So, for any value of κ, this charge associated to a conformal Carrollian Killing of the type (3.37) is manifestly conserved on-shell, but insensitive to the supertranlsation $\mathcal{T}$.

Relationship with the bulk analysis

Finally, in both the non-conformal and conformal case, we can relate $\mathcal{C}_\xi$ to the integrable part of the charges obtained through the covariant phase space formalism in Sec. 2.3. Indeed, consider an asymptotic Killing (f, Y^A) (2.13); as already stated in Sec. 2.3, its projection on the horizon is a generic Carrollian vector field. We can further ask the latter to be a (conformal-)Carrollian Killing, thus considering the subset of asymptotic Killings whose projection on the horizon provides an isometry of the induced Carrollian geometry. If we do so, one can show that

$$\mathcal{C}_{(f,Y^A)} = Q^{\text{int}}_{(f,Y^A)} - \frac{1}{8\pi G}\oint_{S^{D-2}} d^{D-2}x\sqrt{\Omega}f\mathcal{S}, \tag{3.40}$$

where we notice the mysterious appearance of the function $\mathcal{S}$ that defines the conformal state equation (3.36). This equation holds up to boundary terms that are vanishing when integrated on the sphere and if the couple (f, Y^A) satisfies the Carrollian Killing equations

(3.25) or its conformal version (3.31). This equality is off-shell; if we further impose the equations of motion and perform a time derivative we obtain

$$\frac{d}{dv}Q^{\text{int}}_{(f,Y^A)} = \frac{1}{8\pi G}\oint_{S^{D-2}} d^{D-2}x\sqrt{\Omega}\left[f\partial_v + \partial_v f - \nabla_A Y^A\right]\mathcal{S}. \tag{3.41}$$

We conclude that the non-conservation of $Q^{\text{int}}_{(f,Y^A)}$, for (conformal-)Carrollian Killing vectors, will be sourced by the function $\mathcal{S}$. Therefore we have established a connection between the conservation of the charges and the conformality of the Carrollian momenta associated with the horizon. A last remark is that these very compact results are valid for the splitting we have made in Sec. 2.3 between the integrable and non-integrable part of the charge, it would be interesting to determine how they get modified under the change of splitting (2.23).

4 Perspectives

This analysis sets an indubitable connection between Carrollian and near-horizon physics, the main result being that the dynamics of the black hole horizon is given by an ultra-relativistic conservation law. In the membrane paradigm, the "fluid" describing the horizon is supposed to satisfy the Damour-Navier Stokes equation, which a priori is a non-relativistic equation but for a Galilean fluid (*i.e.* when the speed of light is infinite). We want to point out that, instead, the fluid behaves more like a Carrollian one. This observation is emphasized by the fact that the energy conservation satisfied on the horizon seems very different from the one that a usual Galilean fluid would satisfy, as it does not involve the gradient of the heat current (see first equation of (3.17)), while it is perfectly interpreted in terms of an ultra-relativistic energy conservation. All these remarks lead to the conclusion that the ultra-relativistic approach seems to be more appropriate to the study of horizon dynamics. In [16], the authors study the ultra-relativistic limit of a relativistic fluid; it would be interesting to see how this translates in the horizon analysis. One could also study the thermodynamics of such a fluid, especially its entropy current, and see if we can relate it to the black hole entropy.

Another question is the role of the function $\mathcal{S}$ introduced to define the conformal state equation. It would be interesting to understand better its status at the level of the charges. Indeed, the exact same relationship was found in the context of asymptotically flat gravity between the Carrollian charges and the charges obtained through covariant phase space formalism [10]. In that case, the function $\mathcal{S}$ (called σ there) was representing the flux of gravitational radiation through null infinity and was therefore responsible of the non-conservation of the charges. At the level of the horizon, the function $\mathcal{S}$ could have the same kind of physical interpretation which would be worth clarifying.

Finally, let us mention two other interesting directions. The first one would be to add other fields to source the bulk energy–momentum tensor and see how this analysis get modified, in particular their influence on the charges. The second one is the specific case of extremal black holes. We have not mentioned them in this paper since their study would require strong modifications in our analysis (for instance, Carrollian momenta for $\kappa = 0$ would diverge as one can see from (3.15)). The study of Carrollian physics for extremal black holes will be the subject of future works.

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Carroll Structures, Null Geometry and Conformal Isometries

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Abstract

We study the concept of Carrollian spacetime starting from its underlying fiber-bundle structure. The latter admits an Ehresmann connection, which enables a natural separation of time and space, preserved by the subset of Carrollian diffeomorphisms. These allow for the definition of Carrollian tensors and the structure at hand provides the designated tools for describing the geometry of null hypersurfaces embedded in Lorentzian manifolds. Using these tools, we investigate the conformal isometries of general Carrollian spacetimes and their relationship with the BMS group.

1 Introduction

The Carroll group was discovered by Lévy-Leblond in 1965 [1] as a dual contraction of the Poincaré group, operating at vanishing rather than infinite velocity of light. The increasing interest in non-Minkowskian spacetimes possessing nonetheless boost-like isometries, has led to more systematic studies of Carrollian constructions. Besides the intrinsic value of the latter (along with Newton–Cartan), the resurgence in the area has been sustained by the parallel growth of two distinct albeit related fields of application. The first involves codimension-one null hypersurfaces in Lorentzian *i.e.*, hyperbolic pseudo-Riemannian manifolds. The second concerns the development of flat holography.

Carroll structures were introduced in [2–4] as alternatives to Riemannian or Newton–Cartan geometries.[1] According to these authors, Carroll structures consist of a $d+1$-dimensional manifold $\mathcal{C}$ equipped with a degenerate metric g and a vector field E, which defines the kernel of the metric, *i.e.* $g(E, .) = 0$. In this definition, the Carroll group emerges as the isometry group of *flat* Carrollian structures, whereas general diffeomorphisms are always available. Because of the field E, the Carroll structure defines a natural separation between time and space, and a subset of diffeomorphisms arises, the Carrollian diffeomorphisms, which preserves this separation.

Given their defining properties, Carroll structures are expected to arise systematically as geometries on null hypersurfaces of relativistic spacetimes, because the induced metric inherited from the embedding is degenerate (see *e.g.* [6]). There are several notable instances of null hypersurfaces. Generally, null hypersurfaces occur as components of the boundary of causal diamonds and related structures, relevant in the study of entanglement. One also finds null hypersurfaces in other important physical situations, such as black-hole horizons and the hypersurfaces appearing at light-like infinity of asymptotically flat spacetimes (commonly designated as $\mathscr{I}^{\pm}$). The latter makes the bridge with asymptotically flat holography, in which the putative dual degrees of freedom are expected to be defined precisely on this null-infinity hypersurface.

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[1] See also the earlier publication [5] on geometries with degenerate metric.

In fact, asymptotically flat holography has been probably the first arena of application of Carrollian physics [7,8], not so much because of the geometric structure the boundary is endowed with (its Carrollian nature was identified much later), but for the emergence of the BMS symmetry. The BMS group was discovered in 1962 by Bondi, van der Burg, Metzner and Sachs [9,10] as the asymptotic isometry group of asymptotically flat spacetimes towards light infinity, and became popular lately in relation with null hypersurfaces and flat holography (see *e.g.* [11,12]). It was in particular proven [3] that the $\mathfrak{bms}(d+2)$ algebra is isomorphic to the conformal Carroll algebra $\mathfrak{ccarr}(d+1)$ for $d=1,2$. This is yet another sign corroborating the triangle "Carroll–null–BMS".

The aim of the present work is to revisit this web of relationships and provide an alternative perspective to some of its aspects. Our analysis follows two paths. On the one hand, we define a Carrollian spacetime in terms of a fiber bundle accompanied with a Carroll structure. The ingredients are thus an Ehresmann connection, a degenerate metric and a scale factor,[2] all assumed *a priori* time- and space-dependent. This provides us with a geometric understanding of the appearance of Carrollian diffeomorphisms and the reduction of spacetime tensors to Carrollian tensors. Carrollian spacetimes with the above set of ingredients are also naturally revealed in null embedded hypersurfaces. On the other hand, we discuss the conformal isometry algebra of general Carrollian spacetimes. In the *shearless* case (properly defined shortly), we generally recover the familiar algebra of transformations. In two and three dimensions, the algebra coincides with BMS, whereas in arbitrary dimension it appears as the semi-direct product of the conformal isometry group of the metric with supertranslations. The strength of our results resides in their wide validity for shearless but otherwise arbitrary Carrollian geometries. In the literature there have been other proposals made for a notion of geometry defined on null embedded hypersurfaces, the "universal structures", (see *e.g.*, [15]). Different such proposals may lead to different algebras that preserve the given structure, with subsequently a potential choice of partial gauge fixing.

2 Carrollian Spacetimes as Fiber Bundles

The Intrinsic Definition

A $d+1$-dimensional Carrollian spacetime $\mathcal{C}$ is elegantly described in terms of a fiber bundle, with one-dimensional fibers, and a d-dimensional base $\mathcal{S}$ thought of as the space, the fiber being the time. As usual, the bundle structure provides a projection $\pi : \mathcal{C} \to \mathcal{S}$, which defines in turn a surjective linear map between the corresponding tangent bundles, $\mathrm{d}\pi : T\mathcal{C} \to T\mathcal{S}$. It is convenient to choose a local coordinate system $x = \{t, \mathbf{x}\}$ such that the action of the projector simplifies to $\pi : (t, \mathbf{x}) \to \mathbf{x}$, that is, t is the fiber coordinate.

One can define a vertical subbundle as $V = \ker(\mathrm{d}\pi)$. The above coordinate set has been chosen such that V is given by all sections of $T\mathcal{C}$ proportional to ∂_t (vectors of the vertical tangent subspace $V_{(t,\mathbf{x})}$ are of the form $W^t\partial_t$). In order to split the tangent space $T_{(t,\mathbf{x})}\mathcal{C}$ into a direct sum of vertical and horizontal components, $V_{(t,\mathbf{x})} \oplus H_{(t,\mathbf{x})}$, smooth everywhere *i.e.* valid for the tangent bundle, $T\mathcal{C} = V \oplus H$, one needs an *Ehresmann connection.* With this connection, the linear map $\mathrm{d}\pi$ restricted to $H_{(t,\mathbf{x})}$ sets a one-to-one correspondence between $H_{(t,\mathbf{x})}$ and $T_{\mathbf{x}}\mathcal{S}$. This allows to lift vertically vectors $W = W^i\partial_i \in T_{\mathbf{x}}\mathcal{S}$ to $\bar{W} = W^i E_i \in H_{(t,\mathbf{x})}$, where

$$E_i = \partial_i + b_i \partial_t, \quad i = 1, \dots, d \tag{1}$$

provide a basis for $H_{(t,\mathbf{x})}$. The Ehresmann connection is encoded in the one-form field $\boldsymbol{b} = b_i(t,\mathbf{x})\mathrm{d}x^i \in T^*\mathcal{C}$.

The Ehresmann connection has many facets. On the one hand, it provides a lift of curves in $\mathcal{S}$ onto curves in $\mathcal{C}$ such that the tangent vectors to the latter are horizontal. On the other hand, it makes it possible to realize the splitting $T\mathcal{C} = V \oplus H$ through the definition of the projector p acting on $T\mathcal{C}$ with

[2]Note that these ingredients all appear within the context of Carrollian fluids and the fluid-gravity correspondence, as in Refs. [13,14].

image V and kernel H:

$$p = \partial_t \otimes (\mathrm{d}t - b_i(t,\mathbf{x})\mathrm{d}x^i). \tag{2}$$

We will call the fiber bundle $\mathcal{C}$ a Carrollian spacetime, once endowed with a degenerate metric g whose one-dimensional kernel coincides with the vertical subbundle V:

$$g(X,.) = 0, \quad \forall X \in V. \tag{3}$$

In the local coordinate system this imposes the metric be of the form

$$g = g_{ij}(t,\mathbf{x})\,\mathrm{d}x^i \otimes \mathrm{d}x^j. \tag{4}$$

providing a time-dependent notion of distances.

At this point of the presentation, it is worth mentioning that the triple $(\mathcal{C}, V, g)$ corresponds to the definition of a *weak* Carrollian structure given in [2]. Together with this triple, the Ehresmann connection defines a Leibnizian structure [16–18]. From the spacetime viewpoint, the fiber-bundle structure and the accompanying Ehresmann connection are the key ingredients for the intrinsic horizontal versus vertical splitting of the tangent bundle, and more generally of any tensor bundle.

The coordinate system $\{t, \mathbf{x}\}$ is adapted to the splitting at hand, as is any new chart obtained through the transformation

$$t \mapsto t'(t,\mathbf{x}) \quad \text{and} \quad \mathbf{x} \mapsto \mathbf{x}'(\mathbf{x}). \tag{5}$$

The motivation for introducing the fiber-bundle structure is, among others, to make these diffeomorphisms natural, being a reparameterization of the fiber coordinate at each spatial point and a change of coordinates on the base, respectively. With this, the Jacobian matrix $J^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ is upper triangular:

$$\begin{pmatrix} J(t,\mathbf{x}) & J_i(t,\mathbf{x}) \\ 0 & J^j_i(\mathbf{x}) \end{pmatrix}, \tag{6}$$

since

$$\mathrm{d}t' = J(t,\mathbf{x})\mathrm{d}t + J_i(t,\mathbf{x})\mathrm{d}x^i, \quad \mathrm{d}x'^j = J^j_i(\mathbf{x})\mathrm{d}x^i, \tag{7}$$

or equivalently

$$\partial_t = J(t,\mathbf{x})\partial'_t, \quad \partial_i = J_i(t,\mathbf{x})\partial'_t + J^j_i(\mathbf{x})\partial'_j. \tag{8}$$

These diffeomorphisms were called *Carrollian* in [13]. Every spacetime tensor field can be decomposed intrinsically into vertical and horizontal components, the latter transforming tensorially under Carrollian diffeomorphisms. These components are the *Carrollian tensors* introduced in [13]. An example of Carrollian tensor is the degenerate metric (4), whose components transform as

$$g'_{ij} = J^{-1}{}^k_i J^{-1}{}^\ell_j g_{k\ell} \tag{9}$$

i.e., as a rank-$(0,2)$ Carrollian tensor field. In order to maintain p in Eq. (2) invariant, the components of the Ehresmann connection must transform as:

$$b'_j = J^{-1}{}^i_j \left(J b_i + J_i\right). \tag{10}$$

For reasons that will become clear in the course of the paper, it is convenient to introduce a density $\Omega(t,\mathbf{x})$, transforming under Carrollian diffeomorphisms as:[3]

$$\Omega'(t',\mathbf{x}') = J(t,\mathbf{x})^{-1}\Omega(t,\mathbf{x}). \tag{11}$$

[3]Observe that $\boldsymbol{b}$ transforms as a Carrollian connection density. Strictly speaking, the Ehresmann connection is thus $\boldsymbol{b}\Omega$. To avoid confusion, we should mention that the latter combination was precisely used as $\boldsymbol{b}$ in [13, 14].

With this density, one defines a new basis vector of $V_{(t,\mathbf{x})}$ as

$$E = \Omega(t,\mathbf{x})^{-1}\partial_t. \tag{12}$$

Together with the $H_{(t,\mathbf{x})}$ basis vectors E_i defined in (1), we obtain a frame $E_\mu, \mu = 0, \dots, d$, adapted to the split tangent space and transforming canonically under Carrollian diffeomorphisms ($E_0 \equiv E$):

$$E' = E \quad \text{and} \quad E'_i = J^{-1}{}^j_i E_j. \tag{13}$$

The dual coframe, generically referred to as $\boldsymbol{e}^\mu, \mu = 0, \dots, d$, is ($\boldsymbol{e}^0 \equiv \boldsymbol{e}$)

$$\boldsymbol{e} = \Omega\left(\mathrm{d}t - b_j \mathrm{d}x^j\right) \quad \text{and} \quad \boldsymbol{e}^i = \mathrm{d}x^i, \quad i = 1, \dots, d \tag{14}$$

with $\boldsymbol{e}^i$ transforming as in (7) and $\boldsymbol{e}' = \boldsymbol{e}$.

Any vector $W \in T\mathcal{C}$ is decomposed in the above frame as $W = W^0(t,\mathbf{x})E + W^i(t,\mathbf{x})E_i$, while any form $\boldsymbol{\omega} \in T^*\mathcal{C}$ is $\boldsymbol{\omega} = \omega_0(t,\mathbf{x})\boldsymbol{e} + \omega_i(t,\mathbf{x})\boldsymbol{e}^i$. In this basis, the vertical and horizontal components are reduced, *i.e.*, do not mix under Carrollian diffeomorphisms. The vertical components remain invariant, while the horizontal transform tensorially under Carroll diffeomorphisms:

$$W'^0 = W^0, \quad W'^i = J^i_j W^j, \quad \omega'_0 = \omega_0, \quad \omega'_i = J^{-1}{}^j_i \omega_j. \tag{15}$$

From the horizontal perspective W^0 and ω_0 are scalars, and we refer to them as *Carrollian scalars*, whereas W^i and ω_i are components of a *Carrollian vector* and a *Carrollian one-form*. The same reduction properties are valid for rank-(r,s) tensor fields in $T^{(r,s)}\mathcal{C}$. Notice that one can use $g_{ij} = g(E_i, E_j)$ and its inverse g^{ij} for lowering and raising spatial indices $i, j, \dots$ amongst Carrollian tensors.

In terms of the frame (1), (12), and the coframe (14), the action of the exterior derivative on the generic one form $\boldsymbol{\omega}$ reads:

$$\mathrm{d}\boldsymbol{\omega} = (E(\omega_i) - E_i(\omega_0))\boldsymbol{e} \wedge \boldsymbol{e}^i + E_k(\omega_i)\boldsymbol{e}^k \wedge \boldsymbol{e}^i. \tag{16}$$

One can define the Ehresmann curvature as

$$\mathrm{d}\boldsymbol{e} = \boldsymbol{\varphi} \wedge \boldsymbol{e} + \boldsymbol{\varpi} = \varphi_i \boldsymbol{e}^i \wedge \boldsymbol{e} + \tfrac{1}{2}\varpi_{ij}\, \boldsymbol{e}^i \wedge \boldsymbol{e}^j, \tag{17}$$

which exhibits a pair of genuine Carrollian tensors. The purely horizontal piece $\boldsymbol{\varpi}$ is a Carrollian two-form, which we will call the *Carrollian torsion*.[4] It has components

$$\varpi_{ij} = -\Omega\left(E_i(b_j) - E_j(b_i)\right). \tag{18}$$

The vertical-horizontal mixed components

$$\varphi_i = \Omega E(b_i) + E_i(\ln \Omega), \tag{19}$$

define a Carrollian one-form $\boldsymbol{\varphi}$, the *acceleration.* Both appear in the Lie bracket of the basis vectors:

$$[E_i, E_j] = -\varpi_{ij} E, \quad [E_i, E] = -\varphi_i E, \tag{20}$$

which is dual to (17).

A natural question to ask is whether H can be thought of as the tangent bundle of codimension-one hypersurfaces in $\mathcal{C}$. If this holds, $\mathcal{C}$ is foliated by a family of hypersurfaces modeled on $\mathcal{S}$. This is indeed possible whenever H is an *integrable distribution* in $T\mathcal{C}$. The corresponding integrability condition originates from Fröbenius' theorem stating that the Lie bracket of horizontal vectors must be horizontal, or equivalently, that the vorticity of the normal (vertical) vector should vanish: $\varpi_{ij} = 0$. In other words, the Ehresmann curvature should have no horizontal component.

Besides the above Carrollian tensors emanating from the Ehresmann connection, others can be defined using the metric g. Those are of two kinds.

[4]The quantity $-\frac{1}{2}\varpi_{ij}$ is also referred to as the Carrollian vorticity of the vector field E [13].

1. The first is based on first time-derivatives (the metric components are generically functions of both t and $\mathbf{x}$):

$$\theta = E\left(\ln\sqrt{\det g}\right) = \frac{1}{2}g^{ik}E(g_{ki}), \quad \zeta_{ij} = \frac{1}{2}E(g_{ij}) - \frac{\theta}{d}g_{ij}, \tag{21}$$

   referred to as *expansion* and *shear* (g^{ij} are the components of the inverse of g). They are respectively a Carrollian scalar and a Carrollian symmetric and traceless rank-two tensor. The latter vanishes if and only if the time dependence in the metric is factorized: $g_{ij}(t,\mathbf{x}) = \mathrm{e}^{2\sigma(t,\mathbf{x})}\tilde{g}_{ij}(\mathbf{x})$, in which case the expansion reads $\theta = d\,E(\sigma)$. This instance will turn out to play a significant role later in the discussion of BMS symmetry (Sec. 3).

2. The second class is second-order in derivatives, and corresponds to the curvature of a generalized Levi–Civita connection. This is a canonical connection, which defines a horizontal parallel transport *i.e.* a covariant derivative acting on Carrollian tensors and producing new Carrollian tensors. It was introduced in [13] as $\mathrm{D} = \mathrm{E} + \gamma$, dubbed Levi–Civita–Carroll, with γ the Christoffel–Carroll symbols:

$$\gamma^i_{jk}(t,\mathbf{x}) = \frac{1}{2}g^{il}\left(E_j(g_{lk}) + E_k(g_{lj}) - E_l(g_{jk})\right) = \Gamma^i_{jk} + c^i_{jk}, \tag{22}$$

   where Γ^i_{jk} are the ordinary Christoffel symbols and

$$c^i_{jk}(t,\mathbf{x}) = \frac{\Omega}{2}g^{il}\left(b_jE(g_{lk}) + b_kE(g_{lj}) - b_lE(g_{jk})\right). \tag{23}$$

   This connection, also cast as $\mathrm{D} = \nabla + \mathrm{c}$ with ∇ the Levi–Civita connection, is metric-compatible ($D_kg_{ij} = 0$), and since $\gamma^i_{[jk]} = 0$, the torsion is exclusively encoded in the commutator of E_i's, *i.e.* in ϖ_{ij}. Its curvature tensors can be worked out following [13]. As opposed to the ordinary Levi–Civita connection for Riemannian manifolds, the Levi–Civita–Carroll is not the unique metric-compatible and torsionless connection one can define on $T\mathcal{C}$. This question has been addressed *e.g.* in [17, 18].

In the spirit of [2–4], one can introduce the concept of *flat Carrollian spacetime* given in an adapted coordinate system by

$$g_{ij} = \delta_{ij}, \quad \Omega = 1, \quad b_i = \text{const.} \tag{24}$$

For this case, the Ehresmann curvature ϖ_{ij} as well as the acceleration φ_i, the shear ζ_{ij} and the expansion θ vanish, as do the Christoffel–Carroll symbols written above. Carrollian flatness implies the Ehresmann connection being a pure gauge.

Realization on Null Hypersurfaces

We would like now to discuss the appearance of the above structures on null hypersurfaces $\mathcal{C}$ of a Lorentzian spacetime $\mathcal{M}$. The pull-back g of the ambient metric on null hypersurfaces is degenerate with one-dimensional tangent subbundle kernel V, and from this perspective the Carrollian structure encompassed in the triple $(\mathcal{C}, V, g)$ emerges naturally. This feature has been discussed by several authors, the more complete account being in the already quoted Ref. [6]. Our fiber bundle with Ehresmann connection approach, which is designed for separating explicitly Carrollian time and space, emerges naturally in null embeddings. This requires appropriate gauge-fixing in the ambient Lorentzian spacetime.[5]

We will illustrate the above in the case of a $d+2$-dimensional spacetime $\mathcal{M}$ foliated with null hypersurfaces. In this case the ambient metric reads

$$\mathrm{d}s^2_{\mathcal{M}} = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = -2\Omega\Xi\left(\mathrm{d}t - b_i\mathrm{d}x^i + \theta^t\mathrm{d}r - b_i\theta^i\mathrm{d}r\right)\mathrm{d}r + g_{ij}\left(\mathrm{d}x^i + \theta^i\mathrm{d}r\right)\left(\mathrm{d}x^j + \theta^j\mathrm{d}r\right), \tag{25}$$

[5] See [19] for a recent discussion on foliations and symmetries that preserve them.

where Ω, Ξ, b_i, θ^t, θ^i and g_{ij} depend on all the coordinates $(r, t, \mathbf{x})$ and t is a retarded time. The constant-r leaves of the foliation $\mathcal{C}_r$ define $d+1$-dimensional null hypersurfaces because the pull-back of the metric, $g_r = g_{ij}(r, t, \mathbf{x})\mathrm{d}x^i\mathrm{d}x^j$, is indeed degenerate. The diffeomorphisms that preserve the form of this metric are

$$r \mapsto r'(r), \quad t \mapsto t'(r, t, \mathbf{x}), \quad \mathbf{x} \mapsto \mathbf{x}'(r, \mathbf{x}). \tag{26}$$

Defining as usual

$$J^a_b = \frac{\partial x'^a}{\partial x^b}, \tag{27}$$

the various quantities involved transform as

$$\begin{aligned}
\Omega' &= \left(J^t_t\right)^{-1}\Omega & (28)\\
b'_j &= J^{-1i}_{\ \ j}\left(J^t_t b_i + J^t_i\right) & (29)\\
g'_{ij} &= J^{-1k}_{\ \ i}J^{-1\ell}_{\ \ j}g_{k\ell} & (30)\\
\Xi' &= \left(J^r_r\right)^{-1}\Xi & (31)\\
\theta'^t &= \left(J^t_r\right)^{-1}\left(J^t_t\theta^t - J^t_r + J^t_i\theta^i\right) & (32)\\
\theta'^i &= \left(J^t_r\right)^{-1}\left(J^i_j\theta^j - J^i_r\right). & (33)
\end{aligned}$$

Therefore, we see that Ω, b_i and g_{ij} transform on every leaf as they do on a Carrollian spacetime, eqs. (11), (10), and (9). Hence, the diffeomorphisms (26) are interpreted as Carroll diffeomorphisms on each leaf $\mathcal{C}_r$. The other elements Ξ, θ^t and θ^i were not present in the intrinsic definition of the previous section. This is not surprising as they account for the non-trivial r-dependence of the residual gauge symmetry (26). For simplicity we will fix locally $\Xi = 1$ and $\theta^t = \theta^i = 0$. This is achievable using (and therefore fixing) the r-dependence of the diffeomorphism (26). Henceforth the bulk metric simplifies to

$$\mathrm{d}s^2_{\mathcal{M}} = -2\Omega\left(\mathrm{d}t - b_i\mathrm{d}x^i\right)\mathrm{d}r + g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{34}$$

with the residual gauge freedom (5):

$$r \mapsto r, \quad t \mapsto t'(t, \mathbf{x}), \quad \mathbf{x} \mapsto \mathbf{x}'(\mathbf{x}). \tag{35}$$

Indeed, if we were to describe a single null hypersurface, it would also be natural to set, $\Xi = 1$, and θ^i and θ^t to zero in its neighborhood. Under the coordinate change (35), g_{ij}, b_i and Ω still transform according to (11), (10), and (9). One can show that $\mathcal{C}_r$ equipped with these data is a $d+1$-dimensional Carrollian spacetime, in the lines we have discussed earlier. For this we need to exhibit the Ehresmann connection.

The ambient metric (25) allows to define two independent null vector fields,[6] sections of $T\mathcal{M}$:

$$\ell = \frac{1}{\Omega}\partial_t, \quad n = \partial_r, \quad \ell \cdot n = 1. \tag{36}$$

The corresponding forms in $T^*\mathcal{M}$ are

$$\boldsymbol{\ell} = -\mathrm{d}r, \quad \boldsymbol{n} = \Omega\left(\mathrm{d}t - b_i\mathrm{d}x^i\right). \tag{37}$$

Hence, the vector field ℓ is normal to $\mathcal{C}_r$. Since it is null, it is also tangent to $\mathcal{C}_r$ and belongs therefore to $T\mathcal{C}_r$. Being the kernel of the degenerate metric g_r on $\mathcal{C}_r$, it spans the vertical subbundle V_r. The horizontal subbundle H_r is given by the set of vectors X in $T\mathcal{C}_r$ that are orthogonal to n:

$$X \cdot n = 0; \tag{38}$$

[6] Our choice of gauge fixing differs from other works as [6, 20].

but since $X \in H_r$, by definition

$$X \cdot \ell = 0. \tag{39}$$

Thus, writing $X = X^r\partial_r + X^t\partial_t + X^i\partial_i$, Eqs. (38) and (39) lead to $X^r = 0$ and $X^t - b_i X^i = 0$, so that

$$X \in H_r \Leftrightarrow X = X^i\left(\partial_i + b_i\partial_t\right) = X^i E_i. \tag{40}$$

Consequently, the field $b_i(r, t, \mathbf{x})$ plays the role of an Ehresmann connection for each null leave $\mathcal{C}_r$, as one could have anticipated. Notice also that the tensor $p^a{}_b = \ell^a n_b$ has non-zero components $p^t{}_t$ and $p^t{}_i$. These define a Carrollian tensor, which is the vertical Ehresmann projector p introduced in (2).

Given the above embedding of null hypersurfaces $\mathcal{C}_r$, we can determine their extrinsic geometry. This is generally captured by three quantities: the surface gravity, the deformation tensor and the twist, all built with the projector onto $H_r \subset T\mathcal{C}_r$:

$$h^a_b = \delta^a_b - n^a\ell_b - \ell^a n_b. \tag{41}$$

Lowering an index we find that the non-zero components are $h_{ij} = g_{ij}(r, t, \mathbf{x})$ and the surface gravity vanishes with our choice of ℓ. The other extrinsic quantities are respectively given by

$$\begin{aligned} D^{ab} &= \frac{1}{2}h^{ac}h^{bd}\mathcal{L}_\ell\, h_{cd}, \\ \omega_a &= h^b_a n_c \nabla_b \ell^c, \end{aligned} \tag{42}$$

where ∇_a stands for the Levi–Civita connection of g_{ab}. In addition, the deformation tensor is reduced to the expansion and the shear:

$$\begin{aligned} \Theta &= h_{ab}D^{ab} = \frac{1}{2}h^{ab}\mathcal{L}_\ell\, h_{ab}, \\ \sigma^{ab} &= D^{ab} - \frac{\Theta}{d}h^{ab}. \end{aligned} \tag{43}$$

For the geometry at hand, the non vanishing components of the extrinsic tensors, at every r, coincide with the Carrollian tensors defined on $\mathcal{C}_r$ (see (19), (21)):

$$\begin{aligned} \omega_i &= -\frac{1}{2}\partial_t b_i - \frac{1}{2\Omega}\left(\partial_i\Omega + b_i\partial_t\Omega\right) = -\frac{1}{2}\varphi_i, \\ \Theta &= \frac{1}{\Omega}\partial_t \ln\sqrt{g} = \theta, \\ \sigma_{ij} &= \frac{1}{2\Omega}\partial_t g_{ij} - \frac{\Theta}{d}g_{ij} = \zeta_{ij}. \end{aligned} \tag{44}$$

The reduced bulk covariance (26), which preserves the form (25), corresponds precisely to the Carrollian diffeomorphisms (5), for which these objects are genuine tensors.

In conclusion, before we turn to the investigation of conformal isometries, the message is that the definition of Carrollian spacetimes as fibre bundles with Ehresmann connection and a degenerate metric is adapted to the description of families of embedded null hypersurfaces where, on any leaf, the induced geometry is Carrollian.

3 Conformal Carrollian Isometries

Carrollian spacetimes $\mathcal{C}$ have been introduced in Sec. 2 irrespective of any isometry properties. Carrollian diffeomorphisms are not isometries. They are a subgroup of the full diffeomorphism group, compatible with the intrinsic splitting in vertical versus horizontal components of the tangent bundle $T\mathcal{C}$, made possible thanks to the Ehresmann connection. The Carroll group emerges precisely on the tangent space at a point. Suppose indeed that we trade the H basis vectors E_i for a set of vectors $\hat{E}_{\hat{\imath}}$, orthonormal with respect to g: $g\big(\hat{E}_{\hat{\imath}}, \hat{E}_{\hat{\jmath}}\big) = \delta_{\hat{\imath}\hat{\jmath}}$. The tangent space is now everywhere spanned by $\{E, \hat{E}_{\hat{\imath}}, \hat{\imath} = 1, \ldots, d\}$, whereas for the cotangent space the basis is $\{\boldsymbol{e}, \hat{\boldsymbol{e}}^{\hat{\imath}}, \hat{\imath} = 1, \ldots, d\}$ with $\hat{\boldsymbol{e}}^{\hat{\imath}}\big(\hat{E}_{\hat{\jmath}}\big) = \delta^{\hat{\imath}}_{\hat{\jmath}}$. Automorphisms of the tangent space preserving the vertical vector field E and the orthonormal nature of the H basis are generally as follows:

$$\begin{pmatrix} E' & \hat{E}'_{\hat{\imath}} \end{pmatrix} = \begin{pmatrix} E & \hat{E}_{\hat{\jmath}} \end{pmatrix} \begin{pmatrix} 1 & B_{\hat{k}} R^{\hat{k}}{}_{\hat{\imath}} \\ 0 & R^{\hat{\jmath}}{}_{\hat{\imath}} \end{pmatrix} \tag{45}$$

with $R^{\hat{k}}{}_{\hat{\imath}}(t, \mathbf{x})$ the elements of a d-dimensional orthogonal matrix and $B_{\hat{k}}(t, \mathbf{x})$, d numbers. The explicit dependence on the coordinates underlines that this transformation needs not be the same at every point of $\mathcal{C}$. These transformations are the $d+1$-dimensional Carroll boosts (the full Carroll group also includes spacetime translations). They rotate the horizontal frame and coframe, and produce a rotation plus a shift proportional to B on the Ehresmann connection. This latter statement can be made explicit by writing

$$\hat{E}_{\hat{\imath}} = E^j{}_{\hat{\imath}} \partial_j + b_{\hat{\imath}} \partial_t; \tag{46}$$

the transformation (45) thus implies

$$E'^j{}_{\hat{\imath}} = E^j{}_{\hat{k}} R^{\hat{k}}{}_{\hat{\imath}} \quad \text{and} \quad b'_{\hat{\imath}} = \left(b_{\hat{k}} + \Omega^{-1} B_{\hat{k}}\right) R^{\hat{k}}{}_{\hat{\imath}}. \tag{47}$$

The Carroll boosts play for the tangent bundle of a Carrollian spacetime the same role as the Lorentz group does for the tangent bundle of a pseudo-Riemannian manifold.

The Carroll group appears also as the isometry group of the flat Carroll manifold introduced in Eqs. (24). These isometries are diffeomorphisms generated by vectors ξ such that $\mathcal{L}_\xi g = 0$, $\mathcal{L}_\xi E = 0$, and shifting the Ehresmann connection by an arbitrary constant. One finds:

$$\xi^0 = \beta_j x^j + \gamma, \quad \xi^i = \omega^i{}_j x^j + \epsilon^i \tag{48}$$

with all entries constant and $\omega_{kj} = \delta_{ki} \omega^i{}_j$ antisymmetric. These are precisely the $(d+2)(d+1)/2$ generators of the Carroll algebra $\mathfrak{carr}(d+1)$.

We would like to enter now the core of our discussion about conformal Carrollian isometries for generic Carrollian spacetimes. We will first define them, and then solve the associated differential equations under the assumption of the absence of shear. This will enable us to exhibit a rather universal algebra, which gives a generalized version of the infinite-dimensional conformal Carroll algebra $\mathfrak{ccarr}(d+1)$.

We define Carrollian conformal Killing vector fields ξ by imposing

$$\mathcal{L}_\xi g = \lambda g, \tag{49}$$

where $\lambda(t, \mathbf{x})$ is an *a priori* arbitrary function. Setting $\xi = f(t, \mathbf{x}) E + \xi^i(t, \mathbf{x}) E_i$ we obtain:

$$\begin{aligned} \mathcal{L}_\xi g &= \left(2 g_{ij} \partial_t \xi^i\right) \mathrm{d}t \mathrm{d}x^j + \left(\left(\Omega^{-1} f + b_k \xi^k\right) \partial_t g_{ij} + \xi^k \partial_k g_{ij} + g_{ik} \partial_j \xi^k + g_{jk} \partial_j \xi^k \right) \mathrm{d}x^i \mathrm{d}x^j && (50) \\ &= \left(2 \Omega^{-1} g_{ij} \partial_t \xi^i\right) \boldsymbol{e}\boldsymbol{e}^j + \left(2 f \left(\zeta_{ij} + \tfrac{1}{d} \theta g_{ij}\right) + D_i \xi_j + D_j \xi_i\right) \boldsymbol{e}^i \boldsymbol{e}^j, && (51) \end{aligned}$$

where D_i stands for the Levi–Civita–Carroll connection introduced in (22). Observe that the time dependence of the metric enters these expressions explicitly and one might expect it to alter significantly the structure of the conformal isometry algebra. At the same time one should also stress that *in the absence of time dependence*, neither the Ehresmann connection nor the scale factor $\Omega(t, \mathbf{x})$ play a role in the analysis of conformal properties, which would reduce to the analysis in [2–4].[7] The first term of (51) translates through Eq. (49) into

$$\partial_t \xi^i(t, \mathbf{x}) = 0. \tag{52}$$

This imposes that ξ is the generator of a Carrollian diffeomorphism (it ensures the vanishing entry in (6) since it imposes $\xi^i(t, \mathbf{x}) = Y^i(\mathbf{x})$), and this is assumed systematically here. Hence the core of the definition of conformal Carrollian isometries is in the second term of (51), leading to

$$2f\left(\zeta_{ij} + \frac{1}{d}\theta g_{ij}\right) + D_i Y_j + D_j Y_i = \lambda g_{ij}. \tag{53}$$

The trace of this equation determines λ,

$$\lambda(t, \mathbf{x}) = \frac{2}{d}\left(f\theta + D_i Y^i\right)(t, \mathbf{x}), \tag{54}$$

and substitution back into (53) then gives

$$D_i Y_j + D_j Y_i - \frac{2}{d} D_k Y^k g_{ij} = -2f\zeta_{ij}. \tag{55}$$

At the present stage, the equations to be solved for finding the components of the conformal Killing vectors $f(t, \mathbf{x})$ and $Y^i(\mathbf{x})$ are Eqs. (55), which are a set of time-dependent partial differential equations sourced by the Carrollian shear.

In the Carrollian case under consideration, as a consequence of the degenerate nature of the metric, this set – in other words Eq. (49) – is not sufficient for defining conformal Killing fields. In order to proceed, we must refine our definition of the latter. We will further impose vanishing shear for the Carrollian spacetime, and with this the full conformal algebra can be unravelled without any further restriction on the Carrollian data g_{ij}, Ω and b_i, generalizing thereby the range of validity of the results obtained in [2–4].

We note that for $\xi = f(t, \mathbf{x})E + Y^i(\mathbf{x})E_i$, the Lie derivative of the vertical vector field E is itself vertical, satisfying

$$\mathcal{L}_\xi E = \mu E, \tag{56}$$

where

$$\mu(t, \mathbf{x}) = -E(f) - \varphi_i Y^i. \tag{57}$$

A precise definition of the conformal Carrollian Killing vectors is reached by setting a relation among the *a priori* independent functions $\lambda(t, \mathbf{x})$ and $\mu(t, \mathbf{x})$. The guideline for this is Weyl covariance, because a desirable feature for conformal Killing fields is their insensitivity to Weyl rescalings of the metric.

We define Weyl rescalings as $g \mapsto g/\mathcal{B}(t, \mathbf{x})^2$ and $\boldsymbol{b}$ invariant (this is required for the spatial vectors E_i in (1) to remain well-defined), supplemented with $\Omega(t, \mathbf{x}) \mapsto \mathcal{B}(t, \mathbf{x})^{-z}\Omega(t, \mathbf{x})$ for some real number z, the dynamical exponent. Under such rescalings, ξ has Weyl weight zero which implies that Y^i and f have weights zero and $-z$. Therefore $\lambda(t, \mathbf{x})$ and $\mu(t, \mathbf{x})$ transform as

$$\lambda \mapsto \lambda - 2Y^i E_i(\ln \mathcal{B}), \quad \mu \mapsto \mu + zY^i E_i(\ln \mathcal{B}). \tag{58}$$

Thus, the combination $2\mu + z\lambda$ is Weyl covariant (actually invariant). Setting it to zero

$$2\mu(t, \mathbf{x}) + z\lambda(t, \mathbf{x}) = 0 \tag{59}$$

is compatible with the basic expected attributes of Killing vectors, as stressed earlier.

Equations (49) and (59) define our conformal Killing fields. It should be mentioned that (59) was introduced in [2–4] with $z = -2/N$ and N a positive integer, following the requirement that $\mathcal{L}_\xi\left(g \otimes E^{\otimes N}\right) = 0$.

[7] Notice that $\xi = \left(\Omega^{-1} f + b_k \xi^k\right)\partial_t + \xi^i \partial_i$. Equation (50) depends on b_k and Ω only through $\xi^t \equiv \Omega^{-1} f + b_k \xi^k$.

Leaving z arbitrary does not support such a geometrical interpretation, but is nonetheless consistent. The case $z = 1$ (*i.e.*, $N = 2$), where time and space equally dilate, pertains when the Carrollian spacetime emerges on an embedded null hypersurface in a pseudo-Riemannian geometry.

The combination of (54), (57) and (59) leads to[8]

$$D_i Y^i - \frac{d}{z}\varphi_i Y^i - \frac{d}{z}\left(E(f) - \frac{z}{d}\theta f\right) = 0. \tag{60}$$

Summarizing, the conformal isometry group as defined in (49) and (59) for a Carrollian spacetime described in terms of $\Omega(t,\mathbf{x})$, $b_i(t,\mathbf{x})$ and $g_{ij}(t,\mathbf{x})$ is the set of solutions $f(t,\mathbf{x})$ and $Y^i(\mathbf{x})$ of Eqs. (55) and (60) for a given choice of z.

At this point we will restrict our analysis to Carroll spacetimes with vanishing shear, $\zeta_{ij} = 0$, because in this case the system (55, 60) can be solved. As stated previously, ζ_{ij} vanishes if and only if the time dependence of the metric is conformal:

$$g_{ij}(t,\mathbf{x}) = \mathrm{e}^{2\sigma(t,\mathbf{x})}\tilde{g}_{ij}(\mathbf{x}). \tag{61}$$

Recall now that (55, 60) are Weyl covariant. Performing a Weyl rescaling with $\mathcal{B}(t,\mathbf{x}) = \mathrm{e}^{2\sigma(t,\mathbf{x})}$ removes the time-dependence from the metric, while it transforms the other fields as

$$\tilde{\Omega}(t,\mathbf{x}) = \mathrm{e}^{-z\sigma(t,\mathbf{x})}\Omega(t,\mathbf{x}), \qquad \tilde{\varphi}_i(t,\mathbf{x}) = \varphi_i(t,\mathbf{x}) - z(\partial_i + b_i(t,\mathbf{x})\partial_t)\sigma(t,\mathbf{x}), \qquad \tilde{\theta}(t,\mathbf{x}) = 0. \tag{62}$$

The Killing field is invariant, $\tilde{\xi} = \xi = \tilde{f}\tilde{E} + Y^i E_i$ with $\tilde{E} = \mathrm{e}^{z\sigma}E$, and this leads to

$$\tilde{f}(t,\mathbf{x}) = \mathrm{e}^{-z\sigma(t,\mathbf{x})}f(t,\mathbf{x}), \qquad \tilde{Y}^i(\mathbf{x}) = Y^i(\mathbf{x}), \qquad \tilde{Y}_i(\mathbf{x}) = \tilde{g}_{ij}(\mathbf{x})Y^j(\mathbf{x}). \tag{63}$$

Equations (55) and (60) finally become equations for $\tilde{f}(t,\mathbf{x})$ and $Y^i(\mathbf{x})$:

$$\tilde{\nabla}_i Y_j + \tilde{\nabla}_j Y_i = \frac{2}{d}\tilde{\nabla}_k Y^k \tilde{g}_{ij}, \tag{64}$$

$$\tilde{\Omega}^{-1}\partial_t \tilde{f} = \frac{z}{d}\tilde{\nabla}_k Y^k - \tilde{\varphi}_k Y^k, \tag{65}$$

where $\tilde{\nabla}_i$ is the Levi–Civita connection for $\tilde{g}_{ij}$.

The first equation is an ordinary conformal Killing equation, and its solutions $\{Y^i(\mathbf{x})\}$ are the generators of the conformal group for $\mathcal{S}$ equipped with a metric $\tilde{g}_{ij}(\mathbf{x})$. Given any such vector in H solving (64),

$$\bar{\xi}_Y = Y^i(\mathbf{x})E_i = Y^i(\mathbf{x})\left(\partial_i + b_i(t,\mathbf{x})\partial_t\right) \tag{66}$$

(the subscript "Y" stresses that the vector field at hand depends on the set $\{Y^i(\mathbf{x})\}$), Eq. (65) provides a solution for $\tilde{f}(t,\mathbf{x})$:

$$\tilde{f}(t,\mathbf{x}) = T(\mathbf{x}) + \frac{z}{d}\int^t \mathrm{d}t^*\, \tilde{\Omega}\left(t^*,\mathbf{x}\right)\left(\tilde{\nabla}_i Y^i(\mathbf{x}) - \frac{d}{z}\tilde{\varphi}_i\left(t^*,\mathbf{x}\right)Y^i(\mathbf{x})\right). \tag{67}$$

Here $T(\mathbf{x})$ is an arbitrary smooth function of weight $-z$, which specifies any conformal Carrollian Killing field.

Before we further investigate this family of conformal Carrollian Killing vectors, we should pause and make contact with previous results reached in the already quoted literature. The situation that has been studied in [4] corresponds in our language to $\sigma = 0$ and $\Omega = 1$. This means in particular that the metric is time-independent. In Ref. [4] no Ehresmann connection was introduced. We could therefore set it to zero,

[8] The left-hand side of Eq. (60) can actually be recast using Weyl-covariant derivatives, based on the *Weyl connection* $\boldsymbol{A} = \frac{1}{z}\boldsymbol{\varphi} + \frac{1}{d}\theta\boldsymbol{e}$, which transforms as $\boldsymbol{A} \mapsto \boldsymbol{A} - \mathrm{d}\ln\mathcal{B}$.

or better leave $b_i(t,\mathbf{x})$ unspecified, because, as mentioned earlier for a time-independent metric, it is not expected to play any role in the conformal algebra. Indeed, using (64) we find the precise family of vectors $\bar{\xi}_Y$ as in (66), which combined with (67) lead to

$$\xi_{T,Y} = \left(T(\mathbf{x}) + \frac{z}{d}t\tilde{\nabla}_i Y^i(\mathbf{x})\right)\partial_t + Y^i(\mathbf{x})\partial_i \tag{68}$$

irrespective of $b_i(t,\mathbf{x})$ (again the subscript "T,Y" reminds the dependence on $\{T(\mathbf{x}), Y^i(\mathbf{x})\}$). Therefore the corresponding algebra is infinite-dimensional and emerges as the semi-direct product of the conformal group of $g = \tilde{g}(\mathbf{x})$ on $\mathcal{S}$, generated by $Y^i(\mathbf{x})\partial_i$, with supertranslations. For a flat, or conformally flat metric on $\mathcal{S}$, the spatial conformal algebra in d dimensions is $\mathfrak{so}(d+1,1)$, and the conformal Carrollian Killing fields (68) span[9] $\mathfrak{ccarr}_N(d+1) = \mathfrak{so}(d+1,1) \ltimes \mathfrak{T}_N$, where $z = 2/N$. The standard conformal Carrollian algebra $\mathfrak{ccarr}(d+1)$ refers to dynamical exponent $z = 1$ (level $N = 2$): $\mathfrak{ccarr}(d+1) = \mathfrak{ccarr}_2(d+1)$. This algebra emerges as the null-infinity isometry algebra of asymptotically flat $d+2$-dimensional spacetimes in Bondi gauge, $\mathfrak{bms}(d+2)$.[10]

Our general analysis embraces the above case, by including time dependence in the spatial metric g and a general scale factor $\Omega(t,\mathbf{x})$ on top of the Ehresmann connection $b_i(t,\mathbf{x})$. Despite these generalizations, as a direct consequence of the factorized time dependence in the metric (see (61)) due to the requirement of vanishing shear, the structure of the conformal Carrollian Killing vectors remains unaltered *i.e.*, as in (68): their algebra is the semi-direct product of the conformal group of $\tilde{g}(\mathbf{x})$ on $\mathcal{S}$ with supertranslations at dynamical exponent z. This statement is shown as follows.

Using (67), we obtain the general conformal Carrollian Killings as vector fields in $T\mathcal{C}$:

$$\xi_{T,Y} = \left(T(\mathbf{x}) + \frac{z}{d}\int^t \mathrm{d}t^*\, \tilde{\Omega}\left(t^*,\mathbf{x}\right)\left(\tilde{\nabla}_i Y^i(\mathbf{x}) - \frac{d}{z}\tilde{\varphi}_i\left(t^*,\mathbf{x}\right)Y^i(\mathbf{x})\right)\right)\tilde{E} + Y^i(\mathbf{x})E_i. \tag{69}$$

We can unravel the structure of these conformal Carroll Killings and of their algebra by introducing an *invariant local clock*:

$$C(t,\mathbf{x}) \equiv \int^t \mathrm{d}t^*\, \tilde{\Omega}\left(t^*,\mathbf{x}\right). \tag{70}$$

This in fact is a specific instance of $C_\gamma = \int_\gamma \tilde{\Omega}(\mathrm{d}t - b)$ with γ a path in $\mathcal{C}$. In (70), $C(t,\mathbf{x})$ appears as a local function because the path runs along a vertical fibre starting at, say, the zero section, reference to which we have suppressed.[11] Using (19) and (70) we reach the following identity:

$$\int^t \mathrm{d}t^*\, \tilde{\Omega}\left(t^*,\mathbf{x}\right)\tilde{\varphi}_i\left(t^*,\mathbf{x}\right) = E_i\left(C(t,\mathbf{x})\right), \tag{71}$$

which enables us to express (69) as

$$\xi_{T,Y} = \left(T(\mathbf{x}) - Y^i E_i\left(C(t,\mathbf{x})\right) + \frac{z}{d}C(t,\mathbf{x})\tilde{\nabla}_i Y^i(\mathbf{x})\right)\tilde{E} + Y^i(\mathbf{x})E_i. \tag{72}$$

The invariant clock defines a Carrollian diffeomorphism (see (5)) with $t' = C(t,\mathbf{x})$ and $\mathbf{x}' = \mathbf{x}$. Under this diffeomorphism $\tilde{\Omega} \to 1$, $\tilde{E} \to \partial_{t'}$, while (72) reads now precisely as (68) with t traded for t'. This

[9] This algebra is defined in the literature for integer N.

[10] As before, strictly speaking this is valid for $d = 1$ and 2 (where furthermore $\tilde{g}$ is always conformally flat). For higher d, it was presumed to hold by some authors [3]. However, gauge conditions exist for the Bondi-gauge null-infinity behavior of asymptotically flat spacetimes that render $\mathfrak{bms}(d+2)$ finite-dimensional [21], and with this choice $\mathfrak{ccarr}_2(d+1) \neq \mathfrak{bms}(d+2)$. This does not exclude that less restrictive gauge fixing might be considered leading to other, possibly infinite-dimensional $\mathfrak{bms}(d+2)$ algebras for $d \geq 3$.

[11] We refer to $C(t,\mathbf{x})$ as invariant local clock because it defines an integration measure on each one-dimensional fiber, a proper time.

demonstrates the earlier statement about the algebra of conformal Carrollian Killing vectors of a shearless Carroll spacetime.

Summarizing, shearless Carrollian spacetimes, *i.e.* spacetimes equipped with a metric of the form $g_{ij}(t,\mathbf{x}) = \mathrm{e}^{2\sigma(t,\mathbf{x})}\tilde{g}_{ij}(\mathbf{x})$, have a conformal isometry algebra that depends only on $\tilde{g}(\mathbf{x})$, d and z: it is the semi-direct product of the conformal algebra of $\mathcal{S}$ equipped with $\tilde{g}(\mathbf{x})$ and supertranslations at level $N = 2/z$. This conclusion is valid irrespective of $\Omega(t,\mathbf{x})$ and $b_i(t,\mathbf{x})$. On the one hand, $\Omega(t,\mathbf{x})$ can disappear from the expression (72) of the Killings upon an appropriate Carrollian diffeomorphism driven by the invariant local clock. Hence its presence does not affect the algebra. On the other hand, although the Ehresmann connection $b_i(t,\mathbf{x})$ cannot be removed with Carrollian diffeomorphisms (unless its field strength $\boldsymbol{\varpi}$ and acceleration $\boldsymbol{\varphi}$ vanish), it cancels out between the last two terms in (72). This is not insignificant though, and we would like to discuss it in the remaining of the present chapter.

The set of vectors $Y = Y^i(\mathbf{x})\partial_i \in T\mathcal{S}$ with $\{Y^i(\mathbf{x})\}$ solving (64) realize the conformal algebra of $\tilde{g}$:

$$\left[Y, Y'\right] = \left[Y^i\partial_i, Y'^j\partial_j\right] = Y''^k\partial_k = Y'' \tag{73}$$

with

$$Y''^k = Y^i\partial_i(Y'^k) - Y'^i\partial_i(Y^k). \tag{74}$$

These vectors act generally on functions $\phi(\mathbf{x})$. One may instead contemplate a realization in terms of Carrollian vectors $\bar{\xi}_Y \in H$ as in (66) acting on functions $\Phi(t,\mathbf{x})$ of $\mathcal{C}$. In this case,

$$\left[\bar{\xi}_Y, \bar{\xi}_{Y'}\right] = \bar{\xi}_{[Y,Y']} - \boldsymbol{\varpi}(Y,Y')E = \bar{\xi}_{[Y,Y']} - \tilde{\boldsymbol{\varpi}}(Y,Y')\tilde{E} \in V \oplus H, \tag{75}$$

where $\boldsymbol{\varpi}(Y,Y') = \varpi_{ij}Y^iY'^j$ and $\tilde{\boldsymbol{\varpi}} = \mathrm{e}^{-z\sigma}\boldsymbol{\varpi}$. Because of the Ehresmann connection, this realization is not faithfully the conformal algebra (73) of $\tilde{g}$, except if the Carrollian torsion is zero (horizontal piece of the Ehresmann curvature), which coincides with the condition for H to be integrable[12] (or if the action is limited to functions of $\mathbf{x}$ only, which is not what we want). Furthermore the extra V-term is not a central extension, unless the Carrollian acceleration vanishes (in this case E and E_i commute).

The expression in parentheses present in (72) suggests to define, for each set $\{Y^i(\mathbf{x})\}$ associated with a solution of (64), a Carrollian operator M_Y acting on any function $\Phi(t,\mathbf{x})$ of $\mathcal{C}$ as

$$M_Y(\Phi) \equiv Y^iE_i(\Phi) - \frac{z}{d}\Phi\tilde{\nabla}_iY^i. \tag{76}$$

The mapping $Y \to M_Y$ is a representation of the group of conformal Killing vectors of $\tilde{g}$, which however is again not faithful as the commutator exhibits an extra term, similar to the one in (75), possibly vanishing in the same circumstances:

$$\left[M_Y, M_{Y'}\right](\Phi) \equiv M_Y\left(M_{Y'}(\Phi)\right) - M_{Y'}\left(M_Y(\Phi)\right) = M_{[Y,Y']}(\Phi) - \tilde{\boldsymbol{\varpi}}(Y,Y')\tilde{E}(\Phi). \tag{77}$$

Using now the map (76) and $\bar{\xi}_Y \in H$ given in Eq. (66), the conformal Killing field in $T\mathcal{C}$, Eq. (72), is recast as

$$\xi_{T,Y} = \big(T(\mathbf{x}) - M_Y(C)(t,\mathbf{x})\big)\tilde{E} + \bar{\xi}_Y. \tag{78}$$

For vanishing $T(\mathbf{x})$, the representation M_Y defines a lift of $\bar{\xi}_Y = Y^iE_i \in H \to T\mathcal{C}$ through the map

$$\bar{\xi}_Y \mapsto \xi_{0,Y} = \bar{\xi}_Y - M_Y(C)\tilde{E}. \tag{79}$$

This lift provides a *faithful and Carrollian (*i.e., *acting on functions of t and $\mathbf{x}$)* realization of the conformal isometry algebra (73) of $\tilde{g}$ *on* $T\mathcal{C}$, thanks to the cancellation of the extra term appearing in (75) and (77).

[12] Generally, one expects invariants that prevent the horizontal part of the Ehresmann connection from being flat. For example, in $d = 2$, one might have non-zero Chern class $c = \frac{1}{2\pi}\int_{\mathcal{S}}\boldsymbol{\varpi}$.

Even though the Ehresmann connection does not appear ultimately in the conformal algebra, when non-vanishing, it adjusts for making compatible the realization of the algebra with Carrollian diffeomorphism invariance. This is yet another of its numerous facets. For non-vanishing $T(\mathbf{x})$, we obtain the following commutation relations for the complete conformal Carrollian Killing fields (78):[13]

$$\left[\xi_{T,Y}, \xi_{T',Y'}\right] = \xi_{M_Y(T')-M_{Y'}(T),[Y,Y']}. \tag{80}$$

This is the usual pattern for conformal Carrollian and BMS algebras.

4 Conclusions

In this work, we have considered Carrollian geometries from various perspectives: their defining properties, their emergence on embedded null hypersurfaces and their conformal symmetries. We have emphasized the interpretation of Carrollian spacetime as a fiber bundle endowed with an Ehresmann connection. Realized by a one-form field, this connection defines the splitting of the tangent bundle into vertical and horizontal components. The vertical component coincides precisely with the kernel of a degenerate metric, which is the last piece of equipment for a Carroll structure. It is worth stressing that all defining fields (Ehresmann connection, metric and scale factor) have been assumed space *and* time-dependent throughout the paper.

The vertical versus horizontal canonical separation is preserved by the subset of Carrollian diffeomorphisms. These enable the reduction of spacetime tensors into purely spatial components, the paradigm being Carrollian torsion and acceleration, emerging as reduced components of the Ehresmann curvature. Other geometric objects can be introduced using the degenerate metric, such as shear and expansion, and even further based on a horizontal connection, which we only alluded to when discussing the Christoffel–Carroll symbols. Investigating the types of connections that can be defined on the full tangent bundle $T\mathcal{C}$ is an interesting subject that has been discussed in the literature, but remains incomplete and worth pursuing.

The above ingredients (Ehresmann connection, vertical and horizontal subbundles) arise naturally on null hypersurfaces embedded in Lorentzian spacetimes, and specific tensors such as Carrollian shear, acceleration and expansion are inherited from the ambient geometry. Our analysis was here confined to the instance of genuine null foliations, but can be adapted to the case of boundary null hypersurfaces, such as black-hole horizons or null infinities.

The last element of our investigation concerns symmetries, and more specifically conformal isometries of Carrollian spacetimes. Contrary to pseudo-Riemannian geometries, the definition of (conformal) isometries cannot rely solely on the Killing equation for the metric, because the latter is degenerate. Here we complied with the standard definition of the conformal Carrollian Killing vectors, and additionally restricted our analysis to the case of *shearless* Carrollian structures. Although seemingly innocuous, as time dependence remains general both in the scale factor and in the Ehresmann connection, this limitation is quite severe. Indeed time dependence of the metric is factorized and this ultimately drives us to the standard semi-direct product of the conformal isometry algebra of the metric with supertranslations. This is infinite-dimensional and coincides with $\mathfrak{ccarr}_N(d+1)$, for conformally flat spatial metrics. One thus recovers $\mathfrak{bms}(d+2)$ in $d=1$ and 2, and possibly in higher dimension with some appropriate definition of the BMS algebra. Our study has the virtue of sustaining the robustness of the format already known to emerge in static Carrollian spacetimes without scale factor or Ehresmann connection. It stresses the role of the shear, but leaves open the probe of the conformal Carrollian isometries, when the latter is non-zero. It also illustrates another subtle role of the Ehresmann connection, which allows to lift without alteration the conformal isometry algebra of the metric from the basis tangent bundle $T\mathcal{S}$ to the Carrollian tangent bundle $T\mathcal{C}$.

Although relatively confined, our investigation touches upon several timely and perhaps deep issues. Conformal symmetries and in particular the BMS algebra are known to appear as the backbone of conserved

[13] We use here the identity $\tilde{E}\left(M_Y(C)\right) = \tilde{\varphi}_i Y^i - \frac{z}{d}\tilde{\nabla}_i Y^i$.

charges in asymptotically flat spacetimes. Alongside, the role of null hypersurfaces has been appreciated in flat holography, where they are expected to replace the time-like foliations relevant in anti-de Sitter holography. In particular, their symplectic structure should play a significant role in giving an alternative reading of the gravitational degrees of freedom. Clearly, Carrollian spacetimes and their symmetries are the central concepts in all these developments, which deserve further analysis, possibly in the lines of our current work.

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Gravitation in flat spacetime from entanglement

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Abstract: We explore holographic entanglement entropy for Minkowski spacetime in three and four dimensions. Under some general assumptions on the putative holographic dual, the entanglement entropy associated to a special class of subregions can be computed using an analog of the Ryu-Takayanagi formula. We refine the existing prescription in three dimensions and propose a generalization to four dimensions. Under reasonable assumptions on the holographic stress tensor, we show that the first law of entanglement is equivalent to the gravitational equations of motion in the bulk, linearized around Minkowski spacetime.

Contents

1 Introduction

The AdS/CFT correspondence has been a fruitful avenue to understand quantum gravity in asymptotically AdS spacetimes. A question of interest is whether the holographic principle makes sense in more general spacetimes, such as our own universe. Some proposals have been made for de Sitter [1], Kerr [2] or warped AdS [3, 4]. The asymptotically flat case is particularly interesting because it can be obtained as a flat limit of AdS [5, 6]. Other approaches to flat space holography exist, such as applying AdS/CFT on hyperbolic foliations of Minkowski spacetime [7] or using the recently discovered equivalence between BMS Ward identities and Weinberg's soft theorems [8].

The flat space limit of AdS is an ultra-relativistic limit, or Carrollian limit, of the dual field theory. Already at the level of the symmetries, one can show that the conformal Carroll group is the BMS group [9], which is the symmetry group of asymptotically flat gravity [10]. More precisely, the conformal Carroll group associated with the future boundary, i.e. null infinity $\mathcal{I}^+$, is isomorphic to BMS_3 when $\mathcal{I}^+ = \mathbb{R} \times S^1$ and to BMS_4 when $\mathcal{I}^+ = \mathbb{R} \times S^2$. Therefore, the putative dual theory should enjoy a Carrollian symmetry. Recent works have been able to match the gravitational dynamics with ultra-relativistic conservation laws [11, 12]. This suggests that the holographic duals of asymptotically flat spacetimes should be Carrollian CFTs [13].

An important insight from AdS/CFT is the role of entanglement in the emergence of the bulk spacetime from the field theory degrees of freedom. The Ryu-Takayanagi prescription [14], and its covariant generalization [15], have lead to a more precise understanding of bulk reconstruction [16, 17] and a landmark result was the derivation of the gravitational equation, linearized around AdS, from the first law of entanglement in the CFT [18–20]. This suggests that linearized gravity can be understood as the thermodynamics of entanglement. Jacobson's earlier result [21], and its more recent refinements [22, 23], suggest that this connection is very general and goes beyond asymptotically AdS spacetimes. In this paper, we show that a similar result holds for flat space holography in three and four dimensions, under some general assumptions that allow us to use an analog of the Ryu-Takayanagi prescription.

Entanglement entropies in 3d Minkowski spacetime were considered in [24] and were matched with computations in conjectured dual theories. We will follow the geometrical picture proposed in [25], where the authors used a generalization of the CHM transformation [26], to propose an RT prescription for flat spacetime. This requires some assumptions on the putative dual theory which are given in full details below. Under the same working assumptions, we refine their 3d prescription to include perturbations and propose a generalization to 4d.

This paper is organized as follows. In Sec. 2 we detail our working assumptions on flat holography. This allows us to use an analog of the Ryu-Takayanagi prescription in Minkowski spacetimes. We review and generalize the existing 3d prescription in Sec. 3 to include perturbations. In Sec. 4 we prove that the gravitational equations, linearized around 3d Minkowski, follow from the first law of entanglement.[1] In Sec. 5 we perform a flat limit of AdS_3, also considered in [6, 28], to identify the holographic stress tensor associated of 3d Minkowski, a necessary ingredient for the proof. In Sec. 6 we generalize the RT prescription to 4d Minkowski and prove that the first law of entanglement is equivalent to the gravitational equations of motion. Our proof is valid for general theories of gravity.

[1] Before submitting our paper, we learned that another group is currently pursuing similar ideas [27].

2 Working assumptions on flat holography

Holography in asymptotically flat spacetimes is not well understood. The putative dual field theory should be defined on null surfaces and it is not clear how one should understand objects such as local operators or path integrals. Therefore, to obtain a well-defined equivalent of the Ryu-Takayanagi prescription, we need some general assumptions on holography in flat spacetime which are listed below:

- (Assumption 1) There exists a quantum system living on the future boundary $\mathcal{I}^+$, such that we can associate a Hilbert space $\mathcal{H}$ to any slice Σ of constant retarded time u. To any bulk configuration on Σ, we can associate a state in $\mathcal{H}$. For the purpose of this work, we could also weaken this assumption by taking the bulk configurations to be only linear perturbations of Minkowski.

- (Assumption 2) For a subregion A of $\partial\Sigma$ among a special class, we can associate a density matrix ρ_A. If the Hilbert space factorizes on subregions, we expect that $\rho_A = \mathrm{Tr}_{\bar{A}}|0\rangle\langle 0|$ where $\bar{A}$ is the complement of A on the slice and $|0\rangle$ is the Minkowski vacuum. We allow ρ_A to be only defined on some subspace $\mathcal{H}_{\text{code}}$ of $\mathcal{H}$.

The domain of dependence $\mathcal{D}$ of A is defined to be the union of all the images of A under translation along the u direction. This is simply the ultra-relativistic limit of the Lorentzian domain of dependence. Indeed, in this limit, the width of the lightcone vanishes (see Fig. 2 for an illustration). Following [25], we define a generalized Rindler transformation to be a symmetry transformation on $\mathcal{I}^+$ which maps $\mathcal{D}$ to a spacetime which has a thermal circle.[2] The generator ζ_A of the thermal identification, which is called the modular flow generator, is required to annihilate the vacuum and leave $\mathcal{D}$ and $\partial\mathcal{D}$ invariant. A Rindler transformation is a generalization of the CHM conformal transformation [26].

- (Assumption 3) If we can find a Rindler transformation, the density matrix can be written as $\rho_A = U^{-1}e^{-K_A}U$ where K_A is the operator that generate translations along the thermal circle and U is a unitary operator acting on the Hilbert space which implements the symmetry transformation. For this definition to make sense, K_A needs to be bounded from below in $\mathcal{H}_{\text{code}}$.

From the knowledge of the boundary modular flow ζ_A, one can find a bulk modular flow ξ_A. It is the Killing vector field of Minkowski spacetime which asymptotes to ζ_A.

- (Assumption 4) The expectation value $\delta\langle K_A\rangle$ for a linear perturbation of the vacuum is computed by the Iyer-Wald energy δE_A^{grav} associated to the Killing vector ξ_A of the corresponding bulk configuration on Σ.

- (Assumption 5) The von Neumann entropy $S_A = -\mathrm{Tr}\,\rho_A \log \rho_A$ is computed by the area[3] of the special bulk surface $\widetilde{A}$ that is preserved by the bulk modular flow ξ_A and is homologous to A. This is the analog of the Ryu-Takayanagi (RT) prescription and $\widetilde{A}$ will be called the RT surface.

[2] This means that one coordinate of the new spacetime should have an imaginary identification $x \sim x + i\beta$.
[3] Or the adequate functional for other theories than Einstein gravity.

These assumptions can be derived for holographic CFTs with AdS duals. There, the special class of entangling regions are spatial balls in the boundary CFT. Also, Assumptions 3 and 5 were obtained in [26] and Assumption 4 is a consequence of the AdS/CFT holographic dictionary. The RT prescription for more general entangling regions was derived in [29, 30].

In this work, we want to consider the implications of the above assumptions for flat holography. In particular, we will investigate the consequences of the first law of entanglement $\delta S_A = \delta\langle K_A\rangle$ which is valid for any quantum system where these objects can be defined. Paralleling the AdS story [19], we will show that the linearized gravitational equations of motion are equivalent to the first law. We believe that although the microscopic theory is not well understood, this approach can provide valuable insights about holography in non-AdS spacetimes.

The results that we have proven can also be phrased purely in classical gravity. We have shown that for linearized perturbations of Minkowski spacetime, the gravitational equations of motion are *equivalent* to the first law

$$\delta S_A^{\text{grav}} = \delta E_A^{\text{grav}}, \tag{2.1}$$

for a set of boundary regions A among a special class, and where S_A^{grav} is the gravitational entropy of the surface $\widetilde{A}$ defined to be the surface homologous to A and fixed by the Killing vector field ξ_A. The existence of a holographic theory such that $\delta S_A^{\text{grav}} = \delta S_A$ and $\delta E_A^{\text{grav}} = \delta\langle K_A\rangle$ provides a microscopic realization and an interpretation in term of entanglement which renders the first law automatic.

3 Ryu-Takayanagi prescription in 3d Minkowski

We consider three-dimensional flat spacetime in Bondi gauge

$$ds^2 = -du^2 - 2dudr + r^2 d\phi^2, \tag{3.1}$$

where $u = t - r$. The boundary is the null infinity $\mathcal{I}^+$ (at $r = \infty$) and the boundary metric is degenerate:

$$ds^2 = 0 \times du^2 + d\phi^2. \tag{3.2}$$

Let's pick a region A on $\mathcal{I}^+$. We would like to compute the entanglement entropy associated to A in a putative holographic theory living on $\mathcal{I}^+$. This can be computed with an analog of the Ryu-Takayanagi formula, which was proposed in [25]. In this section, we will review and refine this prescription.

3.1 Review of the 3d prescription

In [25], the authors proposed an RT prescription for 3d Minkowski spacetime by using a "generalized Rindler method". This consists of finding a transformation, which satisfies the same properties as the Casini-Huerta-Myers conformal mapping [26]. One should look for a symmetry transformation which maps the domain of dependence $\mathcal{D}$ of a subregion A to a Rindler spacetime characterized by a thermal identification. The modular flow generator, which is the generator of the thermal identification, is required to annihilate the vacuum and to leave $\mathcal{D}$ and $\partial\mathcal{D}$ invariant.

Let's consider an interval A on the boundary, it is characterized by its sizes ℓ_u and ℓ_ϕ in the u and ϕ directions. The authors of [25] were able to find a Rindler transformation for A and to derive a boundary modular flow. Then, the Rindler transformation was extended into the bulk by finding a suitable change of coordinates. The bulk image of the transformation is a flat space cosmological solution [31], which is the flat space analog of the hyperbolic black hole in AdS_3. This maps the entanglement entropy into thermal entropy, which is computed geometrically from the area of the horizon of the flat space cosmological solution. This leads to the following picture: the RT surface is the union of three curves

$$\widetilde{A} = \gamma_+ \cup \gamma \cup \gamma_-, \tag{3.3}$$

where $\gamma_\pm$ are two light rays emanating from the two extremities ∂A of the interval and γ is a bulk curve connecting γ_+ and γ_-. In Einstein gravity, the entanglement entropy is then obtained as

$$S_A = \frac{\mathrm{Length}(\gamma)}{4G}\,. \tag{3.4}$$

We illustrate this procedure in Fig. 1. This prescription is consistent with computations in conjectured dual theories [24]. This RT surface was also shown in [32] to correspond to an extremal surface. See also [33] for a discussion on the replica trick in this context.

We would like to consider more general theories of gravity and derive a first law. In a more general context, the RT configuration is the same but the entanglement entropy is given by Wald's functional

$$S_A = \int_{\widetilde{A}} \mathbf{Q}[\xi_A] \tag{3.5}$$

where ξ_A is the bulk modular flow reviewed below. As we will show, it is important to integrate over $\widetilde{A}$ here, instead of just γ, if we want to have a first law. In Einstein gravity, (3.5) reduces to (3.4) because Wald's functional vanishes when integrated on γ_+ and γ_-.

Generalized Rindler method. We are now going to review how the generalized Rindler method is implemented in [25]. The Rindler transformation in the 2d boundary theory is

$$\begin{aligned} u &= \frac{\sin(\frac{\ell_\phi}{2})}{\cosh\rho + \cos(\frac{\ell_\phi}{2})}\left(\tau + \frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})}\sinh\rho\right)\,, \\ \phi &= \arctan\left(\frac{\sin(\frac{\ell_\phi}{2})\sinh\rho}{1+\cos(\frac{\ell_\phi}{2})\cosh\rho}\right)\,. \end{aligned} \tag{3.6}$$

The thermal identification is given by $\rho \sim \rho + 2\pi i$. The boundary modular flow is the thermal generator $2\pi\partial_\rho$ which is

$$\zeta_A = \frac{2\pi}{\sin(\frac{\ell_\phi}{2})}\left[\left(-u\sin\phi + \frac{\ell_u\cos\phi}{2\tan(\frac{\ell_\phi}{2})} - \frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})}\right)\partial_u + \left(\cos\phi - \cos(\tfrac{\ell_\phi}{2})\right)\partial_\phi\right]. \tag{3.7}$$

This modular flow generates a transformation of BMS_3 since it can be written as

$$\zeta_A = (u\,Y'(\phi) + T(\phi))\partial_u + Y(\phi)\partial_\phi, \tag{3.8}$$

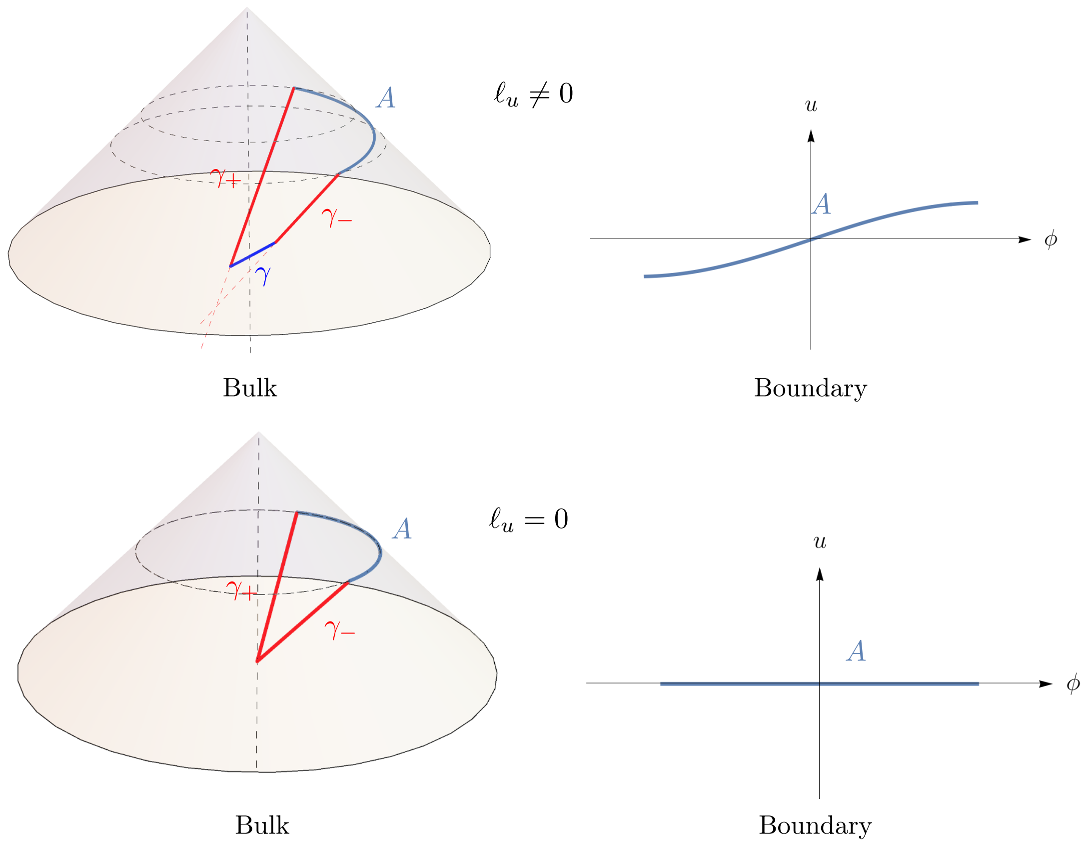


Figure 1: Examples of Ryu-Takayanagi surfaces in 3d Minkowski spacetime

where $Y(\phi)$ corresponds to a superrotation and $T(\phi)$ to a supertranslation. It is depicted together with its Wick rotated version in Fig. 2. A simple shape for the region A when $\ell_u \neq 0$ is a portion of sinusoid with equation

$$u = \frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})}\sin\phi \ , \tag{3.9}$$

although the precise shape doesn't matter in the computation of the entanglement entropy. The bulk modular flow can be found by looking for a Killing vector of 3d Minkowski which asymptotes to ζ_A. It takes the form

$$\begin{aligned}\xi_A = \frac{2\pi}{\sin(\frac{\ell_\phi}{2})}\Bigg[&\left(u\sin\phi + \frac{\ell_u}{2\tan(\frac{\ell_\phi}{2})}\cos\phi - \frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})}\right)\partial_u \\ &+\left(\cos(\tfrac{\ell_\phi}{2}) - \cos\phi - \frac{u}{r}\cos\phi + \frac{\ell_u}{2\tan(\frac{\ell_\phi}{2})}\frac{\sin\phi}{r}\right)\partial_\phi \\ &-\left((u+r)\sin\phi + \frac{\ell_u}{2\tan(\frac{\ell_\phi}{2})}\cos\phi\right)\partial_r\Bigg] \ .\end{aligned} \tag{3.10}$$

The bulk modular flow ξ_A vanishes on the curve γ. It doesn't vanish on the two light rays $\gamma_\pm$ but is tangent to them. This is enough to guarantee the existence of a first law, as explained in Sec. 3.3.

Entanglement entropy as Rindler entropy. To understand better the bulk picture described above, it is useful to go to Cartesian coordinates (t, x, y) defined as

$$t = u + r, \qquad x = r\cos\phi, \qquad y = r\sin\phi\ . \tag{3.11}$$

In these coordinates, the bulk modular flow becomes

$$\xi_A = \frac{2\pi}{\sin(\frac{\ell_\phi}{2})}\left[\left(y + \frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})}\right)\partial_t + \left(y\cos(\tfrac{\ell_\phi}{2}) + \frac{\ell_u}{2\tan(\frac{\ell_\phi}{2})}\right)\partial_x + \left(t - x\cos(\tfrac{\ell_\phi}{2})\right)\partial_y\right], \tag{3.12}$$

which is simply a boost, as can be seen by defining new Cartesian coordinates

$$\tilde{t} = \frac{t}{\sin(\frac{\ell_\phi}{2})} - \cot(\tfrac{\ell_\phi}{2})\, x, \qquad \tilde{x} = \frac{x}{\sin(\frac{\ell_\phi}{2})} - \cot(\tfrac{\ell_\phi}{2})\, t, \qquad \tilde{y} = y + \frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})}. \tag{3.13}$$

In these coordinates, the modular flow is simply

$$\xi_A = 2\pi\left(\tilde{y}\,\partial_{\tilde{t}} + \tilde{t}\,\partial_{\tilde{y}}\right)\ . \tag{3.14}$$

In App. A, we confirm that the Rindler thermal circle is the same as the one appearing in the generalized Rindler transform (3.6).[4] This geometry should be seen as the analog of the hyperbolic black hole in AdS.

We will now review the explicit RT prescription of [25] but in Cartesian coordinates where the description becomes simpler. This will be important in discussing the more general prescription in Sec. 3.2 and the 4d generalization in Sec. 6. As depicted in Fig. 1, we consider two bulk light rays that go to the two extremity points of A on $\mathcal{I}^+$. There is an ambiguity in choosing such light rays, as discussed in Sec. 3.2. The prescription adopted in [25] is to impose that these two light rays pass through the spatial origin $r = 0$, which is natural given a choice of Bondi coordinates. A parametrization of these two light rays is

$$\gamma_+ : \quad \begin{cases} t = -\frac{\ell_u}{2} + s \\ x = s\cos(\frac{\ell_\phi}{2}) \\ y = -s\sin(\frac{\ell_\phi}{2}) \end{cases}, \qquad \gamma_- : \quad \begin{cases} t = \frac{\ell_u}{2} + s \\ x = s\cos(\frac{\ell_\phi}{2}) \\ y = s\sin(\frac{\ell_\phi}{2}) \end{cases}. \tag{3.15}$$

In the limit $r \to +\infty$, we have

$$\gamma_+ : \quad \begin{cases} u \to \frac{\ell_u}{2}, \\ \phi \to \frac{\ell_\phi}{2} \end{cases}, \qquad \gamma_- : \quad \begin{cases} u \to -\frac{\ell_u}{2}, \\ \phi \to -\frac{\ell_\phi}{2} \end{cases}, \tag{3.16}$$

[4] One should remember that in the upper wedge, the Rindler time is spacelike, which is consistent with the boundary picture, see Fig. 2.

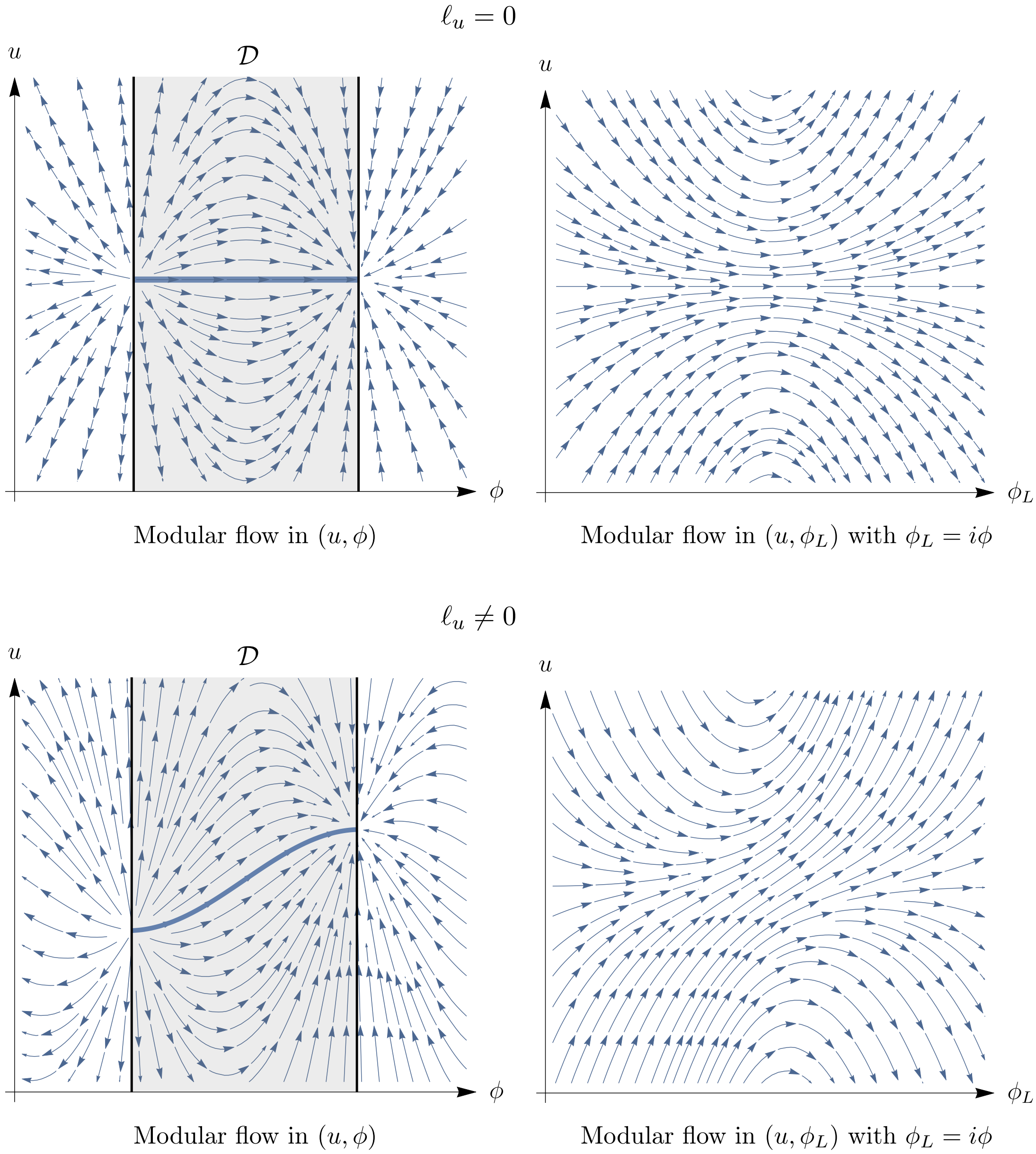


Figure 2: Boundary modular flow for 3d Minkowski. The left pictures represents the modular flow with the entangling region A (in blue) and its domain of dependence $\mathcal{D}$ (shaded) for $\ell_u = 0$ and $\ell_u \neq 0$. The right picture is the Wick rotated version with $\phi_L = i\phi$, where we see that the modular flow circles around a point at infinity. In contrast with the corresponding AdS/CFT picture (which is Fig. 2 in [20]), the modular flow does not "transport" the entangling region A but is parallel to it. This suggests that the density matrix ρ_A is more naturally associated with the domain of dependence $\mathcal{D}$, as argued by [34] in the AdS/CFT context. Since they have the same domain of dependence, this suggests that the case $\ell_u = 0$ is really equivalent to the case $\ell_u \neq 0$, as we will explain in Sec. 3.2.

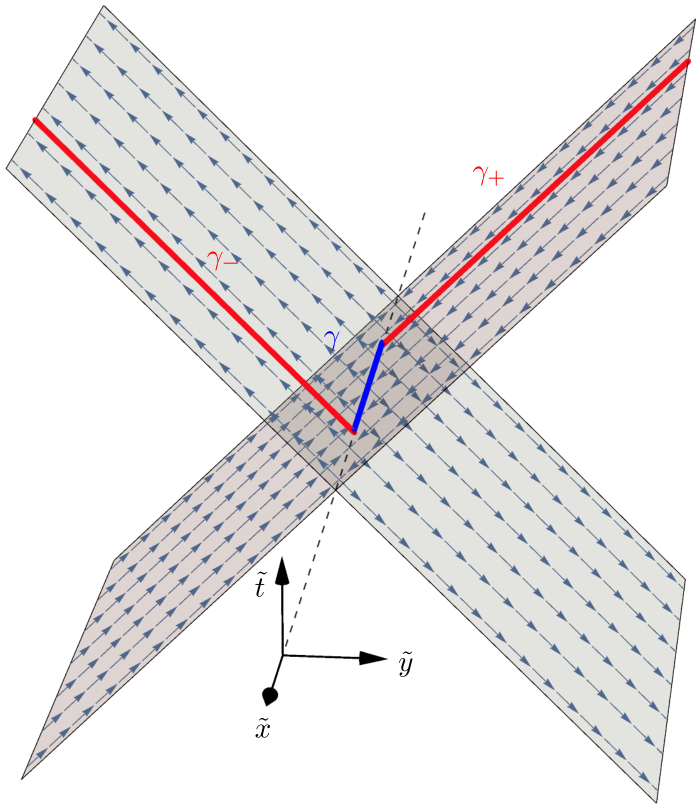


Figure 3: Ryu-Takayanagi surface in coordinates $(\tilde{t}, \tilde{x}, \tilde{y})$ in which the bulk modular flow is a boost. It is given by $\widetilde{A} = \gamma_- \cup \gamma \cup \gamma_+$. The surface γ lies on the Rindler bifurcation surface (the dashed line) and the light rays γ_+ and γ_- are tangent to the modular flow.

so that they intersect the two extremities of A on $\mathcal{I}^+$ as required. The bulk modular flow vanishes on the Rindler bifurcation surface

$$\tilde{t} = \tilde{y} = 0 \ . \tag{3.17}$$

The curve γ should be located where the bulk modular flow vanishes. Therefore, it has to lie on the bifurcation surface. To determine which portion it covers, we should look for the intersection of $\gamma_\pm$ with the bifurcation surface which gives two points P_+ and P_- with coordinates

$$P_\pm : \quad \tilde{t} = \tilde{y} = 0, \quad \tilde{x} = \pm \frac{\ell_u}{2\sin^2(\frac{\ell_\phi}{2})}. \tag{3.18}$$

The curve γ is then the segment $[P_- P_+]$. The resulting RT surface becomes

$$\widetilde{A} = \gamma_+ \cup \gamma_- \cup \gamma \ , \tag{3.19}$$

where it is understood that we only consider the portions of $\gamma_\pm$ that connect γ to A. From the general prescription (3.5), the entanglement entropy of the region A is be given by the integral of Wald's functional on $\widetilde{A}$. For Einstein gravity, this reduces to the length of γ and this leads to

$$S_A = \frac{\ell_u}{4G}\cot(\tfrac{\ell_\phi}{2}) \qquad \text{(Einstein gravity)} \ . \tag{3.20}$$

We illustrate this prescription in Fig. 3 in the coordinates (3.13) where the modular flow is a boost. A success of the prescription of [25] is that this reproduces the entanglement entropies obtained through field theoretic methods in [24]. We can now understand what is going to happen when we will perturb the bulk geometry: the portion of the bifurcation surface in consideration will satisfy a first law on-shell (this is true for any Killing horizon) that will map, through the assumptions we have made earlier, to a first law of entanglement of a putative dual field theory. This is explained in details in Sec. 3.3.

More RT surfaces. The authors of [25] derived a prescription to compute the entanglement entropies for a particular set of boundary regions. The prescription is summarized in Fig. 1 with two qualitatively different cases $\ell_u = 0$ or $\ell_u \neq 0$. There is a simple way to generate the RT surfaces associated to more general regions on $\mathcal{I}^+$. This can be done by acting with bulk isometries on the initial configurations. In Minkowski spacetime, we should act with elements of the Poincaré group. Their actions on $\mathcal{I}^+$ are given by BMS_3 transformations which transform A into a new region A'. This new region will be a more complicated curve. The corresponding RT surface $\widetilde{A}'$ is simply obtained as the image of $\widetilde{A}$ under the bulk isometry. These transformed RT surfaces are depicted in Fig. 4 and play a crucial role in the proof of the linearized gravitational equations of motion from the first law of entanglement.

3.2 General 3d prescription

We will explain an important ambiguity in the RT prescription of [25], which we reviewed above, corresponding to the choice of how the light rays reach infinity. This ambiguity was also considered in [32]. As a result, we will show that additional RT configurations are possible.

Infalling light sheaf. This ambiguity is most apparent when we consider the following fact: the case $\ell_u \neq 0$ can actually be obtained from the case $\ell_u = 0$ by acting with the bulk translation

$$y \to y + \frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})} \ . \tag{3.21}$$

This is apparent from the formula of the bulk modular flow (3.12): the modular flow for $\ell_u \neq 0$ is simply the image of the bulk modular flow for $\ell_u = 0$ under this translation. On the boundary, this translation becomes

$$u \to u + \frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})}\sin\phi \ , \tag{3.22}$$

and maps the boundary interval with $\ell_u = 0$ to the one with $\ell_u \neq 0$, see Fig. 2. This fact is puzzling because it implies that the configuration with $\ell_u = 0$ and the configuration with $\ell_u \neq 0$ are physically equivalent, as they are related by a bulk translation (which should be a true symmetry of the Minkowski vacuum). However, the entanglement entropies computed earlier are not the same for $\ell_u = 0$ and $\ell_u \neq 0$, as seen for (3.20).

In fact, this arises because the RT prescription depends on a choice of how the light rays arrive at infinity, or a choice of *infalling light sheaf.* For a given point on $\mathcal{I}^+$ with coordinates (u, ϕ), there are many inequivalent bulk light rays that go to this point, differing by bulk translations. We define an infalling light sheaf to be a set of light rays whose intersection with $\mathcal{I}^+$ is ∂A. The RT prescription will depend on the choice of such a light sheaf and acting with a bulk translation will modify this choice. To obtain a good RT prescription, we must require that the light sheaf satisfies the following two conditions:

1. Each light ray in the light sheaf must intersect the Rindler bifurcation surface.

2. The bulk modular flow must be tangent to the light sheaf.

The first condition is necessary to be able to define an RT surface (which should contain a portion of the Rindler bifurcation surface) while the second condition ensures the existence of a well-defined first law as we will show in the next section.

Heuristically, the choice of a light sheaf amounts to a choice of cutoff surface at infinity. In more mundane language, we are just saying that the entanglement entropy is cutoff dependent (even though it is finite). It is difficult to be more precise about what we mean by "cutoff" because the dual theory is not well-understood. We believe that this ambiguity reflects some properties of the UV structure of the dual theory.

Generalized 3d prescription. In 3d, the boundary ∂A consists of two points B_+ and B_-. Hence, the choice of infalling light sheaf is the choice of two light rays γ_+ and γ_- that arrive at these points and satisfy the two conditions stated above. An explicit parametrization of this light sheaf can be given as

$$\gamma_+ : \quad \begin{cases} t = \frac{\ell_u}{2} + s + Y_+ \sin(\frac{\ell_\phi}{2}) \\ x = s\cos(\frac{\ell_\phi}{2}) \\ y = s\sin(\frac{\ell_\phi}{2}) + Y_+ \end{cases} , \qquad \gamma_- : \quad \begin{cases} t = -\frac{\ell_u}{2} + s - Y_- \sin(\frac{\ell_\phi}{2}) \\ x = s\cos(\frac{\ell_\phi}{2}) \\ y = -s\sin(\frac{\ell_\phi}{2}) + Y_- \end{cases} \tag{3.23}$$

where $s \in \mathbb{R}$ is a parameter on the light ray and Y_+, Y_- are arbitrary constants. The light rays $\gamma_\pm$ arrive on $\mathcal{I}_+$ respectively at the points $B_\pm$. As required, they intersect the bifurcation surface $\tilde{y} = \tilde{t} = 0$ and are tangent to the bulk modular flow. Note that we have also used the freedom of reparametrization of s to reduce the number of independent parameters. At the end, we obtain a family of light sheaf parametrized by two arbitrary constants Y_+ and Y_-. The light rays $\gamma_\pm$ intersect the bifurcation surface at $\tilde{x} = \tilde{x}_\pm$ with

$$\tilde{x}_+ = -\frac{\ell_u}{2\tan(\frac{\ell_\phi}{2})} - Y_+\cos(\tfrac{\ell_\phi}{2}), \qquad \tilde{x}_- = \frac{\ell_u}{2\tan(\frac{\ell_\phi}{2})} + Y_-\cos(\tfrac{\ell_\phi}{2}) \ . \tag{3.24}$$

The length of γ is therefore given by the separation in $\tilde{x}$ which leads to the entropy

$$S_A = \frac{1}{4G}\left|\ell_u \cot(\tfrac{\ell_\phi}{2}) + (Y_+ + Y_-)\cos(\tfrac{\ell_\phi}{2})\right| \ . \tag{3.25}$$

The case $Y_+ = Y_- = 0$ corresponds to the prescription adopted of [25] described above. This prescription can also be obtained by requiring that the light rays intersect the line $r = 0$, which makes this prescription natural given a choice of Bondi coordinates. Another simple choice is

$$Y_+ = Y_- = -\frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})} \,. \tag{3.26}$$

In this case, the two light rays γ_+ and γ_- intersect at the point

$$\tilde{t} = \tilde{x} = 0, \qquad \tilde{y} = -\frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})} \,. \tag{3.27}$$

This gives a vanishing entropy and it corresponds to the case where we have applied a bulk translation to go from the $\ell_u = 0$ configuration shown in Fig. 1 to a configuration with $\ell_u \neq 0$ in which the light rays γ_+ and γ_- still meet. We can see that the intersection point (3.27) is indeed precisely the image of the origin by this translation. We would like to emphasize that there are no reason to favor one prescription or the other. Instead, we believe that we are free to choose any light sheaf satisfying the two conditions described above, and we interpret this choice as reflecting a choice of regulator in the putative dual theory.

3.3 First law of entanglement

In quantum mechanics, the first law of entanglement is a general property of the von Neumann entropy, which holds whenever we have a well-defined density matrix. It states that under a variation $\rho \to \rho + \delta\rho$, we have

$$\delta S = \delta\langle K\rangle, \tag{3.28}$$

where $S = -\mathrm{Tr}\,\rho\log\rho$ and $K = -\log\rho$. The proof uses simple manipulations on density matrices and is given in [19]. When ρ is the density matrix associated to the boundary region A, we will denote δS_A the entropy variation and $\delta E_A = \delta\langle K\rangle$ the energy variation. The first law of entanglement states that

$$\delta S_A = \delta E_A \,. \tag{3.29}$$

We would like to compute the corresponding gravitational quantities $\delta S_A^{\rm grav}$ and $\delta E_A^{\rm grav}$ under a general perturbation of the metric. Following the general prescription discussed above, we consider the RT surface $\widetilde{A} = \gamma_+ \cup \gamma \cup \gamma_-$ where $\gamma_\pm$ are given in (3.23). In Einstein gravity, the gravitational entropy associated to the RT surface $\widetilde{A}$ is nothing but its area in Planck units. The variation of the entropy is then computed from the variation of the area of $\widetilde{A}$. We want to allow for general theories of gravity so we introduce Wald's Noether charge $\mathbf{Q}[\xi_A]$ associated to the Killing vector field ξ_A. The variation of the gravitational entropy is then given by

$$\delta S_A^{\rm grav} = \int_{\widetilde{A}} \delta\mathbf{Q}[\xi_A]. \tag{3.30}$$

The gravitational energy is defined as the boundary term appearing in the expression of the canonical energy of the region Σ such that $\partial\Sigma = A \cup \tilde{A}$. It has the expression

$$\delta E_A^{\text{grav}} = \int_\Sigma \left(\delta \mathbf{Q}[\xi_A] - \xi_A \cdot \mathbf{\Theta}(\delta\phi)\right) \ , \tag{3.31}$$

where $\mathbf{\Theta}$ is the presymplectic form. Paralleling the AdS story [19], let's define the form

$$\boldsymbol{\chi} = \delta \mathbf{Q}[\xi_A] - \xi_A \cdot \mathbf{\Theta}(\delta\phi) \,, \tag{3.32}$$

we will show that $\boldsymbol{\chi}$ satisfies the same properties as its AdS counterpart. The bulk modular flow ξ_A vanishes on γ. It doesn't vanish on $\gamma^\pm$ where it is tangent, nonetheless, the integral of $\xi_A \cdot \mathbf{\Theta}(\delta\phi)$ on $\gamma^\pm$ vanishes because $\xi_A \cdot (\xi_A \cdot \mathbf{\Theta}(\delta\phi)) = 0$ since $\mathbf{\Theta}$ is a 2-form. This shows that $\int_{\widetilde{A}} \xi_A \cdot \mathbf{\Theta}(\delta\phi) = 0$ and that we have

$$\delta S_A^{\text{grav}} = \int_{\widetilde{A}} \boldsymbol{\chi} \,. \tag{3.33}$$

Using similar manipulations as in Sec. 5.1 of [19], we can also show that

$$\delta E_A^{\text{grav}} = \int_A \boldsymbol{\chi} \,, \tag{3.34}$$

and that

$$d\boldsymbol{\chi} = -2\xi_A^a \delta E_{ab} \boldsymbol{\varepsilon}^b \,, \tag{3.35}$$

where δE_{ab} are the equations of motion. Therefore, the gravitational entropy and energy satisfy a first law for on-shell perturbations

$$\delta S_A^{\text{grav}} = \delta E_A^{\text{grav}} \,, \tag{3.36}$$

which follows from the fact that

$$\delta E_A^{\text{grav}} - \delta S_A^{\text{grav}} = \int_A \boldsymbol{\chi} - \int_{\widetilde{A}} \boldsymbol{\chi} = \int_\Sigma d\boldsymbol{\chi} = 0 \,. \tag{3.37}$$

The goal of our paper is to show that the converse also holds: the first law of entanglement for all the regions A (among a special class) implies the gravitational equations of motion.

Einstein gravity. For pure Einstein gravity, we have

$$\mathbf{\Theta}(\delta g) = \frac{1}{16\pi G}(\nabla_b \delta g^{ab} - \nabla^a \delta g_b{}^b) \,, \qquad \mathbf{Q}[\xi] = -\frac{1}{16\pi G} \nabla^a \xi^b \boldsymbol{\varepsilon}_{ab} \,. \tag{3.38}$$

The expression for $\boldsymbol{\chi}$ reads

$$\begin{aligned} \boldsymbol{\chi}(\delta g) &= \delta \mathbf{Q}[\xi_A](\delta g) - \xi_A \cdot \mathbf{\Theta}(\delta g) \\ &= \frac{1}{16\pi G} \boldsymbol{\varepsilon}_{ab} \left(\delta g^{ac} \nabla_c \xi_A^b - \frac{1}{2} \delta g_c{}^c \nabla^a \xi_A^b + \nabla^b \delta g^a{}_c \xi_A^c - \nabla_c \delta g^{ac} \xi_A^b + \nabla^a \delta g^c{}_c \xi_A^b \right) . \end{aligned} \tag{3.39}$$

We now consider a small perturbation of the metric around Minkowski

$$g_{ab} = \eta_{ab} + \lambda h_{ab}, \tag{3.40}$$

such that $\delta g_{ab} = \lambda h_{ab}$, where λ is small. For instance, one can consider a perturbation in Bondi gauge (see Sec. 5.2 for a complete description),

$$h_{ab}dx^a dx^b = \left(\frac{V}{r} - 2\beta\right) du^2 - 4\beta du dr - 2r^2 U du dr + 2r^2 \varphi\, d\phi^2\,, \tag{3.41}$$

where V, β, U are functions of all coordinates, while φ depends only on u and r. The linearized Einstein equation are obtained for small λ:

$$R_{ab} - \frac{1}{2} R g_{ab} = \delta E_{ab}(h)\lambda + O(\lambda^2)\,. \tag{3.42}$$

Using (3.39), we have computed $\boldsymbol{\chi}$ explicitly and checked that indeed

$$d\boldsymbol{\chi} = -2\xi^a \delta E_{ab} \boldsymbol{\varepsilon}^b\,. \tag{3.43}$$

Note that this formula follows from the general derivation given in [2]. It ensures the validity of the first law for on-shell perturbations. A simple class of asymptotically flat on-shell perturbations is

$$ds^2 = \eta_{ab}dx^a dx^b + \lambda\left(\Theta(\phi)\, du^2 + 2\left(\Xi(\phi) + \frac{u}{2}\partial_\phi \Theta(\phi)\right) du d\phi\right), \tag{3.44}$$

where Θ and Ξ are arbitrary functions of ϕ. They were found in [10] and we show how to obtain them in Sec. 5.2. We focus on an interval A on the slice $u = 0$ (taking $\ell_u = 0$) and with width ℓ_ϕ. We compute explicitly the energy variation

$$\delta E_A = \int_A \boldsymbol{\chi} = \frac{1}{4\sin(\frac{\ell_\phi}{2})} \int_{-\frac{\ell_\phi}{2}}^{\frac{\ell_\phi}{2}} d\phi \left(\cos\phi - \cos(\tfrac{\ell_\phi}{2})\right) \Xi(\phi)\,. \tag{3.45}$$

Note that this can be written in term of the modular flow (3.12) as

$$\delta E_A = \frac{1}{8\pi} \int_A d\phi\, \zeta_A^\phi\, \Xi(\phi)\,. \tag{3.46}$$

We conclude that this perturbation should be accompanied by a variation of the entropy for the first law to be satisfied.

Refined prescription. In [25], the RT prescription was proposed only for Minkowski spacetime. For linearized perturbations at first order, the RT surface $\widetilde{A}$ is unchanged so we expect to be able to use the same prescription for perturbed Einstein gravity:

$$S_A = \frac{\text{Length}(\widetilde{A})}{4G}\,, \tag{3.47}$$

where the length is computed in the perturbed geometry. For the perturbation (3.44), it is easy to see that γ_+ and γ_- are still light rays that intersect at the origin and, since $\tilde{A}$ is the union of them, the prescription would imply that $\delta S_A = 0$.[5] This contradicts the

[5]We are using here the light sheaf prescription where we impose that the light rays pass through the origin $r = 0$. This is the prescription used in [25].

first law of entanglement because $\delta E_A \neq 0$. The resolution of this problem comes from the corner in $\widetilde{A}$ between γ_+ and γ_-. We should regulate it by considering a smooth curve $\widetilde{A}_{\rm reg}$ arbitrarily close to $\widetilde{A} = \gamma_+ \cup \gamma_-$. In other words, the corner has a non-trivial contribution to the integral.[6] The correct prescription is then

$$S_A = \int_{\widetilde{A}_{\rm reg}} \mathbf{Q}[\xi_A] = \lim_{\varepsilon \to 0} \int_{\widetilde{A}_\varepsilon} \mathbf{Q}[\xi_A] \,, \tag{3.48}$$

where $\widetilde{A}_\varepsilon$ is a smooth curve that regulates the corner in $\widetilde{A} = \gamma_+ \cup \gamma_-$ and converges to $\widetilde{A}$ when $\varepsilon \to 0$. From the fact that $d\boldsymbol{\chi} = 0$ on-shell and that $\widetilde{A}_\varepsilon$ is a smooth curve homologous to A, we have

$$\int_{\widetilde{A}_\varepsilon} \boldsymbol{\chi} = \int_A \boldsymbol{\chi} = \delta E_A \ , \tag{3.49}$$

which would not be necessarily true if $\widetilde{A}_\varepsilon$ had corners. From the definition (3.32) of $\boldsymbol{\chi}$, we can see that

$$\delta S_A = \lim_{\varepsilon \to 0} \int_{\widetilde{A}_\varepsilon} (\boldsymbol{\chi} + \xi_A \cdot \boldsymbol{\Theta}) \,. \tag{3.50}$$

In the limit where $\varepsilon \to 0$, the integral of $\xi_A \cdot \boldsymbol{\Theta}$ vanishes because ξ_A is tangent to $\gamma_\pm$ and vanishes at the corner $\gamma_+ \cap \gamma_-$ (while $\boldsymbol{\Theta}$ is finite at the corner). Therefore, we have checked the validity of the first law of entanglement for the RT prescription,

$$\delta S_A = \delta E_A \,. \tag{3.51}$$

Note that for Einstein gravity, (3.48) doesn't reduce to the length of $\widetilde{A}_{\rm reg}$ because $\mathbf{Q}[\xi_A]$ computes only the length of the surface on which ξ_A vanishes. In particular, S_A can become negative for some choices of perturbations. We comment on this in Sec. 3.4.

3.4 Positivity constraints

Let's consider the interval A with $\ell_u = 0$ and use the prescription in which the light rays intersect at the origin, see Fig. 1. In Einstein gravity, the entanglement entropy S_A vanishes. This implies that the state ρ_A is pure. This is unlike any standard quantum field theory, where the vacuum entanglement entropy has a universal divergence. This suggests some form of ultralocality as discussed in [35]: the vacuum factorizes between subregions of a constant u slice of $\mathcal{I}^+$. A perturbation will then create a nonzero entropy

$$S_A = \delta S_A = \delta E_A \,. \tag{3.52}$$

From the explicit expression of (3.46), we can see that this expression can become negative. This is in tension with the fact that von Neumann entropies are always positive. This gives a constraint on perturbations of the form (3.44) that can be described within a quantum system on $\mathcal{I}^+$ satisfying our assumptions. Imposing that

$$S_A = \delta E_A \geq 0 \tag{3.53}$$

[6] There is a similar problem with the origin in polar coordinates. For example, we have $\int_{S^1_\varepsilon} d\theta = 2\pi$ for a circle S^1_ε of radius ε. Stokes theorem implies that this integral doesn't depend on ε. In the limit $\varepsilon \to 0$ though, S^1_ε reduces to a point which suggests that the integral should be set to zero. This is incorrect because $d\theta$ is not defined at the origin.

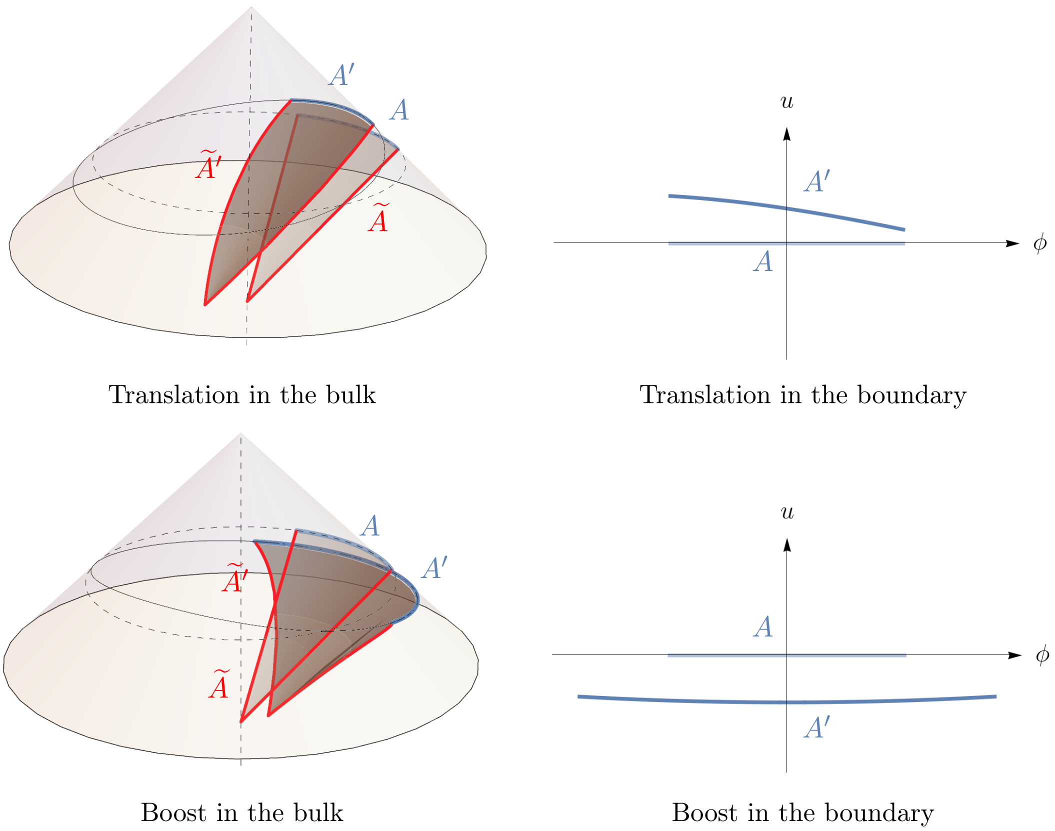


Figure 4: Examples of new RT surfaces obtained by bulk isometries acting on the reference configuration for $\ell_u = 0$.

gives a constraint on $\Xi(\phi)$ according to (3.46). To understand this better, let's restrict the Hilbert space $\mathcal{H}$ that contains only the perturbations (3.44) of 3d Minkowski. The condition (3.53) implies that we should restrict to the subspace $\mathcal{H}_{\text{code}} \subset \mathcal{H}$ on which $\delta\langle K_A \rangle \geq 0$. This implies that the operator K_A is bounded from below on $\mathcal{H}_{\text{code}}$ and hence, that the density operator e^{-K_A} is well-defined there. As a result, positivity of the entropy gives a constraint on the perturbations that can be described within a quantum system satisfying our assumptions. This is similar to the constraints on AdS perturbations coming from quantum information inequalities [36–38].

Sign ambiguity. The generalized Rindler method doesn't fix the sign of the modular flow. If a path integral formulation can eventually be given, the sign would be fixed from the choice of the vacuum state. Choosing the new modular flow $\zeta'_A = -\zeta_A$, with new modular Hamiltonian $K'_A = -K_A$, the condition $S_A \geq 0$ selects a different subspace $\mathcal{H}'_{\text{code}} \subset \mathcal{H}$: the subspace on which K'_A is a positive operator. This ensures that for the modular flow ζ'_A, we have a density operator $e^{-K'_A}$ which is well-defined on $\mathcal{H}'_{\text{code}}$. Hence, changing the sign of the modular flow amounts to selecting a different subspace on which ρ_A is well-defined.

4 Flat 3d gravity from entanglement

In this section, we show that the first law of entanglement implies the gravitational equations of motion, linearized around three-dimensional Minkowski spacetime. Our proof is valid for any theory of gravity, including higher-derivative terms. The generalization to four dimensions is treated in the Sec. 6.

4.1 General strategy

Let's consider a general off-shell perturbation of 3d Minkowski. The one-form $\boldsymbol{\chi}$ satisfies

$$d\boldsymbol{\chi} = -2\xi^a \delta E_{ab} \boldsymbol{\varepsilon}^b, \tag{4.1}$$

where δE_{ab} are the equations of motion for the perturbations and $\boldsymbol{\varepsilon}_a = \frac{1}{2}\varepsilon_{abc} dx^b \wedge dx^c$.[7] As explained in (3.37), the first law of entanglement implies that for all surfaces Σ bounded by A and $\widetilde{A}$, we have

$$\int_\Sigma d\boldsymbol{\chi} = 0\,. \tag{4.2}$$

We would like to show that this implies that $\delta E_{ab} = 0$. This is reasonable because we have a large number of such surfaces Σ. The derivation will be similar to the AdS case [19] although the RT surfaces are more involved here. Bulk isometries will play a crucial role.

The strategy is to start with some reference configuration. By varying the parameters of this configuration, we will obtain constraints on the gravitational equations δE_{ab}. We will then act on this configuration with bulk isometries to obtain new constraints. This amounts to probing the perturbation with new RT surfaces, obtained by applying a bulk isometry to the reference configuration. The new constraint is obtained by replacing δE_{ab} by its image under the transformation. The logic can be phrased as follows: the first law of entanglement gives the equation

$$\int_\Sigma \xi^a \delta E_{ab}(x) \boldsymbol{\varepsilon}^b = 0\,. \tag{4.3}$$

We can consider a new configuration $\widetilde{\Sigma}$ obtained by performing a bulk isometry $x \to \tilde{x}$. The associated bulk modular flow $\tilde{\xi}^a$ and volume form $\tilde{\boldsymbol{\varepsilon}}^b$ can be obtained by applying the transformation to ξ^a and $\boldsymbol{\varepsilon}^b$, which gives

$$\int_{\widetilde{\Sigma}} \tilde{\xi}^a \delta E_{ab}(\tilde{x}) \tilde{\boldsymbol{\varepsilon}}^b = 0\,. \tag{4.4}$$

We are probing the same perturbation δE_{ab} with a different RT surface and we emphasize that $\delta E_{ab}(\tilde{x})$ is now evaluated on the new RT surface $\tilde{\Sigma}$. Now, we can change variables in the integral using the inverse bulk isometry $x \to x'$. This gives

$$\int_\Sigma \xi^c \left(\frac{\partial \tilde{x}^a}{\partial x^c} \frac{\partial \tilde{x}^b}{\partial x^d} \delta E_{ab}(\tilde{x}(x)) \right) \boldsymbol{\varepsilon}^d = 0\,. \tag{4.5}$$

[7] ε_{abc} is a totally antisymmetric tensor such that $\varepsilon_{ur\phi} = \sqrt{-g}$.

This shows that if (4.3) allows us to prove that some functional of the equations of motion vanishes:

$$\mathcal{F}\left[\delta E_{ab}(x)\right]=0, \tag{4.6}$$

then we immediately have that the same functional but applied to the transformed equations of motion vanishes:

$$\mathcal{F}\left[\frac{\partial\tilde{x}^c}{\partial x^a}\frac{\partial\tilde{x}^d}{\partial x^b}\delta E_{cd}(\tilde{x}(x))\right]=0. \tag{4.7}$$

This procedure is made mathematically precise in App. B.

4.2 Linearized gravitational equations

We now describe the proof of the gravitational equations, linearized around 3d Minkowski spacetime. Although the proof is conceptually similar to the AdS case derived in [19], it is rather more challenging in flat space. In particular, we will have to use different RT prescriptions as discussed in Sec. 3.2. Bulk isometries will also play an important role in generating enough constraints on the perturbation.

Reference configuration. The reference configuration is an interval A with $\ell_u=0$ at $u=0$ and with length ℓ_ϕ centered at $\phi=0$. We can parametrize the interval A by

$$A: \quad u=0, \quad \phi\in[-\tfrac{\ell_\phi}{2},\tfrac{\ell_\phi}{2}]\,. \tag{4.8}$$

The RT surface $\widetilde{A}$ consists of two semi-infinite light rays starting at the origin and ending at the extremities ∂A, as in Fig. 1. The surface Σ at $u=0$ which is bounded by A and $\widetilde{A}$ can be parametrized by r and ϕ with

$$\Sigma: \quad u=0, \quad r\geq 0, \quad \phi\in[-\tfrac{\ell_\phi}{2},\tfrac{\ell_\phi}{2}]\,. \tag{4.9}$$

The bulk modular flow (3.10) evaluated on Σ reduces to

$$\xi_A=\frac{2\pi}{\sin(\frac{\ell_\phi}{2})}\Big(r\sin\phi\,\partial_r+(\cos\phi-\cos(\tfrac{\ell_\phi}{2}))\partial_\phi\Big)\,. \tag{4.10}$$

Let's write explicitly the equation (4.1). In Bondi coordinates, we have

$$\boldsymbol{\varepsilon}^r=-\boldsymbol{\varepsilon}_u=-r\,dr\wedge d\phi\,. \tag{4.11}$$

Hence, the pullback of $d\boldsymbol{\chi}$ on Σ is[8]

$$d\boldsymbol{\chi}|_\Sigma=2r\xi^a\delta E_{ar}dr\wedge d\phi\,. \tag{4.12}$$

From (4.1), we obtain[9]

$$\int_{-\frac{\ell_\phi}{2}}^{\frac{\ell_\phi}{2}}d\phi\int_0^{+\infty}dr\,\Big(r^2\sin\phi\,\delta E_{rr}+r\left(\cos\phi-\cos(\tfrac{\ell_\phi}{2})\right)\delta E_{r\phi}\Big)=0\,. \tag{4.13}$$

[8]The 2-form $dr\wedge d\phi$ is singular at $r=0$ so we need to restrict the integration range to $r\geq\varepsilon$ and take $\varepsilon\to 0$ at the end. This is always what we will be doing implicitly.

[9] We thank Hongliang Jiang for pointing out a mistake in the previous version of this formula.

Expanding this equation at small ℓ_ϕ implies that

$$\int_0^{+\infty} dr \, (r^2 \partial_\phi \delta E_{rr}(0,r,0) + r\delta E_{r\phi}(0,r,0)) = 0 \ . \tag{4.14}$$

Rotations and time translations. We can consider new configurations obtained by performing rotations. They are the same as the reference configuration but centered at $\phi = \phi_0$. The new RT suraces are obtained as the image under the bulk isometries

$$\phi \to \phi + \phi_0. \tag{4.15}$$

The Jacobian of this transformation is simply the identity. Therefore, following the logic exposed in the previous section, we obtain that the vanishing of the functional (4.14) but applied to the image of δE_{ab} under this isometry:

$$\int_0^{+\infty} dr \, (r^2 \partial_\phi \delta E_{rr}(0,r,\phi_0) + r\delta E_{r\phi}(0,r,\phi_0)) = 0, \tag{4.16}$$

for any angle ϕ_0. We can do the same with translation $u \to u + u_0$ in retarded time u, to obtain

$$\int_0^{+\infty} dr \, (r^2 \partial_\phi \delta E_{rr}(u_0,r,\phi_0) + r\delta E_{r\phi}(u_0,r,\phi_0)) = 0. \tag{4.17}$$

light sheaf deformation. We consider the same boundary interval A as in the reference configuration (4.8). The latter followed the prescription in which the light rays γ_+ and γ_- intersect the spatial origin $r = 0$. This is not the most general prescription, as discussed in Sec. 3.2. Here, we will use a more general prescriptions to derive more constraints on δE_{ab}. An alternative proof of this step is presented in the App. C.

We consider a more general light sheaf for the interval A. We take the parametrization (3.23) where we set $\ell_u = Y_- = 0$ and $Y_+ = Y$. The two light rays intersect the bifurcation surface at $\tilde{x} = 0$ and $\tilde{x} = -Y\cos(\frac{\ell_\phi}{2})$. The first law tells us that for any Y, we have

$$\int_{\Sigma_Y} d\boldsymbol{\chi} = 0 \ , \tag{4.18}$$

where the surface Σ_Y depends on Y and can be chosen to be any surface such that $\partial\Sigma_Y = \tilde{A} \cup A$. In particular, one can choose $\Sigma_Y = \Sigma_{\{Y=0\}} \cup N_Y$, where N_Y is the strip created by the union of all the half light rays γ_+ given in (3.23) where the parameter Y_+ goes from 0 to Y. From (4.18), it then follows that for any Y, we have

$$\int_{N_Y} d\boldsymbol{\chi} = 0 \ . \tag{4.19}$$

We now take the derivative with respect to Y and evaluate at $Y = 0$. The integral reduces to an integral over the $Y_+ = 0$ light ray and the integrand is contracted with $\partial_{\tilde{x}}$ as the effect of changing Y_+ is to translate the light ray in the $\tilde{x}$-direction. At the end, we get

$$\int_{\gamma_+} \partial_{\tilde{x}} \cdot d\boldsymbol{\chi} = 0. \tag{4.20}$$

where γ_+ is the usual light ray from the origin to the point $(u, \phi) = (0, \frac{\ell_\phi}{2})$. Converting the vector to Bondi coordinates, we obtain

$$\partial_{\tilde{x}} = \frac{1}{\sin(\frac{\ell_\phi}{2})} \left[\left(\cos(\tfrac{\ell_\phi}{2}) - \cos\phi \right) \partial_u + \cos\phi \, \partial_r - \frac{\sin\phi}{r} \partial_\phi \right] . \tag{4.21}$$

The integral is evaluated at $\phi = \frac{\ell_\phi}{2}$ where the expressions for $\partial_{\tilde{x}}$ and for the bulk modular flow (4.10) simplify to

$$\partial_{\tilde{x}} = \cot(\tfrac{\ell_\phi}{2})\partial_r - \frac{1}{r}\partial_\phi, \qquad \xi_A = 2\pi r \, \partial_r \tag{4.22}$$

The pullback on γ_+ only keeps the dr component so in the expression (4.1) for $d\boldsymbol{\chi}$, we only have a contribution from $\boldsymbol{\varepsilon}^r = -r dr \wedge d\phi$. As a result, $\partial_{\tilde{x}} \cdot d\boldsymbol{\chi}|_{\gamma_+} = -4\pi r \, \delta E_{rr} \, dr$ and we obtain

$$\int_0^{+\infty} dr \, r \, \delta E_{rr}(u_0, r, \phi_0) = 0 \ , \tag{4.23}$$

where as above, we have used rotations and time translations to make this expression valid for any u_0 and ϕ_0.

Radial translations. Let's consider a new configuration which is obtained by translating the reference configuration by a distance r_0 in the direction ϕ_0 of the light ray on which (4.23) is integrated. In Cartesian coordinates, such a translation is given by

$$t \to t + r_0, \qquad x \to x + r_0 \cos\phi_0, \qquad y \to y + r_0 \sin\phi_0 \ . \tag{4.24}$$

These configurations are illustrated in Fig. 4. We can apply the reasoning presented in Sec. 4.1 for these new configurations. In Bondi coordinates, the transformation becomes

$$u \to r + r_0 + u - \sqrt{r^2 + 2\, r\, r_0 \cos(\phi - \phi_0) + r_0^2} \ , \tag{4.25}$$

$$r \to \sqrt{r^2 + 2\, r\, r_0 \cos(\phi - \phi_0) + r_0^2} \ , \tag{4.26}$$

$$\phi \to \arctan\left(\frac{r \sin(\phi) + r_0 \sin(\phi_0)}{r \cos(\phi) + r_0 \cos(\phi_0)} \right). \tag{4.27}$$

The constraint (4.17) applied to the image of δE_{ab} under this isometry gives the new constraint

$$\int_{r_0}^{+\infty} dr \, (r - r_0) \, \delta E_{rr}(u_0, r, \phi_0) = 0 \ , \tag{4.28}$$

where we have also performed the change of variable $r \to r - r_0$ in the integral. Taking two derivatives with respect to r_0 shows that

$$\delta E_{rr}(u_0, r_0, \phi_0) = 0 \ , \tag{4.29}$$

for any value of u_0, r_0, ϕ_0. From this, the equation (4.17) simplifies to

$$\int_0^{+\infty} dr \, r \, \delta E_{r\phi}(u_0, r, \phi_0) = 0 \ . \tag{4.30}$$

We use the same radial translation on this equation to obtain the constraint

$$\int_{r_0}^{+\infty} dr \, \frac{(r-r_0)^2}{r} \delta E_{r\phi}(u_0, r, \phi_0) \; . \tag{4.31}$$

Taking three derivatives with respect to r_0 implies that

$$\delta E_{r\phi}(u_0, r_0, \phi_0) = 0 \; , \tag{4.32}$$

which is true for any value of u_0, r_0, ϕ_0. Hence, we have shown that

$$\delta E_{rr} = \delta E_{r\phi} = 0 \; , \tag{4.33}$$

everywhere in the bulk.

General translations. We consider a general bulk translation $\delta x^\mu = v^\mu$. This generates a new family of configurations, illustrated in Fig. 4. Acting with the infinitesimal translation on $\delta E_{r\phi} = 0$ leads to

$$(v_y \cos\phi - v_x \sin\phi)(r^2 \delta E_{ur} + \delta E_{\phi\phi}) = 0 \, , \tag{4.34}$$

which implies that

$$\delta E_{\phi\phi} = -r^2 \delta E_{ur} \; , \tag{4.35}$$

everywhere in the bulk.

Conservation equation. We now consider the conservation equation

$$\nabla_a (\delta E^{ab}) = 0 \; , \tag{4.36}$$

which is always satisfied by the equations of motion. Here, ∇_a is the derivative with respect to the background Minkowski spacetime. We will use this equation together with an additional holographic input to cancel the remaining components. Indeed one should remember that in AdS, the proof requires a holographic input that is the conservation and the tracelessness of the boundary stress tensor. In a radial Hamiltonian perspective, they correspond to initial conditions on the boundary surface. In the flat case, similar initial conditions are required. We will show in the next section how to make sense of a boundary "stress tensor" and derive its constraint equations using a flat limit in AdS.

For $b = u$, the conservation equation implies

$$\partial_r (\delta E_{ur}) = 0 \, , \tag{4.37}$$

which leads to $\delta E_{ur} = C_0(u, \phi)$ and $\delta E_{\phi\phi} = -r^2 C_0(u, \phi)$. We expect that the trace conditions (5.29) and (5.30) imply that $C_0 = 0$ although we have not been able to show it conclusively.[10] Assuming that this is the case, we obtain

$$\delta E_{ur} = \delta E_{\phi\phi} = 0 \, , \tag{4.38}$$

[10]This would be done by turning on an off-shell perturbation in the Bondi gauge such that (5.29) and (5.30) are violated which would allows us to identify the corresponding components of Einstein equations. We leave this analysis for future work.

everywhere in the bulk. The conservation equation for $b = \phi$ then gives

$$\delta E_{u\phi} + r\partial_r(\delta E_{u\phi}) = 0\,. \tag{4.39}$$

The solution of this equation is

$$\delta E_{u\phi} = \frac{C_2(u,\phi)}{r}\,. \tag{4.40}$$

In the next section, we show that the equation $C_2 = 0$ is precisely the conservation equation (5.13) of the boundary stress tensor, so we have $\delta E_{u\phi} = 0$. Finally, the component with $b = r$ gives

$$\delta E_{uu} + r\partial_r(\delta E_{uu}) = 0\,, \tag{4.41}$$

with solution

$$\delta E_{uu} = \frac{C_1(u,\phi)}{r}\,. \tag{4.42}$$

The equation $C_1 = 0$ is the other conservation equation (5.12) of the boundary stress tensor, so we have $\delta E_{uu} = 0$. Hence, we have shown that all the components of the linearized gravitational equation vanish.

5 Holographic stress tensor in flat spacetime

In AdS, the boundary is a timelike hypersurface which allows for the definition of a non-degenerate boundary metric whose dual operator is the boundary stress tensor. In flat space, things are more subtle, because the metric becomes degenerate on the boundary (its determinant vanishes). This is simply because $\mathcal{I}^+$ is a null hypersurface. To have a good understanding of the flat case, it is helpful to start from its AdS counterpart and perform a flat limit sending the AdS radius to infinity, we will see that this amounts to perform a Carrollian limit on the boundary (or ultra-relativistic limit). We will show that the induced geometry on a null hypersurface contains more than a degenerate metric and that additional geometrical objects appear naturally when performing the flat limit. The concept of boundary stress tensor will also have to be modified.

5.1 AdS$_3$ in Bondi gauge

We consider the following metric, written in Bondi gauge:

$$ds^2 = \frac{\tilde{V}}{r} e^{2\tilde{\beta}} du^2 - 2e^{2\tilde{\beta}} du dr + r^2 e^{2\tilde{\varphi}} (d\phi - \tilde{U} du)^2, \tag{5.1}$$

We are going to consider small perturbations around global AdS, the most generic perturbation in Bondi gauge is given by

$$\tilde{\beta} = \lambda\beta, \quad \tilde{V} = -r\left(1 + \frac{r^2}{\ell^2}\right) + \lambda V, \quad \tilde{U} = \lambda U, \quad \tilde{\varphi} = \lambda\varphi, \tag{5.2}$$

where λ is a small parameter. From now on, all the expressions will be linearized in λ. Solving the (r,r), (r,u), (r,ϕ) and (ϕ,ϕ)-components of the linearized Einstein equations, with negative cosmological constant, gives

$$
\begin{aligned}
\beta &= \beta_0(u,\phi), \\
U &= -\frac{N(u,\phi)}{r^2} + U_0(u,\phi) + \frac{2\partial_\phi \beta_0}{r}, \\
V &= rM(u,\phi) + r\left(-\frac{2r^2\beta_0}{\ell^2} - 2r\left(\partial_\phi U_0 + \partial_u \varphi\right)\right).
\end{aligned}
\tag{5.3}
$$

The flat limit was considered for the case $\beta_0 = U_0 = 0$ in [6]. There are two residual equations, given by the (u,u) and (u,ϕ)-components of Einstein equations

$$
\begin{aligned}
\partial_u M &= 2\partial_\phi U_0 + 2\partial_\phi^2 U_0 - 2\partial_u \beta_0 - 4\partial_u\partial_\phi^2\beta_0 + 2\partial_u\varphi + 2\partial_u\partial_\phi^2\varphi + 2\ell^{-2}\partial_\phi N\,, \\
\partial_u N &= \frac{1}{2}\partial_\phi M - \partial_\phi\beta_0\,.
\end{aligned}
\tag{5.4}
$$

The latter can be understood as the conservation of a boundary stress tensor

$$
\nabla_\mu T^{\mu\nu} = 0\,,
\tag{5.5}
$$

where $\mu = \{u,\phi\}$. The boundary metric and the stress tensor are given by

$$
g_{\mu\nu} = \begin{pmatrix} -\frac{1+4\lambda\beta_0}{\ell^2} & -\lambda U_0 \\ -\lambda U_0 & 1+2\lambda\varphi \end{pmatrix}, \quad T^{\mu\nu} = \frac{1}{8G}\tau^{\mu\nu},
\tag{5.6}
$$

where

$$
\begin{aligned}
\tau^{uu} &= -\ell^3\left(-1+\lambda\left(M + 6\beta_0 + 4\partial_\phi^2\beta_0\right)\right), \\
\tau^{u\phi} &= \ell\lambda\left(2N + \ell^2\left(U_0 + 2\partial_\phi^2 U_0 + 2\partial_u\partial_\phi\varphi\right)\right), \\
\tau^{\phi\phi} &= -\ell\left(-1+\lambda\left(M + 2\beta_0 + 2\varphi + 2\ell^2\partial_u\partial_\phi U_0 + 2\ell^2\partial_u^2\varphi\right)\right).
\end{aligned}
\tag{5.7}
$$

This stress tensor can be obtained, for example, through the Brown and York procedure. It is well-known that the boundary theory is a 2d CFT whose central charge is given by [39]

$$
c = \frac{3\ell}{2G}\,,
\tag{5.8}
$$

and this is confirmed by computing the anomalous trace of the stress tensor

$$
T^\mu{}_\mu = -\frac{c}{12}R = -\frac{\ell}{8G}R\,,
\tag{5.9}
$$

where R is the scalar curvature of the boundary metric.

5.2 Flat limit and Carrollian geometry

We have now all the ingredients to perform the flat limit. In the bulk, the $\ell \to \infty$ limit of the metric is given by another metric in the Bondi gauge (5.1) but whose defining functions are

$$
\tilde\beta = \lambda\beta, \quad \tilde V = -r + \lambda V, \quad \tilde U = \lambda U, \quad \tilde\varphi = \lambda\varphi,
\tag{5.10}
$$

we notice that this is now a perturbation around Minkowski. Solving the (r,r), (r,u), (r,ϕ) and (ϕ,ϕ)-components of the linearized Einstein equations, this time without cosmological constant, gives

$$\begin{aligned}\beta &= \beta_0(u,\phi),\\ U &= -\frac{N(u,\phi)}{r^2} + U_0(u,\phi) + \frac{2\partial_\phi\beta_0}{r},\\ V &= rM(u,\phi) + r\left(-2r\left(\partial_\phi U_0 + \partial_u\varphi\right)\right).\end{aligned} \tag{5.11}$$

The two residual equations, the (u,u) and (u,ϕ)-components of Einstein equations, are

$$\partial_u M = 2\partial_\phi U_0 + 2\partial_\phi^2 U_0 - 2\partial_u\beta_0 - 4\partial_u\partial_\phi^2\beta_0 + 2\partial_u\varphi + 2\partial_u\partial_\phi^2\varphi, \tag{5.12}$$

$$\partial_u N = \frac{1}{2}\partial_\phi M - \partial_\phi\beta_0. \tag{5.13}$$

To be more precise, we have that the (u,u) and (u,ϕ)-components of the linearized Einstein equations scale with r as

$$\delta E_{uu} = \frac{C_1(u,\phi)}{r} \quad \text{and} \quad \delta E_{u\phi} = \frac{C_2(u,\phi)}{r}, \tag{5.14}$$

such that $C_1 = 0 \Leftrightarrow$ (5.12) and $C_2 = 0 \Leftrightarrow$ (5.13). These conditions are the holographic input we need for the proof of Sec. 4. The difference with the AdS case is that we cannot recast these two conservation equations as the divergence of a boundary energy–momentum tensor for the simple reason that there is no non-degenerate boundary metric that allows us to build the usual covariant derivative. In the following we will show how to obtain the right geometrical structure to describe the boundary geometry.

To perform the limit on the boundary, it is useful to decompose the boundary metric and energy–momentum tensor with respect to their scaling with ℓ. We start with the metric

$$g_{\mu\nu} = h_{\mu\nu} - \ell^{-2} n_\mu n_\nu, \tag{5.15}$$

where

$$n_\mu = \begin{pmatrix} 1 + 2\lambda\beta_0 \\ 0 \end{pmatrix}, \quad h_{\mu\nu} = \begin{pmatrix} 0 & -\lambda U_0 \\ -\lambda U_0 & 1 + 2\lambda\varphi \end{pmatrix}. \tag{5.16}$$

The inverse metric is

$$g^{\mu\nu} = -\ell^2 v^\mu v^\nu + h^{\mu\nu}, \tag{5.17}$$

where

$$v^\mu = \begin{pmatrix} 1 - 2\lambda\beta_0 \\ \lambda U_0 \end{pmatrix}, \quad h^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & 1 - 2\lambda\varphi \end{pmatrix}. \tag{5.18}$$

This decomposition allows us to define properly the geometry on the null infinity. It will be composed of a degenerate metric $h_{\mu\nu}$ (which induces a real metric on the boundary circle) whose kernel is given by the vector field v^μ which represents the time direction, a temporal one-form n_μ and the pseudo-inverse metric $h^{\mu\nu}$ (indeed, as $h_{\mu\nu}$ is degenerate, it does not enjoy a true inverse). These are the ingredients of a Carrollian geometry [40, 41]. One can check that they satisfy the following relations

$$h_{\mu\nu}v^\nu = 0, \quad h^{\mu\nu}n_\nu = 0, \quad v^\mu n_\mu = 1 \quad \text{and} \quad h^{\mu\sigma}h_{\sigma\nu} = \delta^\mu_\nu - v^\mu n_\nu\,, \tag{5.19}$$

at first order in λ. These can be taken as the defining relations of a Carrollian geometry. We will also make use of the scalings of the Christoffel symbols with ℓ:

$$\Gamma^\mu_{\nu\rho} = \ell^2 X^\mu_{\nu\rho} + Y^\mu_{\nu\rho} + \ell^{-2} Z^\mu_{\nu\rho}, \tag{5.20}$$

where $X^\mu_{\nu\rho}$, $Y^\mu_{\nu\rho}$ and $Z^\mu_{\nu\rho}$ can be written in terms of the Carrollian geometry as[11]

$$X^\mu_{\nu\rho} = -\frac{1}{2}v^\nu v^\sigma(\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\sigma\nu} - \partial_\sigma h_{\nu\rho}), \tag{5.21}$$

$$Y^\mu_{\nu\rho} = \gamma^\mu_{\nu\rho} + v^\mu v^\sigma\left(n_{(\nu}\partial_{\rho)}n_\sigma - (\partial_\sigma n_{(\nu})n_{\rho)}\right) + v^\mu\partial_{(\nu}n_{\rho)}, \tag{5.22}$$

$$Z^\mu_{\nu\rho} = h^{\mu\sigma}\left((\partial_\sigma n_{(\nu})n_{\rho)} - n_{(\nu}\partial_{\rho)}n_\sigma\right), \tag{5.23}$$

where $\gamma^\mu_{\nu\rho} = \frac{1}{2}h^{\mu\sigma}(\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\sigma\nu} - \partial_\sigma h_{\nu\rho})$ is the Levi-Civita of the pseudo metric $h_{\mu\nu}$.

The boundary energy–momentum tensor scales with ℓ as

$$T^{\mu\nu} = \ell^3 T_1^{\mu\nu} + \ell\, T_0^{\mu\nu}, \tag{5.24}$$

so the boundary dynamical data decomposes in two pieces, $T_0^{\mu\nu}$ and $T_1^{\mu\nu}$, defined on $\mathcal{I}^+$. For the perturbation in Bondi gauge, they are given by

$$T_0^{\mu\nu} = \frac{1}{8G}\begin{pmatrix} 0 & 2\lambda N \\ 2\lambda N & 1-\lambda(M+2\beta_0+2\varphi)\end{pmatrix}, \tag{5.25}$$

$$T_1^{\mu\nu} = \frac{1}{8G}\begin{pmatrix} 1-\lambda(M+6\beta_0+4\partial_\phi^2\beta_0) & \lambda(U_0+2\partial_\phi^2 U_0+2\partial_\phi\partial_u\varphi) \\ \lambda(U_0+2\partial_\phi^2 U_0+2\partial_\phi\partial_u\varphi) & -\lambda(2\partial_u\partial_\phi U_0+2\partial_u^2\varphi)\end{pmatrix}. \tag{5.26}$$

We can now take the $\ell\to\infty$ limit of the conservation equations. We obtain the two following conservation laws, a scalar one and a vector one

$$n_\sigma\left(\partial_\mu T_1^{\mu\sigma} + Y^\mu_{\mu\rho}T_1^{\rho\sigma} + Y^\sigma_{\mu\rho}T_1^{\mu\rho} + X^\mu_{\mu\rho}T_0^{\rho\sigma} + X^\sigma_{\mu\rho}T_0^{\mu\rho}\right) = 0\,, \tag{5.27}$$

$$h_{\nu\sigma}\left(\partial_\mu T_0^{\mu\sigma} + Y^\mu_{\mu\rho}T_0^{\rho\sigma} + Y^\sigma_{\mu\rho}T_0^{\mu\rho} + Z^\mu_{\mu\rho}T_1^{\rho\sigma} + Z^\sigma_{\mu\rho}T_1^{\mu\rho}\right) = 0\,. \tag{5.28}$$

In three dimensions, the vector conservation corresponds only to one equation since its projection on v^μ vanishes by definition. These two equations are the analog of the conservation of the stress tensor in AdS_3 and reproduce perfectly the two equations (5.12) and (5.13). They are the holographic input that we need in the proof in Sec. 4 to cancel the integration constants C_1 and C_2.

[11] One can check that $Y^\mu_{\nu\rho}$ is a torsionless "compatible" Carrollian connection [40], which means that it parallel transports v^μ and $h_{\mu\nu}$.

There is also a Carrollian equivalent of the relation between the trace of $T^{\mu\nu}$ and the scalar curvature. It is obtained simply by taking the $\ell \to \infty$ of the formula (5.9) which splits into two equations:

$$h_{\mu\nu}T_0^{\mu\nu} - n_\mu n_\nu T_1^{\mu\nu} = -\frac{R_0}{8G}\,, \tag{5.29}$$
$$h_{\mu\nu}T_1^{\mu\nu} = -\frac{R_1}{8G}\,, \tag{5.30}$$

where R_0 and R_1 are two Carrollian scalar curvatures defined as

$$\begin{aligned} R_0 &= R_Y - 2v^\mu v^\nu \left(\partial_{[\alpha} Z^\alpha_{\nu]\mu} + Y^\beta_{\mu[\nu} Z^\alpha_{\alpha]\beta} + Z^\beta_{\mu[\nu} Y^\alpha_{\alpha]\beta}\right) + 2h^{\mu\nu}\left(Z^\beta_{\mu[\nu} X^\alpha_{\alpha]\beta} + X^\beta_{\mu[\nu} Z^\alpha_{\alpha]\beta}\right), \\ R_1 &= -2v^\mu v^\nu \left(\partial_{[\alpha} Y^\alpha_{\nu]\mu} + Y^\beta_{\mu[\nu} Y^\alpha_{\alpha]\beta} + Z^\beta_{\mu[\nu} X^\alpha_{\alpha]\beta} + X^\beta_{\mu[\nu} Z^\alpha_{\alpha]\beta}\right) \\ &\quad + 2h^{\mu\nu}\left(\partial_{[\alpha} X^\alpha_{\nu]\mu} + Y^\beta_{\mu[\nu} X^\alpha_{\alpha]\beta} + X^\beta_{\mu[\nu} Y^\alpha_{\alpha]\beta}\right), \end{aligned} \tag{5.31}$$

and R_Y is the scalar curvature associated with $Y^\mu_{\nu\rho}$:

$$R_Y = h^{\mu\nu}\left(\partial_\alpha Y^\alpha_{\nu\mu} - \partial_\nu Y^\alpha_{\alpha\mu} + Y^\beta_{\mu\nu} Y^\alpha_{\alpha\beta} - Y^\beta_{\mu\alpha} Y^\alpha_{\nu\beta}\right). \tag{5.32}$$

Equations (5.29) and (5.30) are the third holographic input that we have to impose for the proof in Sec. 4. They are the equivalent of the tracelessness condition for the holographic stress tensor in AdS, that one has to impose on top of its conservation. For the Bondi perturbation, R_0 and R_1 are given by

$$R_0 = -4\partial_\phi^2 \beta_0\,, \tag{5.33}$$
$$R_1 = 2(\partial_\phi \partial_u U_0 + \partial_u^2 \varphi)\,. \tag{5.34}$$

Finally, we can focus on the case $\beta_0 = U_0 = \varphi = 0$, which is the space of solutions considered in Sec. 3.3 (see [10]). The two cuvature elements R_0 and R_1 vanish, therefore it corresponds to a "flat" Carrollian geometry on the boundary (we also have that $X^\mu_{\nu\rho}$, $Y^\mu_{\nu\rho}$ and $Z^\mu_{\nu\rho}$ vanish). Moreover, the two pieces of boundary dynamical data simplify to

$$T_0^{\mu\nu} = \frac{1}{8G}\begin{pmatrix} 0 & 2\lambda N \\ 2\lambda N & 1-\lambda M \end{pmatrix}, \tag{5.35}$$

$$T_1^{\mu\nu} = \frac{1}{8G}\begin{pmatrix} 1-\lambda M & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.36}$$

and their two conservation laws become

$$\partial_u M = 0\,, \tag{5.37}$$
$$\partial_\phi M = 2\partial_u N\,. \tag{5.38}$$

The solutions are given by $M = \Theta(\phi)$ and $N = \frac{u}{2}\partial_\phi \Theta + \Xi(\phi)$. One can check that with these defining functions, together with $\beta_0 = U_0 = \varphi = 0$, the line element (5.1) becomes (3.44):

$$ds^2 = \eta_{ab}dx^a dx^b + \lambda\left(\Theta(\phi)\,du^2 + 2\left(\Xi(\phi) + \frac{u}{2}\partial_\phi\Theta(\phi)\right) du d\phi\right) + O(\lambda^2), \tag{5.39}$$

which is the metric perturbation we have used for exact on-shell computations.

6 Generalization to 4d

In this section, we give the Ryu-Takayanagi prescription in 4d that follows from the assumptions given in Sec. 2. We find a Rindler transformation and describe the corresponding entangling regions and RT surfaces. We show that the general RT prescription depends on the choice of an infalling light sheaf, i.e. a choice of bulk light rays which intersects $\mathcal{I}^+$ at the boundary ∂A of the entangling region. Using these RT surfaces, we show that the gravitational equations of motion are equivalent to the first law of entanglement, assuming that the constraints on the boundary stress tensor imply the vanishing of δE_{ua} at infinity. Our proof is valid for any theory of gravity, including higher-derivative terms.

6.1 Ryu-Takayanagi prescription in 4d Minkowski

Rindler transformation. We describe a transformation which satisfies the assumptions of the generalized Rindler method. It maps the coordinates (u, θ, ϕ) on $\mathcal{I}^+$ into the coordinates (τ, ρ, η) according to

$$\begin{aligned} u &= \frac{\tau}{\cosh\rho}\,, \\ \theta &= \arctan(\sinh\rho) + \frac{\pi}{2}\,, \\ \phi &= \eta\,. \end{aligned} \tag{6.1}$$

This can be compared with the 3d case (3.6). It is in fact a BMS_4 superrotation, which maps the round sphere into a conformally flat space

$$d\theta^2 + \sin^2\theta\, d\phi^2 = \frac{1}{\cosh^2\rho}(d\rho^2 + d\eta^2)\,. \tag{6.2}$$

It is a Rindler transformation because the space that we obtain has a thermal identification

$$\rho \sim \rho + 2\pi i\,. \tag{6.3}$$

The modular flow ζ_A is the generator of this thermal circle, given by

$$\zeta_A = 2\pi\partial_\rho = -2\pi\left(u\cos\theta\,\partial_u + \sin\theta\,\partial_\theta\right)\,. \tag{6.4}$$

This vector belongs to the BMS_4 algebra and hence annihilates the vacuum, as required for a boundary modular flow. To obtain the bulk modular flow, we can look for a Killing of 4d Minkowski which asymptotes to ζ_A. We obtain

$$\xi_A = -2\pi\left(u\cos\theta\,\partial_u - (r+u)\cos\theta\,\partial_r + \frac{(r+u)}{r}\sin\theta\,\partial_\theta\right)\,. \tag{6.5}$$

Note that this is much simpler than trying to find the gravitational solution which is dual to a thermal state, i.e. the flat space analog of the hyperbolic black hole, which is what we do in App. A.

Watermelons. We focus on entangling regions that lie on the slice $u = 0$ as other configurations can be obtained by acting with bulk isometries. The entangling regions A are given by patches on the sphere at infinity that are invariant under the flow. They are "watermelon slices" whose boundaries follow the flow and with width ℓ_ϕ. They can be parametrized as

$$-\tfrac{\ell_\phi}{2} \le \phi \le \tfrac{\ell_\phi}{2}, \qquad 0 \le \theta \le \pi\,, \tag{6.6}$$

and are represented in Fig. 5. The domain of dependence $\mathcal{D}$ and its boundary $\partial\mathcal{D}$ can be checked to be invariant under the flow.[12]

Generalized Rindler transformations. When the sphere is written in complex coordinates

$$z = e^{i\phi}\cot(\tfrac{\theta}{2}), \qquad \bar{z} = e^{-i\phi}\cot(\tfrac{\theta}{2})\,, \tag{6.7}$$

we observe that the Rindler transformation (6.1) can be written as

$$z \to e^{-w}, \qquad \bar{z} \to e^{-\bar{w}}, \tag{6.8}$$

where $w = \rho - i\eta$, $\bar{w} = \rho + i\eta$. This suggests a way to obtain more general Rindler transformations, obtained by acting with a Möbius transformation on the sphere. Let's consider the following transformation

$$\begin{aligned} u &\to \frac{\cos\theta_0}{\cosh\rho + \cos\eta\sin\theta_0}\tau\,, \\ z &\to \frac{\sin\theta_0 + e^{w}(1+\cos\theta_0)}{\sin\theta_0\, e^{w} + (1+\cos\theta_0)}, \\ \bar{z} &\to \frac{\sin\theta_0 + e^{\bar{w}}(1+\cos\theta_0)}{\sin\theta_0\, e^{\bar{w}} + (1+\cos\theta_0)}\,, \end{aligned} \tag{6.9}$$

which is a BMS$_4$ transformation. The boundary modular flow is the vector $2\pi\partial_\rho$ given by

$$\zeta_A = -\frac{2\pi}{\cos\theta_0}\left(u\cos\theta\,\partial_u + k\right)\,, \tag{6.10}$$

where k is a conformal Killing of the sphere given by

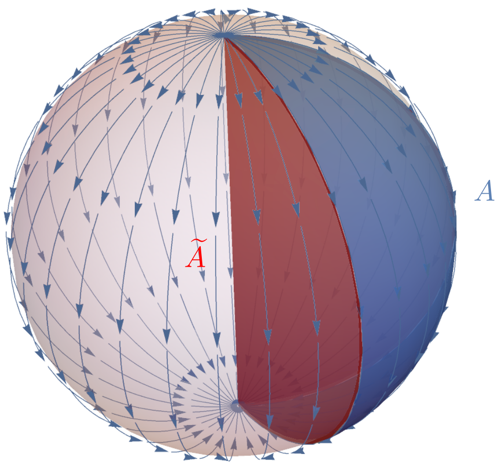
$$k = (\sin\theta - \sin\theta_0\cos\phi)\,\partial_\theta + \sin\theta_0\cot\theta\sin\phi\,\partial_\phi\,. \tag{6.11}$$

The bulk modular flow is

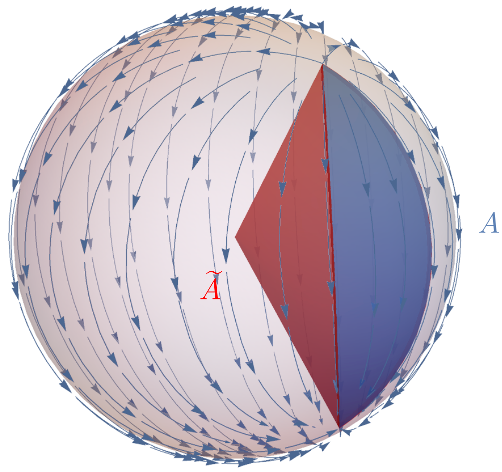
$$\xi_A = \frac{2\pi}{\cos\theta_0}\left((u+r)\cos\theta\,\partial_r - \frac{u}{r}\sin\theta\,\partial_\theta\right) + \zeta_A\,. \tag{6.12}$$

It is obtained as the Killing vector of 4d Minkowski spacetime which matches with ζ_A on the boundary. The transformation described in (6.9) has also the thermal identification $\rho \sim \rho + 2\pi i$. It is a one-parameter generalization of the previous Rindler transformation (6.8), obtained by considering a more general conformal Killing k of the sphere.

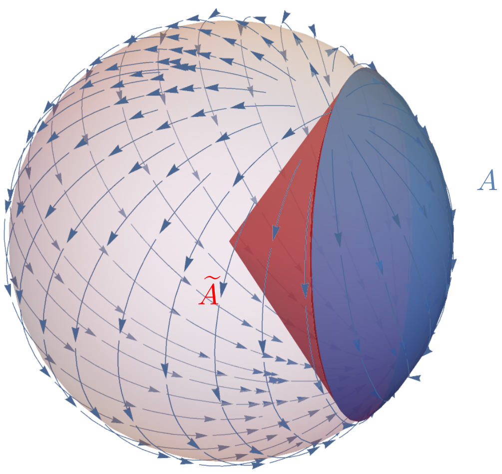
[12] The boundary ∂A is not fixed pointwise by the flow, which is different from the AdS case or in 3d Minkowski. This is inevitable for 4d Minkowski because there is no conformal Killing on the sphere which admits a one-dimensional set of fixed points [42].



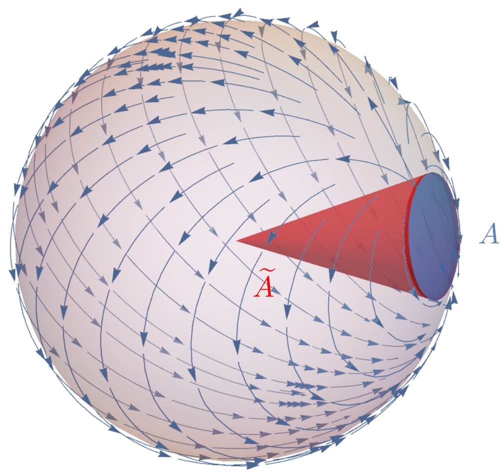
"Watermelon slice" with $\theta_0 = 0$



Deformed watermelon with $\theta_0 > 0$



Disk at $\ell_\phi = \pi$



Smaller disk at $\ell_\phi = \pi$

Figure 5: Examples of entangling regions (in blue) and associated RT surfaces (in red) for 4d Minkowski on the constant $u = 0$ slice. They are associated to the modular flow (6.10) and its bulk extension (6.12).

Generalized watermelons. To understand the entangling regions associated to this modular flow, we should look at regions on S^2 that are preserved under k. There are two fixed points given by

$$P_- : (\theta, \phi) = (\theta_0, 0), \quad P_+ : (\theta, \phi) = (\pi - \theta_0, 0)\,. \tag{6.13}$$

The vector field k is a flow from P_- to P_+. The entangling regions are deformed "watermelons slices" whose boundaries are tangent to this flow, as depicted in Fig. 5. The domain of dependence $\mathcal{D}$ and its boundary $\partial\mathcal{D}$ can be checked to be invariant under the flow. An entangling region A can be parametrized by

$$-\ell(\theta) \leq \phi \leq \ell(\theta), \qquad \theta_0 \leq \theta \leq \pi - \theta_0\,, \tag{6.14}$$

where $\ell(\theta)$ satisfies the condition

$$\ell'(\theta) = (k^\phi/k^\theta)\Big|_{\phi=\ell(\theta)}\,, \tag{6.15}$$

which ensures that the boundary ∂A is tangent to the vector field k. This makes sure that A and ∂A are preserved under the modular flow. Explicitly, we obtain

$$\tan\ell(\theta) = \frac{\cos(2\theta_0) - \cos(2\theta)}{2\sin\theta\left(\cot(\frac{\ell_\phi}{2})\sin\theta + \sin\theta_0\sqrt{1+\cot^2(\frac{\ell_\phi}{2}) - \frac{\sin^2\theta_0}{\sin^2\theta}}\right)}\,, \tag{6.16}$$

where ℓ_ϕ parametrizes the width of the entangling region. For $\theta_0 = 0$, we have $\ell(\theta) = \ell_\phi/2$. At small ℓ_ϕ, we have

$$\ell(\theta) = \left(1 - \frac{\sin\theta_0}{\sin\theta}\right)\frac{\ell_\phi}{2} + O(\ell_\phi^2)\,. \tag{6.17}$$

At the special value $\ell_\phi = \frac{\pi}{2}$, the watermelon becomes a disk on the sphere. This is illustrated in Fig. 5. The opening angle of the disk is $\pi - 2\theta_0$.

Ryu-Takayanagi surfaces. The entangling regions described above are the generalization of the 3d story with $\ell_u = 0$. The bulk modular flow (6.12) is very similar to the bulk modular flow in three dimensions (3.10). The RT surfaces associated to the above regions are easy to describe, they lie on the slice $u = 0$ and are the union of all light rays starting at the origin and ending on ∂A. We illustrate this prescription in Fig. 5 by representing the sphere at infinity on the slice $u = 0$. The entangling regions A are in blue and the RT surfaces $\widetilde{A}$ are in red. We also represent the boundary modular flow on the sphere. The entanglement entropy of the region A is then given by

$$S_A = \int_{\widetilde{A}} \mathbf{Q}[\xi_A]\,. \tag{6.18}$$

For Einstein gravity in the Minkowski vacuum, the areas of all these RT surfaces vanish because they have a null tangent vector everywhere.

Perturbations. As an illustration, we can consider on-shell perturbations of 4d Minkowski in the Bondi gauge. The flat metric is given by

$$ds^2 = -du^2 - 2dudr + r^2\gamma_{ij}dx^i dx^j\,, \qquad \gamma_{ij}dx^i dx^j = d\theta^2 + \sin^2\theta\, d\phi^2\,, \tag{6.19}$$

we consider the linearized on-shell perturbations studied in [43] with $C_{ij} = 0$, which corresponds to setting the gravitational wave aspect to zero. Asymptotically, the perturbation reads

$$h_{uu} = \frac{2}{r}\mathcal{M}(x^i) + O(r^{-2}), \qquad h_{ui} = \frac{1}{r}\mathcal{N}_i(x^i) + O(r^{-2}), \qquad h_{ij} = O(1)\,. \tag{6.20}$$

The subleading pieces in r should not contribute to the charges at infinity. This allows us to compute δE_A in a similar way as in the previous section. We obtain on a slice $u=0$

$$\delta E_A = \frac{3}{8\cos\theta_0}\int_A d\theta d\phi\,[(\cos\phi\sin\theta_0 - \sin\theta)\mathcal{N}_\theta(\theta,\phi) - \cot\theta\sin\phi\sin\theta_0\,\mathcal{N}_\phi(\theta,\phi)]\,, \quad (6.21)$$

which can be written in terms of the boundary modular flow (6.10) as

$$\delta E_A = \frac{3}{16\pi}\int_A d\theta d\phi\,\zeta_A^i\mathcal{N}_i(\theta,\phi)\,. \quad (6.22)$$

Exactly as in the 3d case, the entropy has to be computed using the refined prescription (3.48) where we regulate the corner of the RT surface. The fact that $\delta E_A = S_A$, which has to be positive, gives some constraints on the perturbations that can be described by a quantum system on $\mathcal{I}^+$ satisfying our assumptions, similar to the discussion in Sec. 3.4. These constraints impose the functions $\mathcal{N}_i$ in the perturbation to be such that (6.22) is positive for a given region A. This selects a subspace $\mathcal{H}_{\text{code}}$ on which K_A is bounded from below and this makes the density operator e^{-K_A} is well-defined.

6.2 General 4d prescription

In this section, we discuss the general RT prescription in 4d, in the same spirit as the 3d discussion of Sec. 3.2. Given a boundary entangling region, we will describe the most general choice of light sheaf that satisfies the requirements to give a good RT configuration. That is, the light sheaf must connect ∂A to the Rindler bifurcation surface and the modular flow must be tangent to it. As explained in the 3d case, the first condition ensures that we can define an RT surface (as a portion of the Rindler bifurcation surface) and the second condition is required to have a well-defined first law.

Modular flow for non-zero ℓ_u. In Cartesian coordinates (t,x,y,z), the bulk modular flow given (6.5) takes the following form

$$\xi_A = \frac{2\pi}{\cos\theta_0}\left[z\,\partial_t + z\sin\theta_0\,\partial_x + (t - x\sin\theta_0)\,\partial_z\right]\ . \quad (6.23)$$

We note that this it is similar to the 3d bulk modular flow at $\ell_u = 0$. This suggests the following generalization for $\ell_u \neq 0$ in 4d, obtained by performing a bulk translation

$$z \to z + \frac{\ell_u}{2\cos\theta_0}\ , \quad (6.24)$$

which leads to

$$\xi_A = \frac{2\pi}{\cos\theta_0}\left[\left(z + \frac{\ell_u}{2\cos\theta_0}\right)\partial_t + \left(z\sin\theta_0 + \frac{\ell_u\tan\theta_0}{2}\right)\partial_x + (t - x\sin\theta_0)\,\partial_z\right]\ . \quad (6.25)$$

Going back to Bondi coordinates (u,r,θ,ϕ) and taking the limit $r\to+\infty$, we obtain the corresponding 4d boundary modular flow, which reads

$$\begin{aligned}\zeta_A \;=\; \frac{2\pi}{\cos\theta_0}\Bigg[&\left(-u\cos\theta + \frac{\ell_u}{2\cos\theta_0}(1 - \sin\theta_0\sin\theta\cos\phi)\right)\partial_u \\ &+ (\sin\theta_0\cos\phi - \sin\theta)\,\partial_\theta + \sin\theta_0\cot\theta\sin\phi\,\partial_\phi\Bigg]\ .\end{aligned} \quad (6.26)$$

One can check that this modular flow follows from a generalized Rindler transform, which is the previous Rindler transform (6.9) with a different transformation for u

$$
\begin{aligned}
u &\rightarrow \frac{\cos\theta_0}{\cosh\rho\ +\cos\eta\sin\theta_0}\left(\tau+\frac{\ell_u}{2\cos\theta_0}\sinh\rho\right), \\
z &\rightarrow \frac{\sin\theta_0+e^{w}(1+\cos\theta_0)}{\sin\theta_0\, e^{w}+(1+\cos\theta_0)}, \\
\bar{z} &\rightarrow \frac{\sin\theta_0+e^{\bar{w}}(1+\cos\theta_0)}{\sin\theta_0\, e^{\bar{w}}+(1+\cos\theta_0)},
\end{aligned}
\tag{6.27}
$$

and which remains a BMS_4 transformation. The generator of the thermal circle $2\pi\partial_\rho$ reproduces the boundary modular flow given above. This was guaranteed to work because, as in 3d, the case $\ell_u \neq 0$ is simply the image of the case $\ell_u = 0$ by a bulk translation, which becomes on the boundary

$$
u \rightarrow u + \frac{\ell_u}{2\cos\theta_0}\cos\theta\ . \tag{6.28}
$$

On the boundary, this bulk translation changes the shape of the region A which is the same as before but with an extension in u:

$$
u = \frac{\ell_u}{2\cos\theta_0}\cos\theta\ , \qquad \theta\in[\theta_0,\pi-\theta_0]\ . \tag{6.29}
$$

Similarly to 3d, the bulk modular flow (6.25) is simply a boost. This can be seen explicitly by defining new coordinates

$$
\tilde{t} = \frac{1}{\cos\theta_0}t - \tan\theta_0\, x, \qquad \tilde{x} = \frac{1}{\cos\theta_0}x - \tan\theta_0\, t, \qquad \tilde{z} = z + \frac{\ell_u}{2\cos\theta_0}\ , \tag{6.30}
$$

in which the modular flows is given by

$$
\xi_A = 2\pi\left(\tilde{z}\,\partial_{\tilde{t}} + \tilde{t}\,\partial_{\tilde{z}}\right)\ . \tag{6.31}
$$

In App. A we show that, exactly like in the 3d case, there exists a change of coordinates in the bulk defined on the exterior of a Rindler horizon that maps to the transformation (6.27) on the boundary.

RT prescription. In 4d, the prescription where we impose that the light rays pass through the origin $r = 0$ is inconsistent in the case $\ell_u \neq 0$ because most light rays won't have an intersection with the bifurcation surface. Instead, we should consider the most general light sheaf which satisfies the requirements necessary for a good RT configuration, as was done in Sec. 3.2 for the 3d case. We will take all these choices of light sheaf to be equally physical, reflecting a choice of UV cutoff in the putative dual theory.

The boundary of A on $\mathcal{I}^+$ has two pieces $\partial A = B_+ \cup B_-$ which can be parametrized as

$$
\begin{aligned}
B_+ :&\quad \phi = \ell(\theta), &\qquad \theta_0 \leq \theta \leq \pi-\theta_0\ , \\
B_- :&\quad \phi = -\ell(\theta), &\qquad \theta_0 \leq \theta \leq \pi-\theta_0\ .
\end{aligned}
\tag{6.32}
$$

where $\ell(\theta)$ is defined in (6.16), while their extension in the $u-$direction is given by (6.29). The most general light rays that arrive at a point $(\theta,\phi) = (\theta,\pm\ell(\theta))$ on $\mathcal{I}^+$ can be parametrized as follows in Cartesian coordinates

$$\gamma_+(\theta) : \begin{cases} t = s + T_+(\theta) \\ x = s\sin\theta\cos\ell(\theta) + X_+(\theta) \\ y = s\sin\theta\sin\ell(\theta) + Y_+(\theta) \\ z = s\cos\theta + Z_+(\theta) \end{cases} , \qquad \gamma_-(\theta) : \begin{cases} t = s + T_-(\theta) \\ x = s\sin\theta\cos\ell(\theta) + X_-(\theta) \\ y = s\sin\theta\sin\ell(\theta) + Y_-(\theta) \\ z = s\cos\theta + Z_-(\theta) \end{cases} , \tag{6.33}$$

and the arbitrary functions $T_\pm(\theta), X_\pm(\theta), Y_\pm(\theta), Z_\pm(\theta)$ reflect the ambiguity in choosing these light rays. This ambiguity will be partially fixed by imposing the necessary requirements. Firstly, the light rays $\gamma_+(\theta)$ and $\gamma_-(\theta)$ should intersect $\mathcal{I}^+$ at ∂A, so that the value of u at infinity is given by (6.29). Then we should impose that all these light rays intersect the bifurcation surface of the Rindler horizon associated with the bulk modular flow, i.e. $\tilde{t} = \tilde{z} = 0$. To do this, we impose that after transforming (6.33) to the new Cartesian coordinates (6.30), $\tilde{z}$ and $\tilde{t}$ become proportional. This also imposes the relation

$$\tilde{z} = f(\theta)\,\tilde{t}, \qquad f(\theta) = \frac{\cos\theta\cos\theta_0}{1-\cos\ell(\theta)\sin\theta\tan\theta_0} \ . \tag{6.34}$$

Denoting the two light sheafs

$$\begin{aligned} \gamma_+ &= \{\gamma_+(\theta) \mid \theta \in [\theta_0, \pi-\theta_0]\} \ , \\ \gamma_- &= \{\gamma_-(\theta) \mid \theta \in [\theta_0, \pi-\theta_0]\} \ , \end{aligned} \tag{6.35}$$

we see that γ_+ and γ_- span over the quadrant $\tilde{t} \geq |\tilde{z}|$ because the function $f(\theta)$ is a bijection between the interval $[\theta_0, \pi-\theta_0]$ and the interval $[-1,1]$. To find the region γ, which is a 2d surface in 4d, we should consider the intersection of $\gamma_\pm$ with the bifurcation surface, which is the plane $(\tilde{x},\tilde{y})$. From the explicit parametrization, we find that the intersection of $\gamma_\pm$ with this plane is restricted to the lines

$$\tilde{x} \pm \cos\theta_0\tan(\tfrac{\ell_\phi}{2})\,\tilde{y} = 0 \ . \tag{6.36}$$

Lastly, we should impose that the modular flow is tangent to the light sheaf $\gamma_+ \cup \gamma_-$ which is required to have a well-defined first law. This is necessary because we need $\xi_A \cdot \Theta$ to vanish when integrated on the light sheaf, see the paragraph below for more details. To do this, we consider the two tangent vectors

$$\frac{\partial x^\mu}{\partial\theta}\partial_\mu, \qquad \frac{\partial x^\mu}{\partial s}\partial_\mu \ , \tag{6.37}$$

and we require that the modular flow ξ_A can be written as a linear combination of those. For the light sheaf γ_+, we find that this is only possible if the light sheaf γ_+ intersects the bifurcation surface at a single point P_+. That is, we need all the light rays in γ_+ to converge to the same point P_+ on the bifurcation surface. We have a similar condition on γ_- which should intersect the bifurcation surface at a single point P_-. These points cannot be arbitrary in the plane $(\tilde{x},\tilde{y})$ since they have to belong to the lines given in (6.36). Importantly, P_+ and P_- don't have to be the same. Enforcing all these constraints, we are able to fix the functions $T_\pm(\theta), X_\pm(\theta), Y_\pm(\theta), Z_\pm(\theta)$ and we can write the following simpler parametrization for the light sheafs

$$\gamma_\pm(\theta) : \begin{cases} t = \mp\tilde{y}_\pm \sin\theta_0 \tan(\frac{\ell_\phi}{2}) + s, \\ x = \mp\tilde{y}_\pm \tan(\frac{\ell_\phi}{2}) + s\sin\theta\cos\ell(\theta) \\ y = \tilde{y}_\pm \pm s\sin\theta\sin\ell(\theta) \\ z = -\frac{\ell_u}{2\cos\theta_0} + s\cos\theta \end{cases} . \tag{6.38}$$

In this parametrization, the light sheafs $\gamma_\pm$ intersect the bifurcation surface $\tilde{t} = \tilde{z} = 0$ at P_+ and P_- whose coordinates are given by

$$P_\pm : \quad \tilde{t} = \tilde{z} = 0, \qquad \tilde{y} = \tilde{y}_\pm, \qquad \tilde{x} = \mp\cos\theta_0 \tan(\tfrac{\ell_\phi}{2})\tilde{y}_\pm . \tag{6.39}$$

The simplest choice is to take $P_- = P_+$. The RT configuration that we obtain is the one described in the previous section (up to a bulk isometry) and the RT surface has a conical shape. We can also have configurations where P_- and P_+ are separated. In this case, we should add additional light rays to close the light sheaf. To do this, we define new light sheafs γ_N and γ_S consisting of light rays that go from the two poles of ∂A given by

$$N : (\theta,\phi) = (\theta_0, 0), \qquad S : (\theta,\phi) = (\pi - \theta_0, 0) , \tag{6.40}$$

and intersect the bifurcation surface. It turns out that it is possible to make such a light ray intersect an arbitrary point on the bifurcation surface. For example, a parametrization of γ_N and γ_S can be given as

$$\gamma_N(v) : \begin{cases} \tilde{t} = s\cos\theta_0, \\ \tilde{x} = X_N(v) \\ \tilde{y} = Y_N(v) \\ \tilde{z} = s\cos\theta_0 \end{cases} , \qquad \gamma_S(v) : \begin{cases} \tilde{t} = s\cos\theta_0, \\ \tilde{x} = X_S(v) \\ \tilde{y} = Y_S(v) \\ \tilde{z} = -s\cos\theta_0 , \end{cases} \tag{6.41}$$

where v parametrizes the different light rays in the light sheafs γ_N and γ_S. These light sheafs satisfy our requirements: they intersect $\mathcal{I}_+$ at the two poles N and S (with the required value of u) and the bulk modular flow is tangent to them. The intersection of γ_N with the bifurcation surface is at $s = 0$ and gives a curve $C_N : (\tilde{x},\tilde{y}) = (X_N(v), Y_N(v))$ parametrized by v. Similarly, γ_S intersects the bifurcation surface at the curve $C_S : (\tilde{x},\tilde{y}) = (X_S(v), Y_S(v))$. Both of those curves must connect P_+ to P_-. The total light sheaf is given by $\gamma_+ \cup \gamma_N \cup \gamma_S \cup \gamma_-$. This configuration is illustrated in Fig. 6.

The surface γ is the portion of the bifurcation surface which is in the interior of the contour formed by C_N and C_S. It is depicted in the plane $(\tilde{x},\tilde{y})$ in Fig. 7. The RT surface $\widetilde{A}$ is the union of the total light sheaf with γ. The entanglement entropy of A is given by

$$S_A = \int_{\widetilde{A}} \mathbf{Q}[\xi_A] . \tag{6.42}$$

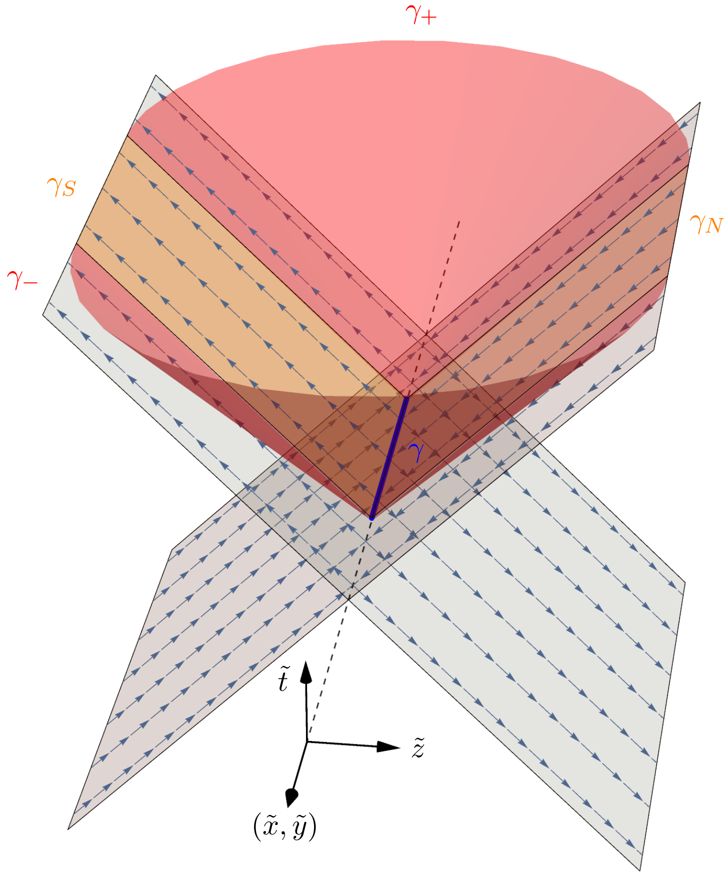


Figure 6: Ryu-Takayanagi configuration in coordinates $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ in which the bulk modular flow is a boost. The RT surface $\widetilde{A}$ is given by the union of the light sheaf $\gamma_- \cup \gamma_N \cup \gamma_S \cup \gamma_+$ with the surface γ on the Rindler bifurcation surface $(\tilde{x}, \tilde{y})$. See Fig. 7 for an illustration of γ in the $(\tilde{x}, \tilde{y})$-plane. The modular flow is tangent to the light sheafs γ_N, γ_S because they are portions of the Rindler horizons and to γ_-, γ_+ because they are half-cones whose transverse sections are hyperbolas which are tangent to the boost.

In Einstein gravity, the integration of Wald's functional on the light sheaf vanishes so the entanglement entropy of A is given by the area of the region γ

$$S_A = \frac{\text{Area}(\gamma)}{4G} \ . \tag{6.43}$$

The possible regions γ can be obtained by the following procedure: put two points $P_\pm$ on the two lines (6.36) (depicted in grey in Fig. 7). Then, connect them by two arbitrary curves C_N and C_S so that their union has a well-defined interior. This interior is the region γ and the entropy is given by the area of γ (in Einstein gravity). We see that as in 3d, the entropy is sensitive to the choice of light sheaf, which should reflect a choice of UV cutoff in the putative dual field theory.

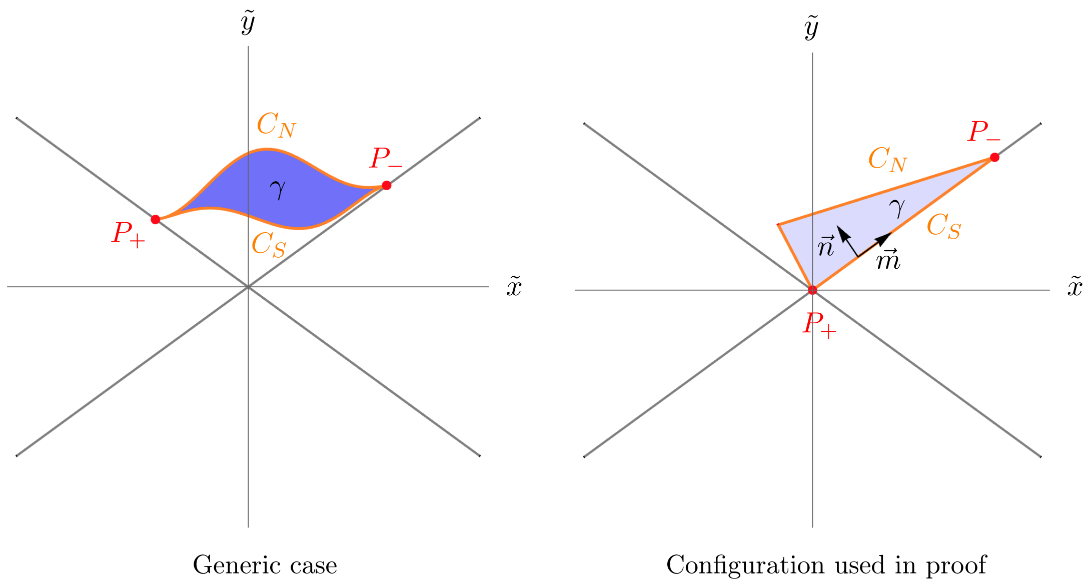


Figure 7: Ryu-Takayanagi configuration in the Rindler bifurcation surface $(\tilde{x}, \tilde{y})$. This surface intersects the light sheafs γ_+ and γ_- at the points P_+ and P_-, which are restricted to lie on the lines (6.36) (in gray). The light sheafs γ_N and γ_S intersect the bifurcation surface at the curves C_N and C_S (in orange). These light sheafs are represented in Fig. 6.

First law of entanglement. We have the following definitions

$$\delta S_A = \int_{\widetilde{A}} \delta \mathbf{Q}[\xi_A], \qquad \delta E_A = \int_A \boldsymbol{\chi} \ , \tag{6.44}$$

the first law states that these two expressions are equal on-shell. The 3d derivation of Sec. 3.3 can be carried out in 4d. In this derivation, the first law follows from the fact that

$$\int_{\widetilde{A}} \xi_A \cdot \mathbf{\Theta}[\delta g] = 0 \ , \tag{6.45}$$

which holds whenever ξ_A is tangent to $\widetilde{A}$. This is the case here since ξ_A vanishes on γ and is tangent to the light sheaf (this was one of our requirements). As a result, all the RT surfaces described here satisfy a first law for perturbations.

6.3 Linearized gravitational equations

In this section, we prove that the four-dimensional linearized gravitational equations follow from the first law of entanglement. The proof is very similar to the three-dimensional case described in Sec. 4, to which we refer for more details.

Reference configuration. We consider a watermelon A at $u = 0$ with $\ell_u = 0$. The first law of entanglement gives the equation

$$\int_{\theta_0}^{\pi-\theta_0} d\theta \int_{-\ell(\theta)}^{\ell(\theta)} d\phi \int_0^{+\infty} dr\, \xi^a \delta E_{ab} \varepsilon^b = 0 \, , \tag{6.46}$$

where $\varepsilon_a = \frac{1}{6}\varepsilon_{abcd}dx^b \wedge dx^c \wedge dx^d$ and $\ell(\theta)$ is defined in (6.16) and contains the parameter ℓ_ϕ which parametrizes the width of A. The dependence on ℓ_ϕ enters in a complicated fashion. However, we can differentiate with respect to ℓ_ϕ at $\ell_\phi = 0$, where we can use the expansion (6.17). This leads to

$$\int_{\theta_0}^{\pi-\theta_0} d\theta \left(1 - \frac{\sin\theta_0}{\sin\theta}\right) \int_0^{+\infty} dr\, \xi^a \delta E_{ab} \boldsymbol{\varepsilon}^b = 0\,, \tag{6.47}$$

where the LHS is evaluated at $\phi = 0$. In Bondi coordinates, we have

$$\boldsymbol{\varepsilon}^r = -\boldsymbol{\varepsilon}_u = -r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi\,. \tag{6.48}$$

The bulk modular flow (6.12) evaluated at $u = 0$ and $\phi = 0$ is given by

$$\xi_A = \frac{2\pi}{\cos\theta_0}\left(r\cos\theta\,\partial_r - (\sin\theta - \sin\theta_0)\,\partial_\theta\right)\,, \tag{6.49}$$

so the integral becomes

$$0 \;=\; \int_{\theta_0}^{\pi-\theta_0} d\theta\,(\sin\theta - \sin\theta_0) \int_0^{+\infty} dr\,\left(-r^3\cos\theta\,\delta E_{rr} + r^2(\sin\theta - \sin\theta_0)\,\delta E_{r\theta}\right)\,. \tag{6.50}$$

The expansion around $\theta_0 = \frac{\pi}{2}$ implies that

$$\int_0^{+\infty} dr\,\left(r^3\partial_\theta\delta E_{rr} + 2\,r^2\delta E_{r\theta}\right)\big|_{(u,\theta,\phi)=(0,\frac{\pi}{2},0)} = 0\,. \tag{6.51}$$

Rotations and time translations. As in the 3d case, we can consider new configurations obtained by performing rotations. They are the same as the reference configuration but centered at $\phi = \phi_0$. We can also consider a translation $u \to u + u_0$ in retarded time u. The Jacobians of these transformations are the identity which implies that the expression (6.51) becomes

$$\int_0^{+\infty} dr\,\left(r^3\partial_\theta\delta E_{rr} + 2\,r^2\delta E_{r\theta}\right)\big|_{(u,\theta,\phi)=(u_0,\frac{\pi}{2},\phi_0)} = 0\,, \tag{6.52}$$

for any u_0 and ϕ_0.

light sheaf deformation. We consider the same boundary region A but with the more general configuration described in Sec. 6.2. For the proof, we consider the configuration depicted on the right of Fig. 7. We put P_+ at the origin and P_- at a distance ℓ from P_+ on one of the axis and we connect them by the two curves C_N and C_S, as represented on the figure. The configuration is parametrized by the length ℓ of the segment $[P_+P_-]$ and the overture angle α at P_-. The first law of entanglement gives

$$I(\alpha,\ell) = \int_{\Sigma(\alpha,\ell)} d\boldsymbol{\chi} = 0\,, \tag{6.53}$$

where Σ, the interior of the RT surface, depends on these two parameters α and ℓ. Let's denote by n the vector normal to the segment C_S

$$n = \cos(\tfrac{\ell_\phi}{2})\partial_{\tilde{x}} + \cos\theta_0\sin(\tfrac{\ell_\phi}{2})\partial_{\tilde{y}}\,, \tag{6.54}$$

Taking the derivative of (6.53) with respect to α and evaluating at $\alpha = 0$, we obtain

$$\int_0^{+\infty} ds \int_0^{\ell} dv\, (n \cdot d\boldsymbol{\chi}) = 0 \ , \tag{6.55}$$

where we have used the fact that γ_N (6.41) can be parametrized by s and v. The vector tangent to C_S is given by

$$m = \cos(\tfrac{\ell_\phi}{2})\partial_{\tilde{y}} - \cos\theta_0 \sin(\tfrac{\ell_\phi}{2})\partial_{\tilde{x}} \ . \tag{6.56}$$

We can now take the derivative with respect to ℓ and evaluate at $\ell = 0$.

$$\int_0^{+\infty} dr\, (m \cdot (n \cdot d\boldsymbol{\chi}))|_{(\theta,\phi)=(\theta_0,0)} = 0 \tag{6.57}$$

We have reduced the integral to a light ray going from the origin to the point of ∂A with $(\theta, \phi) = (\theta_0, 0)$ (and $u = 0$ as we are considering a region A with $\ell_u = 0$). We can then go to Bondi coordinates. From the change of coordinates, we can compute

$$\begin{aligned} m &= -\sin\theta_0 \sin(\tfrac{\ell_\phi}{2})\,\partial_r - \frac{\cos\theta_0 \sin(\frac{\ell_\phi}{2})}{r}\partial_\theta + \frac{\cos(\frac{\ell_\phi}{2})}{r\sin\theta_0}\partial_\phi \ , \\ n &= \cos(\tfrac{\ell_\phi}{2})\tan\theta_0\,\partial_r + \frac{\cos(\frac{\ell_\phi}{2})}{r}\partial_\theta + \frac{\cot\theta_0 \sin(\frac{\ell_\phi}{2})}{r}\partial_\phi \ , \end{aligned} \tag{6.58}$$

when evaluated at $\theta = \theta_0$ and $\phi = 0$. In the definition (4.1) of $d\boldsymbol{\chi}$, the non-trivial contribution comes from

$$\boldsymbol{\varepsilon}^r = -\boldsymbol{\varepsilon}_u = -r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi \ . \tag{6.59}$$

Hence, we obtain that

$$m \cdot (n \cdot d\boldsymbol{\chi})|_{\gamma_N} = 2\left(1 - \sin^2\!\theta_0 \sin^2(\tfrac{\ell_\phi}{2})\right)\, \xi_A^a \delta E_{ar} dr \ . \tag{6.60}$$

The bulk modular flow at $(u, \theta, \phi) = (0, \theta_0, 0)$ is simply given by

$$\xi_A = 2\pi r\, \partial_r \tag{6.61}$$

Hence, we obtain

$$\int_0^{+\infty} dr\, r \delta E_{rr}(0, r, \theta_0, 0) = 0 \ , \tag{6.62}$$

As previously, we can act with rotations and time translations to show that we have

$$\int_0^{+\infty} dr\, r \delta E_{rr}(u_0, r, \theta_0, \phi_0) = 0 \ , \tag{6.63}$$

for arbitrary u_0, θ_0, ϕ_0.

Radial translations. Let's consider a new configuration which is obtained by translating the reference configuration by a distance r_0 in the direction (θ_0, ϕ_0) of the light ray on which (6.63) is integrated. In Cartesian coordinates, such a translation is given by

$$t \to t + r_0, \qquad x \to x + r_0 \cos\theta_0 \cos\phi_0, \qquad y \to y + r_0 \cos\theta_0 \sin\phi_0, \qquad z \to z + r_0 \sin\theta_0 \ . \tag{6.64}$$

This leads to the new constraint

$$\int_{r_0}^{+\infty} dr\, (r - r_0)\, \delta E_{rr}(u_0, r, \theta_0, \phi_0) = 0 \ , \tag{6.65}$$

where we have also performed the change of variable $r \to r - r_0$ in the integral. Taking two derivatives with respect to r_0 shows that

$$\delta E_{rr}(u_0, r_0, \theta_0, \phi_0) = 0 \ , \tag{6.66}$$

for any value of $u_0, r_0, \theta_0, \phi_0$. From this, the equation (6.52) simplifies to

$$\int_0^{+\infty} dr\, r^2\, \delta E_{r\theta}(u_0, r, \tfrac{\pi}{2}, \phi_0) = 0 \ . \tag{6.67}$$

We use the same radial translation on this equation to obtain the constraint

$$\int_{r_0}^{+\infty} dr\, \frac{(r - r_0)^2}{r} \delta E_{r\theta}(u_0, r, \tfrac{\pi}{2}, \phi_0) \ . \tag{6.68}$$

Taking three derivatives with respect to r_0 implies that

$$\delta E_{r\theta}(u_0, r_0, \tfrac{\pi}{2}, \phi_0) = 0 \ , \tag{6.69}$$

which is true for any value of u_0, r_0, ϕ_0.

Vanishing of $\delta E_{r\theta}$ everywhere. The equation (6.69) at $\phi_0 = 0$ shows that $\delta E_{r\theta}$ vanishes on the semi-infinite line L_P given by $(\theta, \phi) = (\frac{\pi}{2}, 0)$. Let's now consider rotations in the plane (x, z). Under such rotations, L_P covers the full disk in the $y = 0$ plane, shown in orange in Fig. 8. The Jacobian of this transformation, when evaluated at $\phi = 0$, is diagonal in Bondi coordinates because it simply corresponds to a shift in θ. It is given explicitly by

$$\frac{\partial x^a}{\partial \tilde{x}^b} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \cos\alpha - \cot\theta \sin\alpha \end{pmatrix} , \tag{6.70}$$

so we obtain $\delta E_{r\theta} = 0$ when evaluated on this disk. For any point on this disk, we can then consider a rotation in the (x, y)-plane, whose Jacobian is the identity. This shows that $\delta E_{r\theta} = 0$ vanishes everywhere inside the ball. This implies that

$$\delta E_{r\theta} = 0 \, , \tag{6.71}$$

everywhere in the bulk. This procedure is illustrated in Fig. 8.

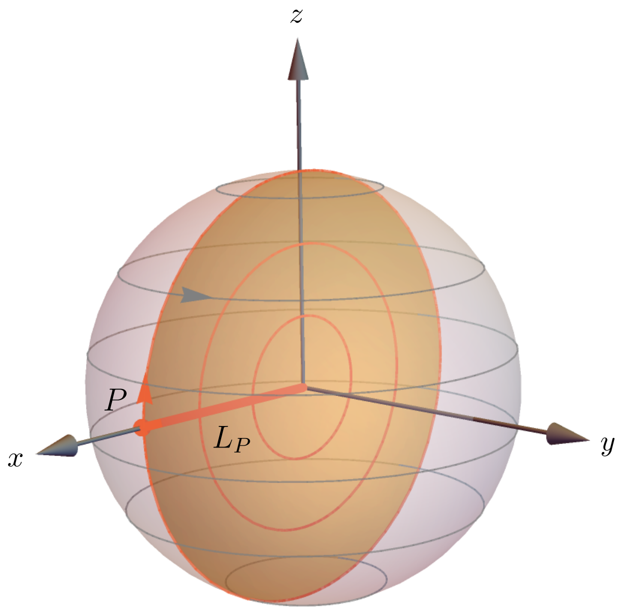


Figure 8: Illustration of the proof that $\delta E_{r\theta} = 0$. After showing that $\delta E_{r\theta}$ vanishes on the orange line L_P, we use the (x,z)-rotation (orange arrow) to show that $\delta E_{r\theta} = 0$ in the orange disk. With the (x,y)-rotation (gray arrow), we can show that $\delta E_{r\theta} = 0$ everywhere in the ball. These transformations all have diagonal Jacobians where they are evaluated so they don't mix $\delta E_{r\theta}$ with other components.

Boosts and rotations. We now act with boosts and rotations on the previous configurations to generate more constraints on δE_{ab}. Transforming the equation $\delta E_{r\theta} = 0$ under the infinitesimal (x,z)-rotation, the (t,y)-boost and the (t,x)-boost, we obtain

$$\delta E_{r\phi} = \delta E_{\theta\phi} = \delta E_{rr} = 0, \qquad \delta E_{\theta\theta} = -r^2 \delta E_{ur} \,. \tag{6.72}$$

Then, the image of $\delta E_{\theta\phi} = 0$ under the (x,z)-rotation implies that

$$\delta E_{\phi\phi} = -r^2 \sin^2\theta \, \delta E_{ur} \,. \tag{6.73}$$

Conservation equation. As in 3d, we consider the conservation equation

$$\nabla_a(\delta E^{ab}) = 0 \,, \tag{6.74}$$

which is always satisfied by the equations of motion. For $b = r$, this implies that

$$\partial_r(\delta E_{ur}) = 0 \,, \tag{6.75}$$

which leads to

$$\delta E_{ur} = C_0(u,\phi), \qquad \delta E_{\theta\theta} = -r^2 C_0(u,\phi), \qquad \delta E_{\phi\phi} = -r^2 \sin^2\theta \, C_0(u,\phi) \,. \tag{6.76}$$

We expect that an analysis similar to the 3d one in Sec. 5 can be performed in 4d and that it will lead to a trace condition and three conservation equations for the holographic stress tensor in 4d Minkowski. A proof of this statement will require a detailed analysis of the flat limit of perturbed AdS_4 in Bondi gauge, which we leave for future work. From now on, we will assume that these boundary conditions ensure the vanishing of the components δE_{ua} at leading asymptotic order. The trace condition, similar to (5.29) and (5.30) in 3d, should imply that $C_0 = 0$, leading to

$$\delta E_{ur} = \delta E_{\theta\theta} = \delta E_{\phi\phi} = 0\,, \tag{6.77}$$

everywhere in the bulk. The conservation equation (6.74) for $b = \theta, \phi$ gives

$$\partial_r(\delta E_{u\theta}) + \frac{2}{r}\delta E_{u\theta} = 0, \qquad \partial_r(\delta E_{u\phi}) + \frac{2}{r}\delta E_{u\phi} = 0\,. \tag{6.78}$$

The solutions of these equations are

$$\delta E_{u\theta} = \frac{C_1(u,\theta)}{r^2}, \qquad \delta E_{u\phi} = \frac{C_2(u,\phi)}{r^2}\,. \tag{6.79}$$

We expect that the conservation of the boundary stress tensor implies that $C_1 = C_2 = 0$, leading to $\delta E_{u\theta} = \delta E_{u\phi} = 0$. Finally, the conservation equation (6.74) for $b = u$ gives

$$\partial_r(\delta E_{uu}) + \frac{2}{r}\delta E_{uu} = 0\,. \tag{6.80}$$

which is solved by

$$\delta E_{uu} = \frac{C_3(u,\phi)}{r^2}\,, \tag{6.81}$$

and $C_3 = 0$ is expected to follow from the conservation of the boundary stress tensor. Thus, we have shown that all the components of the linearized gravitational equation vanish.

7 Conclusion

In this paper, we have considered holographic entanglement entropy in asymptotically flat spacetimes. Under some general assumptions on the dual field theory, an analog of the Ryu-Takayanagi formula was obtained in [25] to compute the entanglement entropies of 3d Minkowski spacetime. We have refined and generalized this prescription and showed that it satisfies a first law when perturbations are considered. Using this RT prescription, we have shown that the first law of entanglement is equivalent to the linearized gravitational equations of motion. We have also extended all these results to 4d.

This result could have also been phrased purely in classical gravity, although it is natural to motivate it from the perspective of holography. It will be important to understand better the dual field theory, and try to prove the assumptions detailed in Sec. 2. Some recent progress in this direction include [13, 44–52].

Another line of research would be to push further the consequences of the RT prescription described here. One could hope to get some hints on the microscopic definition of the dual field theory, or show that one of the assumptions was incorrect. An important feature of our analysis is the importance of the choice of an infalling light sheaf. We believe that this is a hint towards the UV structure of the dual theory, which we hope to investigate in future work. The RT formula in AdS has given rise to a wealth of results connecting quantum information to the emergence of spacetime. It would be interesting to investigate these ideas in asymptotically flat spacetimes, using the RT prescription described here.

Acknowledgments

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A Bulk Rindler transformation

In this appendix, we describe the bulk extension of the generalized Rindler transform (3.6) on the boundary. The image of Minkowski spacetime under this bulk transformation turns out to be the upper wedge of a Rindler spacetime.

Bulk Rindler transformation in 3d. We describe the change of coordinates that brings the metric in Bondi coordinates to the upper wedge of a Rindler spacetime. The Cartesian coordinates are related to Bondi coordinates using

$$t = u + r, \qquad x = r\cos\phi, \qquad y = r\sin\phi, \tag{A.1}$$

and the coordinates in which the modular flow is a boost are

$$\tilde{t} = \frac{t}{\sin(\frac{\ell_\phi}{2})} - \cot(\tfrac{\ell_\phi}{2})\, x, \qquad \tilde{x} = \frac{x}{\sin(\frac{\ell_\phi}{2})} - \cot(\tfrac{\ell_\phi}{2})\, t, \qquad \tilde{y} = y + \frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})}. \tag{A.2}$$

We define new coordinates $(\tilde{\tau}, \rho)$ satisfying

$$\tilde{t} = e^{\tilde{\tau}}\cosh\rho, \quad \tilde{y} = e^{\tilde{\tau}}\sinh\rho. \tag{A.3}$$

These coordinates only cover the upper wedge $\tilde{t}^2 - \tilde{y}^2 > 0$. In these coordinates the bulk metric and modular flow are given by

$$\xi_A = 2\pi\partial_\rho, \quad ds^2 = e^{2\tilde{\tau}}(-d\tilde{\tau}^2 + d\rho^2) + d\tilde{x}^2. \tag{A.4}$$

We recognize the Rindler metric and the bulk modular flow generates the (spacelike) Rindler evolution. The Rindler horizon is situated at $\tilde{\tau} = -\infty$. To obtain the bulk extension of the generalized Rindler transform, consider the new coordinates $\{\tau, \tilde{x}, \rho\}$ satisfying

$$\tau = e^{\tilde{\tau}} - \tilde{x}, \tag{A.5}$$

defined only for $\tau > -\tilde{x}$. The metric becomes

$$ds^2 = -d\tau^2 - 2d\tau d\tilde{x} + (\tau + \tilde{x})^2 d\rho^2, \tag{A.6}$$

and the bulk modular flow is still $\xi_A = 2\pi\partial_\rho$. The Rindler horizon is at $\tau = -\tilde{x}$. Finally, the bulk transformation is obtained by writing the new coordinates in terms of Bondi coordinates $\{u, r, \phi\}$:

$$\begin{aligned}
\tau &= -\tilde{x} + \left[\frac{1}{\sin^2(\frac{\ell_\phi}{2})} \left(r + u - r\cos\left(\frac{\ell_\phi}{2}\right)\cos\phi \right)^2 - \frac{1}{4}\left(\frac{\ell_u}{\sin(\frac{\ell_\phi}{2})} + 2r\sin\phi \right)^2 \right]^{1/2}, \\
\tilde{x} &= \frac{r\cos\phi}{\sin(\frac{\ell_\phi}{2})} - \cot(\tfrac{\ell_\phi}{2})\,(r+u), \\
\rho &= \operatorname{arccoth}\left(\frac{r + u - r\cos(\frac{\ell_\phi}{2})\cos\phi}{\frac{\ell_u}{2} + r\sin(\frac{\ell_\phi}{2})\sin\phi} \right). \qquad \text{(A.7)}
\end{aligned}$$

This coordinate system allows us to perform an asymptotic limit $r \to \infty$, which gives

$$\tau = \frac{2u\sin(\frac{\ell_\phi}{2}) - \ell_u \sin\phi}{2\cos\phi - 2\cos(\frac{\ell_\phi}{2})}, \qquad \text{(A.8)}$$

$$\rho = \operatorname{arccoth}\left(\frac{1 - \cos(\frac{\ell_\phi}{2})\cos\phi}{\sin(\frac{\ell_\phi}{2})\sin\phi} \right). \qquad \text{(A.9)}$$

One can check that this is exactly the inverse of the boundary generalized Rindler transformation (3.6), reproduced below

$$\begin{aligned}
u &= \frac{\sin(\frac{\ell_\phi}{2})}{\cosh\rho + \cos(\frac{\ell_\phi}{2})} \left(\tau + \frac{\ell_u}{2\sin(\frac{\ell_\phi}{2})}\sinh\rho \right), \qquad \text{(A.10)} \\
\phi &= \arctan\left(\frac{\sin(\frac{\ell_\phi}{2})\sinh\rho}{1 + \cos(\frac{\ell_\phi}{2})\cosh\rho} \right).
\end{aligned}$$

Bulk Rindler transformation in 4d. The same procedure can be carried out in 4d. Again, consider the bulk transformation from Bondi coordinates to Rindler coordinates in the upper wedge:

$$t = u + r, \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \qquad \text{(A.11)}$$

followed by

$$\tilde{t} = \frac{t}{\cos\theta_0} - \tan\theta_0\, x, \quad \tilde{x} = \frac{x}{\cos\theta_0} - \tan\theta_0\, t, \quad \tilde{y} = y, \quad \tilde{z} = z + \frac{\ell_u}{2\cos\theta_0}, \qquad \text{(A.12)}$$

and then

$$\tilde{t} = e^{\tilde{\tau}}\cosh\rho, \quad \tilde{z} = e^{\tilde{\tau}}\sinh\rho, \quad \tilde{x} = \mu\cos\eta, \quad \tilde{y} = \mu\sin\eta, \qquad \text{(A.13)}$$

where the last two spacelike coordinates are mapped to polar coordinates: $\mu \in [0, \infty[$ and $\eta \in [0, 2\pi[$. In these coordinates, the metric and the bulk modular flow become

$$\xi_A = 2\pi\partial_\rho, \quad ds^2 = e^{2\tilde{\tau}}(-d\tilde{\tau}^2 + d\rho^2) + d\mu^2 + \mu^2 d\eta^2. \qquad \text{(A.14)}$$

Exactly like in 3d, we recognize the Rindler metric and the bulk modular flow generates the (spacelike) Rindler evolution. The Rindler horizon is at $\tilde{\tau} = -\infty$. To obtain the bulk extension of the boundary generalized Rindler transform, we consider the new coordinates $\{\tau, \mu, \rho, \eta\}$, such that

$$\tau = e^{\tilde{\tau}} - \mu, \tag{A.15}$$

defined only for $\tau > -\mu$. The metric becomes

$$ds^2 = -d\tau^2 - 2d\tau d\mu + (\tau+\mu)^2 d\rho^2 + \mu^2 d\eta^2, \tag{A.16}$$

while the bulk modular flow is still given by $\xi_A = 2\pi\partial_\rho$. The new radial coordinate is μ and by taking the limit $\mu \to \infty$ we confirm that the boundary metric is indeed the degenerate flat metric $d\rho^2 + d\eta^2$. The Rindler horizon is at $\tau = -\mu$. Finally, the bulk transformation is obtained by writing the new coordinates in Bondi coordinates:

$$\begin{aligned}
\tau &= \sqrt{\left(\frac{r+u}{\cos\theta_0} - r\sin\theta\tan\theta_0\cos\phi\right)^2 - \frac{1}{4}\left(\frac{\ell_u}{\cos\theta_0} + 2r\cos\theta\right)^2} && \text{(A.17)}\\
&\quad - \sqrt{r^2\sin^2\!\theta\sin^2\!\phi + \left(\frac{r\sin\theta\cos\phi}{\cos\theta_0} - \tan\theta_0\,(r+u)\right)^2}, && \text{(A.18)}\\
\mu &= \sqrt{r^2\sin^2\!\theta\sin^2\!\phi + \left(\frac{r\sin\theta\cos\phi}{\cos\theta_0} - \tan\theta_0\,(r+u)\right)^2}, && \text{(A.19)}\\
\rho &= \operatorname{arctanh}\left(\frac{\frac{\ell_u}{2} + r\cos\theta_0\cos\theta}{(r+u) - r\sin\theta_0\sin\theta\cos\phi}\right), && \text{(A.20)}\\
\eta &= \arctan\left(\frac{r\cos\theta_0\sin\theta\sin\phi}{r\sin\theta\cos\phi - \sin\theta_0\,(r+u)}\right). && \text{(A.21)}
\end{aligned}$$

This allows us to perform the asymptotic limit $r \to \infty$ which gives

$$\begin{aligned}
\tau &= \frac{u\cos\theta_0 - \frac{\ell_u}{2}\cos\theta}{\sqrt{(\sin\theta_0\sin\theta\cos\phi - 1)^2 - \cos^2\!\theta_0\cos^2\!\theta}}\,, && \text{(A.22)}\\
\rho &= \operatorname{arctanh}\left(\frac{\cos\theta_0\cos\theta}{1 - \sin\theta_0\sin\theta\cos\phi}\right)\,, && \text{(A.23)}\\
\eta &= \arctan\left(\frac{\cos\theta_0\sin\theta\sin\phi}{\sin\theta\cos\phi - \sin\theta_0}\right)\,.
\end{aligned}$$

One can check that this is precisely the inverse of the boundary generalized Rindler transformation (6.27), reproduced below

$$\begin{aligned}
u &\to \frac{\cos\theta_0}{\cosh\rho\, + \cos\eta\sin\theta_0}\left(\tau + \frac{\ell_u}{2\cos\theta_0}\sinh\rho\right), && \text{(A.24)}\\
z &\to \frac{\sin\theta_0 + e^{w}(1+\cos\theta_0)}{\sin\theta_0\, e^{w} + (1+\cos\theta_0)},\\
\bar{z} &\to \frac{\sin\theta_0 + e^{\bar{w}}(1+\cos\theta_0)}{\sin\theta_0\, e^{\bar{w}} + (1+\cos\theta_0)},
\end{aligned}$$

where $z = e^{i\phi}\cot\left(\frac{\theta}{2}\right)$ and $w = \rho - i\eta$.

B Precisions on the general strategy

In this appendix, we make precise the general strategy explained in Sec. 4.1. Let $g : M \to M$ be a bulk isometry, $i : \Sigma \to M$ the original RT surface and $i_g = g \circ i : \Sigma \to M$ the image of this surface through isometry. The original RT surface is associated to a bulk modular flow ξ to which corresponds a two-form $d\boldsymbol{\chi}[\xi]$. The pullback of this two-form on Σ is

$$i^*(d\boldsymbol{\chi}[\xi]) = \xi^a(i(\sigma))\delta E_{ab}(i(\sigma))\frac{1}{2}\varepsilon^b_{cd}(i(\sigma))\frac{\partial x^c}{\partial \sigma^\alpha}\frac{\partial x^d}{\partial \sigma^\beta}d\sigma^\alpha \wedge d\sigma^\beta, \tag{B.1}$$

where σ stands for the coordinates on the two-dimensional manifold Σ. Suppose that from the vanishing of the integral of this two-form on Σ, we have been able to derive that some functional of δE_{ab} vanishes at $i(\sigma)$,

$$\mathcal{F}\left[\delta E_{ab}(i(\sigma))\right] = 0 \ , \tag{B.2}$$

for some $\bar{a}, \bar{b}$. We can now consider another surface, $(g \circ i)(\Sigma)$ in M and we call its associated bulk modular flow ξ_g. We should consider the pullback on the corresponding two-form $d\boldsymbol{\chi}[\xi_g]$ because

$$\int_{(g\circ i)(\Sigma)} d\boldsymbol{\chi}[\xi_g] = \int_\Sigma i_g^* d\boldsymbol{\chi}[\xi_g] \ . \tag{B.3}$$

The pullback is given by

$$i_g^*(d\boldsymbol{\chi}[\xi_g]) = \xi_g^a(g\circ i(\sigma))\delta E_{ab}(g\circ i(\sigma))\frac{1}{2}\varepsilon^b_{cd}(g\circ i(\sigma))\frac{\partial g^c}{\partial x^e}\frac{\partial g^d}{\partial x^f}\frac{\partial x^e}{\partial \sigma^\alpha}\frac{\partial x^f}{\partial \sigma^\beta}d\sigma^\alpha \wedge d\sigma^\beta. \tag{B.4}$$

Now we can insert the identity matrix $\delta^a_b = \frac{\partial g^a}{\partial x^c}\frac{\partial x^c}{\partial g^b}$ to impose the equality of two b-index, leading to

$$\begin{aligned} i_g^*(d\boldsymbol{\chi}[\xi_g]) &= \xi_g^a(g\circ i(\sigma))\delta E_{ab}(g\circ i(\sigma))\frac{\partial g^b}{\partial x^g}\left(\frac{\partial x^g}{\partial g^h}\frac{1}{2}\varepsilon^h_{cd}(g\circ i(\sigma))\frac{\partial g^c}{\partial x^e}\frac{\partial g^d}{\partial x^f}\right) \\ &\quad \times\frac{\partial x^e}{\partial \sigma^\alpha}\frac{\partial x^f}{\partial \sigma^\beta}d\sigma^\alpha \wedge d\sigma^\beta. \end{aligned} \tag{B.5}$$

Now we can use the fact that g is an isometry, while ε^h_{cd} is the volume form to obtain than the parenthesis is actually $\frac{1}{2}\varepsilon^g_{ef}(i(\sigma))$. Moreover we know that the modular flow for the image surface is the image of the modular flow of the initial surface under the g-transformation: $\xi_g^a(g\circ i(\sigma)) = \frac{\partial g^a}{\partial x^b}\xi^b(i(\sigma))$. Finally, we obtain

$$i_g^*(d\boldsymbol{\chi}[\xi_g]) = \xi^i(i(\sigma))\left(\frac{\partial g^a}{\partial x^i}\frac{\partial g^b}{\partial x^g}\delta E_{ab}(g\circ i(\sigma))\right)\frac{1}{2}\varepsilon^g_{ef}(i(\sigma))\frac{\partial x^e}{\partial \sigma^\alpha}\frac{\partial x^f}{\partial \sigma^\beta}d\sigma^\alpha \wedge d\sigma^\beta, \tag{B.6}$$

which, is exactly (B.1) with the replacement

$$\delta E_{ab}(i(\sigma)) \to \frac{\partial g^c}{\partial x^a}\frac{\partial g^d}{\partial x^b}\delta E_{cd}(g\circ i(\sigma)) \ , \tag{B.7}$$

which implies that (B.2) ensures that

$$\mathcal{F}\left[\frac{\partial g^c}{\partial x^a}\frac{\partial g^d}{\partial x^b}\delta E_{cd}(g\circ i(\sigma))\right]=0\,. \tag{B.8}$$

For example, if we can show that some components of δE_{ab} vanish using a set of RT surfaces, we immediately obtain that other components, obtained by applying bulk isometries according to (B.8), will also vanish.

C Alternative proof in 3d

In this appendix, we provide an alternative to the step in the 3d proof of Sec. 4.2 where we used the light sheaf deformation. Here, we insist on doing this step using only RT configurations where the light rays γ_+ and γ_- pass through the spatial origin $r=0$. We will consider such configurations with $\ell_u\neq 0$ described in (3.1) which is the prescription used in [25]. Although a better and equivalent[13] derivation is presented in the main text, it is instructive to perform this step as presented here.

We should note that if we consider only the surfaces with $\ell_u=0$ (and with light rays passing through $r=0$), together with their image under bulk isometries, then the first law does *not* imply the gravitational equations: these surfaces don't provide enough constraints. Indeed, the only constraint that we obtain is

$$\delta E_{r\phi}+r\partial_r\delta E_{r\phi}-r\partial_\phi\delta E_{rr}=0\ , \tag{C.1}$$

and its image under bulk isometries. This does not imply that $\delta E_{ab}=0$ as it's possible to find explicit counterexamples.

Hence, we need to consider RT surfaces with $\ell_u\neq 0$ (still requiring that the light rays pass through $r=0$). The computation becomes simpler in the limit of small ℓ_u. More precisely, we consider

$$\ell_u=\lambda\,\varepsilon^2,\qquad \ell_\phi=\varepsilon\ , \tag{C.2}$$

where we take ε to be small. We would like to compute

$$I=\int_\Sigma \xi^a\delta E_{ab}\boldsymbol{\varepsilon}^b \tag{C.3}$$

in an expansion around $\varepsilon=0$. The first law of entanglement will constrain δE_{ab} to be such that $I=0$. It turns out that $\lim_{\varepsilon\to 0}I=0$ for any perturbation, so we don't get any constraint at zero order in ε. To compute I at first order in ε, it is enough to consider the surface Σ at first order in ε[14]. The configuration simplifies because the points B_+ and B_-

[13] This is because all the configurations described in Sec. 3.2 can be transformed with a bulk translation to a configuration where the two light rays pass through the line $r=0$.

[14] This can be justified as follows. Denoting $i_\varepsilon: S\to M$ the embedding of Σ_ε in M, we have

$$I=\int_{\Sigma_\varepsilon}\xi^a_\varepsilon\delta E_{ab}\boldsymbol{\varepsilon}^b=\frac{1}{2}\int_S \xi^a_\varepsilon(i_\varepsilon(\sigma))\delta E_{ab}(i_\varepsilon(\sigma))\varepsilon^b_{cd}(i_\varepsilon(\sigma))(J_\varepsilon)^c{}_\alpha(J_\varepsilon)^d{}_\beta d\sigma^\alpha\wedge d\sigma^\beta \tag{C.4}$$

where $(J_\varepsilon)^c{}_\alpha$ is the Jacobian of the embedding. This shows that, to compute the leading non-trivial term of I, it is enough to take i_ε at first order in ε, which corresponds to taking the surface Σ_ε at first order in ε.

are at $u = O(\varepsilon^2)$. Hence, we have

$$B_+ : \quad (u,\phi) = \Big(0, \frac{\varepsilon}{2}\Big) , \qquad B_- : \quad (u,\phi) = \Big(0, -\frac{\varepsilon}{2}\Big) \ , \tag{C.5}$$

to first order in ε. We also have the following parametrization for the light rays

$$\begin{aligned} \gamma_+ \quad &: \quad (t,x,y) = \Big(-2\eta + s, -2\eta + s, \frac{\varepsilon}{2}(-2\eta + s)\Big) , \qquad s \geq 0 \\ \gamma_- \quad &: \quad (t,x,y) = \Big(2\eta + s, 2\eta + s, -\frac{\varepsilon}{2}(2\eta + s)\Big) , \qquad s \geq 0 \ , \end{aligned} \tag{C.6}$$

where we only kept the terms at first order in ε. The curve γ is simply a straight line connecting the two points

$$P_+ : \quad (t,x,y) = (-2\eta, -2\eta, -\eta\epsilon), \quad P_- : \quad (t,x,y) = (2\eta, 2\eta, -\eta\epsilon). \tag{C.7}$$

We can show that γ_- stays at $u = 0$ everywhere and that γ_+ is at $u = 0$ for $s \geq 2\eta$, which corresponds to all its points before it crosses the origin. Let's call $\tilde{\gamma}_-$ the segment that connects the origin to P_-, which , which is in the continuation of γ_- past P_-. The plane surface bounded by $\gamma_-, \tilde{\gamma}_-$ and γ_+ (up to the origin) lies on the constant slice $u = 0$. It has the same shape as the RT surface for $\ell_u = 0$ depicted in Fig. 1.

The additional piece consists in another triangle, bounded by γ, $\tilde{\gamma}_-$ and $\tilde{\gamma}_+$, where $\tilde{\gamma}_+$ is the piece of γ_+ connecting the origin to P_+. This is the triangle $T = P_- P_+ O$. Let's introduce coordinates

$$x_+ = t + x, \qquad x_- = t - x \tag{C.8}$$

In these coordinates, we have (at first order)

$$\begin{aligned} P_+ \quad &: \quad (x_+, x_-, y) = (-4\lambda, 0, -\lambda\varepsilon) \\ P_- \quad &: \quad (x_+, x_-, y) = (4\lambda, 0, -\lambda\varepsilon) \end{aligned} \tag{C.9}$$

We see that the triangle $T = P_+ P_- O$ can be parametrized as follows

$$x_- = 0, \qquad -\lambda\varepsilon \leq y \leq 0, \qquad |x_+| \leq -\frac{4y}{\varepsilon} \tag{C.10}$$

The integration over the triangle is

$$I = \int_{-\lambda\varepsilon}^{0} dy \int_{4y/\varepsilon}^{-4y/\varepsilon} dx_+ \, F(x_+, x_-, y) \ , \tag{C.11}$$

where F is the appropriate integrand. We can redefine $y = \eta\varepsilon\tilde{y}$ so that it becomes

$$I = \lambda\varepsilon \int_{-1}^{0} d\tilde{y} \int_{4\lambda\tilde{y}}^{-4\lambda\tilde{y}} dx_+ F(x_+, x_-, \lambda\varepsilon y) \ . \tag{C.12}$$

We now come back to the full integral

$$I = \int_\Sigma \xi^a \delta E_{ab} \boldsymbol{\varepsilon}^b \, , \tag{C.13}$$

which we want to evaluate at first order in ε. The integral splits in an integral over the pizza slice and an integral over the triangle

$$I = I_P + I_T \ . \tag{C.14}$$

The integral over the pizza slice is

$$I_P \quad = \quad \int_{-\varepsilon/2}^{\varepsilon/2} d\phi \int_0^{+\infty} dr\, r\, \xi^a \delta E_{ar} \ . \tag{C.15}$$

The integral over the triangle is found by looking at the metric in the (x_+, x_-, y) coordinates. We have $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$ so that

$$g_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g^{\mu\nu} = 2\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{C.16}$$

The volume form on the triangle is

$$\boldsymbol{\varepsilon}^{x_+} = -2\boldsymbol{\varepsilon}_{x_-} = dy \wedge dx_+ \tag{C.17}$$

this implies that

$$I_T \quad = \quad \int_{-\eta\varepsilon}^{0} dy \int_{4y/\varepsilon}^{-4y/\varepsilon} dx^+\, \xi^a \delta E_{ax_+} \ . \tag{C.18}$$

Both integrals I_P and I_T can be computed explicitly at first order in ε. We now take derivatives of the result with respect to η. The first law gives $I = 0$ so for any η we have

$$\partial_\eta^3 I|_{\eta=0} = 0. \tag{C.19}$$

On the other hand, one find that

$$\begin{aligned} \partial_\eta^3 I_P|_{\eta=0} \quad &= \quad O(\varepsilon^2) \ , \\ \partial_\eta^3 I_T|_{\eta=0} \quad &= \quad -16\pi\varepsilon\left(\delta E_{rr}(0,0,0) - 2\delta E_{ur}(0,0,0) + 2\delta E_{uu}(0,0,0)\right) + O(\varepsilon^2) \ , \end{aligned} \tag{C.20}$$

which provides the new constraint

$$\delta E_{rr}(0,0,0) - 2\delta E_{ur}(0,0,0) + 2\delta E_{uu}(0,0,0) = 0 \ . \tag{C.21}$$

Following the general strategy, we obtain a new constraint by acting with the translation

$$\tilde{t} = t + r_0, \qquad \tilde{x} = x + r_0 \cos\phi_0, \qquad \tilde{y} = y + r_0 \sin\phi_0 \ . \tag{C.22}$$

Evaluating the result at $\phi = \phi_0$, we obtain

$$\delta E_{rr}(0, r_0, \phi_0) - 2\delta E_{ur}(0, r_0, \phi_0) + 2\delta E_{uu}(0, r_0, \phi_0) = 0 \ , \tag{C.23}$$

for any r_0, ϕ_0. We can then consider time translations to show that this relation holds at any u. Finally, acting with a boost in the (t, x)-plane and evaluating at $\phi = 0$ leads to

$$\delta E_{rr}(u_0, r_0, \phi_0) = 0 \ , \tag{C.24}$$

for any u_0, r_0, ϕ_0. The rest of the proof follows.

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ECOLE
DOCTORALE

Titre : Structures de bord et fluides holographiques en gravité

Mots clés : Gravité, Holographie, Correspondence Fluide/Gravité, Symétries Asymptotiques

Résumé : Cette thèse est dédiée à l'étude de certains aspects des espaces-temps dynamiques à bord. Une attention particulière est portée sur les bords asymptotiques comme le bord conforme d'AdS ou l'infini nul de l'espace plat. Le bord d'AdS est de genre temps et donc pseudo-Riemannien. Nous verrons que l'infini nul de l'espace plat est lui décrit par une géométrie de Carroll. Cette dernière apparaît comme la limite ultra-relativiste, ou $c \to 0$, d'une géométrie pseudo-Riemannienne. En particulier, la limite plate dans l'intérieur de l'espace-temps correspond à cette limite ultra-relativiste sur le bord. Nous verrons aussi comment les symétries de la gravité asymptotiquement plate se traduisent par des symétries globales de cette géométrie exotique de bord. Cette analyse est d'une importance capitale pour la correspondence fluide/gravité car le fluide vit sur le bord. Dans ce contexte nous imposons des conditions d'intégrabilité sur le fluide du bord qui permettent une resommation de l'expansion aux dérivées en AdS. La limite plate produit la notion de fluide Carrollien sur le bord dont l'expansion hydrodynamique se traduit par une expansion aux dérivées dans l'intérieur, ce qui donne une notion de correspondence fluide/gravité en espace plat. Un deuxième type de bord que nous étudions est celui formé par l'horizon d'un trou noir. Ici, un autre genre de correspondence fluide/gravité existe : le paradigme des membranes. Nous revisitons ce concept et proposons une interprétation nouvelle des équations de Damour–Navier–Stokes en terme de lois de conservation ultra-relativistes.

Title : Boundary structures and holographic fluids in gravity

Keywords : Gravity, Holography, Fluid/Gravity correspondence, Asymptotic Symmetries

Abstract : This thesis is devoted to the study of several aspects of dynamical spacetimes with boundaries. An emphasis is put on asymptotic boundaries such as the conformal boundary of AdS or the null infinity in flat space. In AdS the geometry of the conformal boundary is pseudo-Riemannian since the boundary is time-like. In flat space, we will show how the geometry of the null infinity can be described in terms of Carroll structure. The latter emerges as the ultra-relativistic limit, or $c \to 0$ limit, of a pseudo-Riemannian geometry. In particular, the flat limit in the bulk maps to this ultra-relativistic limit on the boundary. We will also see how the symmetries of asymptotically flat gravity translate into global symmetries of this exotic boundary geometry. This analysis is of central importance in fluid/gravity correspondence since the fluid is expected to live on the boundary. In this context we find integrability conditions on the boundary fluid that allow for a resummation of the so-called Derivative Expansion in AdS. The flat limit gives rise to the notion of Carrollian fluid on the boundary whose hydrodynamical expansion maps to a flat version of the Derivative expansion in the bulk, thus providing a notion of fluid/gravity correspondence in flat space. A second type of boundary that we study is the one formed by the horizon of a black hole. There, another type of fluid/gravity correspondence exists : the membrane paradigm. We revisit this concept and propose a novel interpretation of the Damour–Navier–Stokes equation in terms of ultra-relativistic conservation laws.

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