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Surface Charges in Gravity and Gauge Theories

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Abstract

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1 Introduction

1.1 Conventions and notations

We describe here our conventions and notations for differential geometry. Let $\mathscr{M}$ a n-dimensional differentiable manifold, provided with a set of coordinates $\{x^\mu\}$ (at least in a sufficiently large open neighborhood of the events of interest). The tangent structure of $\mathscr{M}$ is formed by a countable tower of vector spaces populated by multilinear k-forms, which we write $\Omega^k(\mathscr{M})$, for $k \in \mathbb{N}$. Differential forms are decomposed in the holonomic basis $\{\mathrm{d}x^\mu\}$. The natural basis of codegree-p forms populating $\Omega^{n-p}(\mathscr{M})$ is defined as

$$(\mathrm{d}^{n-p}x)_{\mu_1\dots\mu_p} := \frac{1}{p!(n-p)!}\,\varepsilon_{\mu_1\dots\mu_n}\,\mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}, \tag{1.1}$$

where $\varepsilon_{\mu_1\dots\mu_n}$ denotes the (numerically invariant) Levi-Civita symbol in n dimensions and $\wedge$ is the antisymmetrised tensor product on $\Omega^1(\mathscr{M})$, *i.e.* $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 2\mathrm{d}x^{[\mu} \otimes \mathrm{d}x^{\nu]}$. We have

$$\mathbf{A} = A^{\mu_1\dots\mu_p}(\mathrm{d}^{n-p}x)_{\mu_1\dots\mu_p} \in \Omega^{n-p}(\mathscr{M}). \tag{1.2}$$

The exterior derivative $\mathrm{d} = \mathrm{d}x^\mu \frac{\partial}{\partial x^\mu} : \Omega^k(\mathscr{M}) \longrightarrow \Omega^{k+1}(\mathscr{M})$ increments the form degree and acts as

$$\mathrm{d}\mathbf{A} = \frac{\partial}{\partial x^\sigma} A^{[\mu_1\dots\mu_{p-1}\sigma]}(\mathrm{d}^{n-p+1}x)_{\mu_1\dots\mu_{p-1}} \in \Omega^{n-p+1}(\mathscr{M}). \tag{1.3}$$

The algebra of multilinear forms on $\mathscr{M}$ equipped with the exterior derivative d forms the *de Rham complex*:

$$\big(\Omega^\bullet(\mathscr{M}), \mathrm{d}\big) : 0 \to \Omega^0(\mathscr{M}) \to \Omega^1(\mathscr{M}) \to \Omega^2(\mathscr{M}) \to \cdots \to \Omega^{n-1}(\mathscr{M}) \to \Omega^n(\mathscr{M}) \to 0, \tag{1.4}$$

where $\Omega^0(\mathscr{M}) = \mathscr{C}^\infty(\mathscr{M})$ is the set of scalar fields. By reflexivity, vector fields $\boldsymbol{\xi} \in T\mathscr{M}$ can be identified with functions on $\Omega^1(\mathscr{M})$ thanks to the interior product $\iota_{\boldsymbol{\xi}} : \Omega^1(\mathscr{M}) \to \Omega^0(\mathscr{M}) : \boldsymbol{\omega} \mapsto \iota_{\boldsymbol{\xi}}\boldsymbol{\omega} = \xi^\mu \omega_\mu$. This amounts to set $\iota_{\boldsymbol{\xi}} \coloneqq \xi^\mu \frac{\partial}{\partial \mathrm{d}x^\mu}$ as it formally replaces $\mathrm{d}x^\mu$ by ξ^μ in $\boldsymbol{\omega} = \omega_\mu \mathrm{d}x^\mu$. This operator can be extended to higher-degree forms: $\iota_{\boldsymbol{\xi}} : \Omega^k(\mathscr{M}) \to \Omega^{k-1}(\mathscr{M})$, by acting as

$$\iota_{\boldsymbol{\xi}}\mathbf{A} = A^{[\mu_1 \dots \mu_p}\xi^{\sigma]}(\mathrm{d}^{n-p-1}x)_{\mu_1 \dots \mu_p \sigma} \in \Omega^{n-p-1}(\mathscr{M}). \tag{1.5}$$

We can therefore navigate on the following ladder climbing down by means of ι and climbing up with d. Finally, one computes the Lie derivative of multilinear forms using Cartan's magic formula,

$$\mathscr{L}_{\boldsymbol{\xi}}\boldsymbol{\omega} = \mathrm{d}\iota_{\boldsymbol{\xi}}\boldsymbol{\omega} + \iota_{\boldsymbol{\xi}}\mathrm{d}\boldsymbol{\omega}. \tag{1.6}$$

2 Variational bi-complex

2.1 Infinite jet bundle

Back to the pioneering works of Lagrange, dynamics of classical fields has been thought as an application of the variational calculus. For the latter to be well-defined, the main conceptual step amounts to treat the fields living on $\mathscr{M}$ as abstract coordinates with respect to which one can ultimately derive. The issue is to take into account that these fields can still vary on $\mathscr{M}$ in some sense, *i.e.*, to encode their derivatives as independent objects. To do so, the natural structure is that of a fibre bundle: the *jet bundle*.

We denote as $\mathscr{F}$ the space of fields on $\mathscr{M}$ and we assume it to be finite-dimensional, in order to avoid non-locality issues in the physical theories we shall consider later on. Elements of $\mathscr{F}$ are fibrating over $\mathscr{M}$ which underpins a fibre-bundle structure $\pi : \mathscr{F} \to \mathscr{M}$, for which the spacetime $\mathscr{M}$ is considered as the base manifold. Typical examples of $\mathscr{F}$ are $T\mathscr{M}$ for vector fields or $T^*\mathscr{M}$ for one-form fields. We endow (sufficiently large open sets of) $\mathscr{M}$ and $\mathscr{F}$ with coordinates, respectively denoted as $x = (x^\alpha)$ and $\phi = (\phi^i)$. Note that the fine structure of the field label i is irrelevant to our discussions: it suffices to recall that it belongs to a finite-dimensional countable index. Local coordinates on the fibre bundle $\mathscr{F} \to \mathscr{M}$ are then (x^α, ϕ^i). At this stage, both coordinates are completely independent entities, as they cover different spaces: for instance, computing the derivative of ϕ^i with respect to x^α is meaningless. The operation that anchors a given element in $\mathscr{F}$ of coordinates ϕ^i to a point in $\mathscr{M}$ of coordinates x^α, and evaluates it there, is referred to as a *local section* of the fibre bundle $\mathscr{F} \to \mathscr{M}$, which is a map

$$s : \mathscr{M} \to \mathscr{F} : x^\alpha \mapsto \big(x^\alpha, \phi^i(x^\alpha)\big). \tag{2.1}$$

The second projection $s^i(x^\alpha) \equiv \phi^i(x^\alpha)$ is what one usually refers to as a field, seen as a function over spacetime. Therefore, one usually says that a field is a section of $\mathscr{F} \to \mathscr{M}$ and one denotes s by ϕ^i quite abusively. The space of sections of this fibre bundle is conventionally written $\Gamma(\mathscr{F})$.

After taking the section, one can define the partial derivatives of $\phi^i(x^\alpha)$ as

$$\phi^i_{\mu_1\mu_2\dots\mu_k}(x^\alpha) := \frac{\partial^k s^i(x^\alpha)}{\partial x^{\mu_1}\dots\partial x^{\mu_k}} = \frac{\partial^k \phi^i(x^\alpha)}{\partial x^{\mu_1}\dots\partial x^{\mu_k}}, \tag{2.2}$$

which are fully symmetric in $(\mu_1,\dots,\mu_k)$ because the fields are assumed to be smooth. We are now seeking for a generalisation of the fibre-bundle construction above that takes the successive derivatives of the fields into account. Let $P \in \mathscr{M}$, $k \in \mathbb{N}_*$ and choose a field $\phi \in \Gamma(\mathscr{F})$: we begin by introducing the *jet of order k, source P and target ϕ* as the data of the value of ϕ and its partial derivatives up to the k-th order at point P, which can be written as

$$J^k_P\phi := \left(\phi^i(x^\alpha), \phi^i_{\mu_1}(x^\alpha), \phi^i_{\mu_1\mu_2}(x^\alpha), \phi^i_{\mu_1\mu_2\mu_3}(x^\alpha), \dots, \phi^i_{\mu_1\mu_2\dots\mu_k}(x^\alpha)\right) \tag{2.3}$$

in the coordinate patch (x^α, ϕ^i). These pieces of information allow to reconstruct all the coefficients of the Taylor approximative polynomial of ϕ around P up to k-th order. The *jet bundle* of order k related to $\pi : \mathscr{F} \to \mathscr{M}$ is given by $\pi^k_{\mathscr{M}} : \mathscr{J}^k(\mathscr{F}) \to \mathscr{M}$ of jets (2.3) of order k of local sections (2.1) of this fibre bundle.

Two remarks are in order. First, as the Taylor polynomial is not specific to a given function, multiple fields can have the same Taylor approximation up to a given order k. Therefore, jets are strictly speaking defined as equivalence classes for the equivalence relation mapping fields with the same set of numbers (2.3) onto each other. Hence, by definition, for any $P \in \mathscr{M}$, the fibre $(\pi^k_{\mathscr{M}})^{-1}(P)$ over P in $\mathscr{J}^\infty(\mathscr{F})$ consists of equivalence classes of local sections ϕ on $\mathscr{F}$, for the equivalence relation stipulating that two sections ϕ and ϕ' are equivalent if their Taylor approximations coincide at P up to k-order. However, to not burden our discussion here, we are going to pass over in silence these mathematical subtleties in the following. Second, the $k \to +\infty$ limit of (2.3) is well-defined as provides the so-called *infinite jet* of source P and target ϕ. It contains the countable tower of iterated partial derivatives of ϕ. Since, as physicists, we are agnostic of the differential order up to which we can define interesting (usually conserved) quantities, we are somewhat forced to work with infinite jets from the get-go: we shall therefore assume in the following that jets have infinite cardinality and shall be written in our coordinate patch as

$$J^\infty_P\phi := \left(\phi^i(x^\alpha), \phi^i_{\mu_1}(x^\alpha), \phi^i_{\mu_1\mu_2}(x^\alpha), \phi^i_{\mu_1\mu_2\mu_3}(x^\alpha), \dots\right) \tag{2.4}$$

We now define the *jet bundle* as the fibre-bundle $\pi^\infty_{\mathscr{M}} : \mathscr{J}^\infty \to \mathscr{M}$ over spacetime where

$$\mathscr{J}^\infty := \left\{J^\infty_P\phi : P \in \mathscr{M},\ \phi \in \Gamma(\mathscr{F})\right\}. \tag{2.5}$$

This construction corresponds to moving around the infinite jet (2.4) over the whole manifold $\mathscr{M}$. For the sake of simplicity, we shall consider in what follows that $\mathscr{J}^\infty$ is a differentiable manifold with trivial topology, to be able to confuse the notions of exact and closed forms. This is a widespread assumption, which has only be relaxed in a very small number of exotic situations, see [**?**]. The geometric configuration is depicted in Fig. 1: fields and the related tower of partial derivatives fibrate over the spacetime manifold.

A local trivialisation of the jet bundle is $(x^\alpha, \phi^i, \phi^i_{\mu_1}, \phi^i_{\mu_1\mu_2}, \dots)$ where the coordinates are now all independent, up to the fact that $\phi^i_{\mu_1\dots\mu_k}$ is still required to be symmetric in $(\mu_1,\dots,\mu_k)$ in order to encode the same amount of information that the anchored partial derivatives (2.2). To ease the notation, we shall denote coordinates on the fibres by $(\phi^i_{(\mu)})$ where $(\mu) = (\mu_1, \mu_2, \dots, \mu_k)$ is a multi-index notation for a k-tuple of symmetrised indices, whose cardinal shall be written as $|\mu| = k$. We shall use an Einstein summation rule

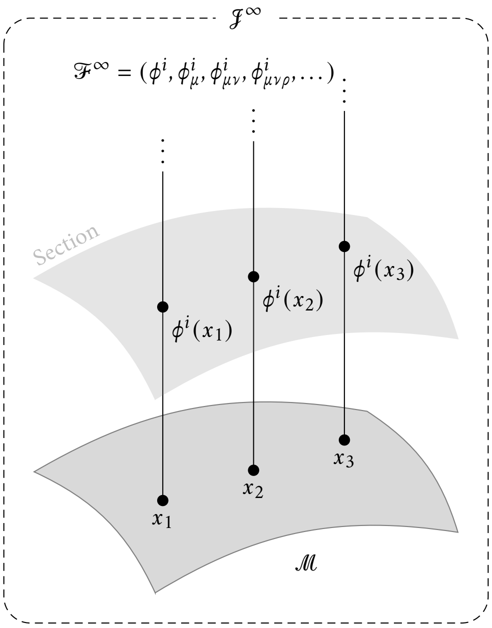

Figure 1: *Schematic depiction of the infinite jet bundle.*

on repeated multi-indices. Denoting by $\delta^{(\mu)}_{(\nu)} := \delta^{\mu_1}_{(\nu_1} \dots \delta^{\mu_k}_{\nu_k)}$ the fully symmetrised Kronecker symbol, we have by construction

$$\frac{\partial \phi^i_{(\mu)}}{\partial \phi^j_{(\nu)}} = \begin{cases} \delta^{(\mu)}_{(\nu)} \delta^i_j & \text{if} \quad |\mu| = |\nu| = k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

In what follows, we shall always work in coordinate systems employing a trivialisation: this amounts to assuming that $\mathcal{J}^\infty \simeq \mathcal{M} \times \mathcal{F}^\infty$ locally, where the second factor is the jet-enhancement of the field space $\mathcal{F}$ previously defined covered by coordinates $(\phi^i_{(\mu)})$. As physicists, we shall pay a particular attention to *local functions* over the jet bundle: such a function is defined as depending on the base coordinates, the fields and *a finite number* of their derivatives. We denote by $\mathrm{Loc}(\mathcal{J}^\infty)$ the space of local functions in $\mathcal{C}^\infty(\mathcal{J}^\infty)$. They are qualified as "local" in reference to locality of physical theories, which depends by definition on a finite number of derivatives.

2.2 Vertical and horizontal derivatives

In the above coordinate chart on $\mathcal{J}^\infty$, a holonomic basis of tangent co-vectors is given by $\mathrm{d}x^\mu$, $\delta\phi^i$, $\delta\phi^i_\mu$, $\delta\phi^i_{\mu\nu}$ and so on. Due to the fibre-bundle structure, a multilinear form over $\mathcal{J}^\infty$ is a bi-degree (k, ℓ), one with respect to the base manifold and one with respect to the field fibres. One defines $\Omega^{(k,\ell)}(\mathcal{J}^\infty)$, the set of (k, ℓ)-forms over the jet bundle, as the set of linear combinations of a finite number of terms of the form

$$F \left(\mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_k}\right) \wedge \left(\delta\phi^{i_1}_{(\nu_1)} \wedge \dots \wedge \delta\phi^{i_\ell}_{(\nu_\ell)}\right) \tag{2.7}$$

where $F \in \mathrm{Loc}(\mathcal{J}^\infty)$. The operator δ is the natural exterior derivative on the fibres $\mathcal{F}^\infty$, defined as

$$\delta := \delta\phi^i_{(\mu)} \frac{\partial}{\partial \phi^i_{(\mu)}} = \delta\phi^i \frac{\partial}{\partial \phi^i} + \delta\phi^i_\mu \frac{\partial}{\partial \phi^i_\mu} + \delta\phi^i_{\mu\nu} \frac{\partial}{\partial \phi^i_{\mu\nu}} + \cdots \tag{2.8}$$

which is perfectly reminiscent of the definition $\mathrm{d} = \mathrm{d}x^\mu \frac{\partial}{\partial x^\mu}$ of exterior derivative on the base manifold. By definition, it obeys $\delta^2 = 0$: it has been coined as *variation operator* or *vertical derivative*, to recall the fact that it acts on the fibres spanned by the fields.

From the base manifold, the exterior derivative $\mathrm{d} = \mathrm{d}x^\mu \frac{\partial}{\partial x^\mu}$, built up from the holonomic basis $\{\frac{\partial}{\partial x^\mu}\} \subset \mathfrak{X}(\mathcal{M})$, can be prolonged onto the jet bundle as the *horizontal derivative* $\mathrm{d} = \mathrm{d}x^\mu \partial_\mu$, keeping the same notation for convenience. The vector field denoted by $\partial_\mu \in \mathfrak{X}(\mathcal{J}^\infty)$ is the *total derivative* with respect to x^μ, given by

$$\partial_\mu := \frac{\partial}{\partial x^\mu} + \phi^i_{\mu(\nu)} \frac{\partial}{\partial \phi^i_{(\nu)}} = \frac{\partial}{\partial x^\mu} + \phi^i_\mu \frac{\partial}{\partial \phi^i} + \phi^i_{\mu\nu} \frac{\partial}{\partial \phi^i_\nu} + \cdots \tag{2.9}$$

It constitutes the extension of the total derivative of mechanics to higher-dimensional base manifolds. The first term extracts the explicit dependence on the coordinate x^μ, while all the others allow to obtain the implicit dependence on the base coordinates through the fields. The horizontal derivative of a codegree-p form $\mathbf{A} \in \Omega^{n-p}(\mathcal{M})$ depending on the fields and therefore prolonged onto $\mathcal{F}^\infty$ is now given by the same formula: $\mathrm{d}\mathbf{A} = \partial_\nu A^{[\mu_1 \dots \mu_p \nu]} (\mathrm{d}^{n-p+1}x)_{\mu_1 \dots \mu_{p-1}}$. It is direct to show that $[\partial_\mu, \partial_\nu] = 0$, hence d still squares to zero after the prolongation. We are left with a bi-complex with two exterior derivative operators, d and δ, with respect to which differential form have a defined degree:

$$\left(\Omega^{\bullet,\bullet}(\mathcal{J}^\infty), \mathrm{d}, \delta\right), \qquad \mathrm{d} : \Omega^{(k,\ell)} \longrightarrow \Omega^{(k+1,\ell)}, \qquad \delta : \Omega^{(k,\ell)} \longrightarrow \Omega^{(k,\ell+1)}, \tag{2.10}$$

As δ has the meaning of variational operator on the fields, one has been referring to it as *variational bi-complex*. Depicted by Fig. 2, it is the natural framework to address Lagrangian theory of fields. When the base is one-dimensional, *i.e.* $\mathcal{M} \simeq \mathbb{R}$, one recovers the technical framework to formulate Lagrangian mechanics in a mathematically robust way. The exterior derivative on $\mathcal{J}^\infty$ is given by $\mathsf{d} = \mathrm{d} + \delta$, and expectedly squares to zero because

$$\left\{\mathrm{d}, \delta\right\} = 0, \tag{2.11}$$

as it can be checked from (2.8) and (2.9). On any $f \in \mathrm{Loc}(\mathcal{J}^\infty)$, we have

$$\mathsf{d}f = \frac{\partial f}{\partial x^\mu} \mathsf{d}x^\mu + \frac{\partial f}{\partial \phi^i_{(\nu)}} \mathsf{d}\phi^i_{(\nu)} \equiv \partial_\mu f \,\mathrm{d}x^\mu + \frac{\partial f}{\partial \phi^i_{(\nu)}} \delta\phi^i_{(\nu)}. \tag{2.12}$$

The identification yields $\mathrm{d}x^\mu \equiv \mathsf{d}x^\mu$ and $\delta\phi^i_{(\mu)} = \mathsf{d}\phi^i - \phi^i_{(\mu)\nu} \mathsf{d}x^\nu$. The second equation shows evidently that the $\delta\phi^i_{(\mu)}$ are truly vertical, in the sense that their pullback onto $\mathcal{M}$ vanish.

2.3 Evolutionary vector fields and transformations

We discuss more technical results about the field space and the related vertical derivative. First, we are interested in modeling transformations to be able to discuss symmetries. For a transformation affecting the fields only, we can write $\delta_Q \phi^i = Q^i$ for Q^i a set of local functions. On field space, this induces a vector field

$$
\begin{tikzcd}
 & \vdots & \vdots & & \vdots & \vdots & \\
0 \arrow[r] & \Omega^{(0,2)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \Omega^{(1,2)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \cdots \arrow[r, "\mathrm{d}"] & \Omega^{(n-1,2)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \Omega^{(n,2)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & 0 \\
0 \arrow[r] & \Omega^{(0,1)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \Omega^{(1,1)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \cdots \arrow[r, "\mathrm{d}"] & \Omega^{(n-1,1)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \Omega^{(n,1)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & 0 \\
0 \arrow[r] & \Omega^{(0,0)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \Omega^{(1,0)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \cdots \arrow[r, "\mathrm{d}"] & \Omega^{(n-1,0)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & \Omega^{(n,0)} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & 0
\end{tikzcd}
$$

Figure 2: *Variational bi-complex.*

$Q^i \frac{\partial}{\partial \phi^i}$ referred to as *evolutionary vector of characteristic* Q^i. Its prolongation onto the jet bundle is

$$V_Q \equiv \partial_{(\mu)} Q^i \frac{\partial}{\partial \phi^i_{(\mu)}} = Q^i \frac{\partial}{\partial \phi^i} + \partial_\mu Q^i \frac{\partial}{\partial \phi^i_\mu} + \partial_\mu \partial_\nu Q^i \frac{\partial}{\partial \phi^i_{\mu\nu}} + \cdots \in \mathfrak{X}(\mathcal{J}^\infty), \tag{2.13}$$

where $\partial_{(\mu)} := \partial_{(\mu_1}\partial_{\mu_2} \dots \partial_{\mu_k)}$ if $|\mu| = k$, which acts on local functions $f \in \mathrm{Loc}(\mathcal{J}^\infty)$ as

$$\delta_Q f := V_Q[f] = \partial_{(\mu)} Q^i \frac{\partial f}{\partial \phi^i_{(\mu)}}. \tag{2.14}$$

Like the interior product on spacetime multilinear differential forms $\iota_{\boldsymbol{\xi}} = \xi^\mu \frac{\partial}{\partial \mathrm{d}x^\mu}$, aimed at replacing $\mathrm{d}x^\mu$ by ξ^μ to obtain the contraction with the vector field $\boldsymbol{\xi}$, we define the interior product of $\boldsymbol{\omega} \in \Omega^{(k,\ell)}(\mathcal{J}^\infty)$ with the evolutionary vector field V_Q of characteristic Q^i on the jet bundle by

$$i_Q \boldsymbol{\omega} := \partial_{(\mu)} Q^i \frac{\partial \boldsymbol{\omega}}{\partial \delta \phi^i_{(\mu)}}. \tag{2.15}$$

This operator replaces the arbitrary variations $\delta \phi^i_{(\mu)}$ by the transformation of characteristic Q^i and ascends the chain of (k,ℓ)-forms as $i_Q : \Omega^{(k,\ell)}(\mathcal{J}^\infty) \to \Omega^{(k,\ell-1)}(\mathcal{J}^\infty)$. In particular, $i_Q \delta \phi^i = \delta_Q \phi^i$, we can then also refer to δ_Q as the *contracted variation* along the evolutionary vector of characteristic Q. Its action on any (k,ℓ)-form $\boldsymbol{\omega}$ is prolonged thanks to Cartan magic formula on the jet bundle, *i.e.*

$$\delta_Q \boldsymbol{\omega} := \delta i_Q \boldsymbol{\omega} + i_Q \delta \boldsymbol{\omega} = \partial_{(\mu)} Q^i \frac{\partial \boldsymbol{\omega}}{\partial \phi^i_{(\mu)}} + \partial_{(\mu)} \delta Q^i \frac{\partial \boldsymbol{\omega}}{\partial \delta \phi^i_{(\mu)}}. \tag{2.16}$$

Importantly, the operator δ_Q is nothing but the Lie derivative on field space on the flow of the evolutionary vector V_Q. We could indeed write it as $\mathscr{L}_{V_Q}$ but we like to keep the more usual notation of "variation of parameter Q." From (2.8), (2.9) and (2.16), we can show that

$$\left[\delta_Q, \mathrm{d}\right] = 0 = \left[\delta_Q, \delta\right] \tag{2.17}$$

if we assume that $\delta_Q \delta\phi^i = \delta Q^i$. The contracted variation allows us to construct a Lie bracket on the set of characteristics as

$$\left[Q_1, Q_2\right]^i \coloneqq \delta_{Q_1} Q_2^i - \delta_{Q_2} Q_1^i. \tag{2.18}$$

The characteristics form an subalgebra of $\mathrm{Loc}(\mathcal{J}^\infty)$ under this Lie bracket: indeed, if we calculate the Lie bracket of two evolutionary vector fields, we find

$$\begin{aligned}\left[V_{Q_1}, V_{Q_2}\right] = \delta_{Q_1} V_{Q_2} &= \partial_{(\nu)}\left(\partial_{(\mu)} Q_1^i \frac{\partial Q_2^j}{\partial \phi^i_{(\mu)}} - (1 \leftrightarrow 2)\right)\frac{\partial}{\partial \phi^j_{(\nu)}} \\ &= \partial_{(\nu)}\left(\delta_{Q_1} Q_2^j - \delta_{Q_2} Q_1^j\right)\frac{\partial}{\partial \phi^j_{(\nu)}} = \partial_{(\nu)}\left[Q_1, Q_2\right]^j \frac{\partial}{\partial \phi^j_{(\nu)}} = V_{[Q_1,Q_2]}.\end{aligned} \tag{2.19}$$

As a simple exercise using all the previous definitions, we can show that the bracket of characteristics (2.18) obeys the following useful identities:

$$i_{[Q_1,Q_2]} = \left[i_{Q_1}, \delta_{Q_2}\right], \tag{2.20}$$

$$\delta_{[Q_1,Q_2]} = \left[\delta_{Q_1}, \delta_{Q_2}\right]. \tag{2.21}$$

To conclude, let us introduce a very convenient notation. In the course of these lectures, we shall be led to consider many infinite-dimensional differential operators and perform inverse Leibniz rules on them. To ease the notation, for any such operator

$$D(f) \coloneqq D^{(\mu)} \partial_{(\mu)} f \tag{2.22}$$

acting on a local function f, we define the related adjoint operator by

$$D^\dagger(g) \coloneqq (-\partial)_{(\mu)}\left(D^{(\mu)} g\right), \tag{2.23}$$

where $g \in \mathrm{Loc}(\mathcal{J}^\infty)$ and $(-\partial)_{(\mu)}$ is a shorthand notation for $(-1)^{|\mu|}\partial_{(\mu)}$, and such that

$$D(f) g = f D^\dagger(g) + \partial_\mu B^\mu, \tag{2.24}$$

for some codegree-one of components B^μ. The adjoint operator thus corresponds to performing inverse Leibniz rules in linear combinations where the operator F appears and forgetting about the boundary term.

2.4 Variational principle and Euler-Lagrange derivatives

All the geometric tools we have reviewed up to now are, in the parlance of physicists, encoding the *kinematics* of the physical theory we are aiming to describe. It remains to prescribe the *dynamics*, and to that end, we look again by to Lagrange, who was among the first to mathematically establish the intuition that the physical trajectories $\phi^i(x)$ must extremise a functional of the fields and a finite number of their derivatives (to ensure locality) under suitable boundary conditions. This functional is called the *action* and is written as

$$S[\phi] = \int_{\mathcal{M}} \mathbf{L}\left[\phi^i, \phi^i_\mu, \phi^i_{\mu\nu}, \dots, \phi^i_{\mu_1\mu_2\dots\mu_k}\right]. \tag{2.25}$$

In the above expression, $\mathbf{L} \in \Omega^{(n,0)}$ is an horizontal top form referred to as the *Lagrangian form*. It can always be written as $\mathbf{L} = L\, \mathrm{d}^n x$ where the *Lagrangian density* L is a local function by design.

In the jet-bundle viewpoint, computing an arbitrary variation δS of the action amonts to compute its vertical derivative, hence the same notation for the two operators. The question of implementing the boundary conditions for the variation to be physically motivated shall be addressed later on. From Eq. (2.8) and after several inverse Leibniz rules, we get the *first-variation formula*:

$$\boxed{\delta \mathbf{L} = \mathbf{E}_i \delta\phi^i - \mathrm{d}\mathbf{\Theta}.} \tag{2.26}$$

The first term contains the response $\mathbf{E}_i \in \Omega^{(n,0)}$ to the variation of the field ϕ^i and given in terms of the *Euler–Lagrange derivatives*

$$\boxed{\mathbf{E}_i := \frac{\delta \mathbf{L}}{\delta \phi^i} = (-\partial)_{(\mu)} \left(\frac{\partial \mathbf{L}}{\partial \phi^i_{(\mu)}} \right) = \frac{\partial \mathbf{L}}{\partial \phi^i} - \partial_\mu \left(\frac{\partial \mathbf{L}}{\partial \phi^i_\mu} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathbf{L}}{\partial \phi^i_{\mu\nu}} \right) + \cdots} \tag{2.27}$$

acting on the horizontal top form $\mathbf{L}$. The latter enjoy the remarkable property to vanish identically if and only if their argument is a total derivative, *i.e.*

$$\frac{\delta \mathbf{L}}{\delta \phi^i} \equiv 0 \quad \Leftrightarrow \quad \mathbf{L} = \mathrm{d}\boldsymbol{\ell}, \tag{2.28}$$

for some $\boldsymbol{\ell} \in \Omega^{(n-1,0)}$. Since $\mathbf{E}_i$ is an horizontal top-form, it has only one independent component that we shall write $E_i := \frac{\delta L}{\delta \phi^i}$, *i.e.* $\mathbf{E}_i = E_i(\mathrm{d}^n x)$.

Remark. First-variation formula (2.26) is obtained by fully integrating by parts to isolate $\delta\phi^i$. It is also possible to partially integrate by parts to isolate the variation of higher-order coordinates on the jet bundle, like $\delta\phi^i_{(\mu)}$ for a given multi-index (μ) of cardinal $|\mu| = k$. The pre-factor will then involve the higher-order Lie–Euler operators, defined as

$$\frac{\delta \boldsymbol{\omega}}{\delta \phi^i_{(\mu)}} := \sum_{(\nu)} C^{|\mu|}_{|\mu|+|\nu|} (-\partial)_{(\nu)} \frac{\partial \boldsymbol{\omega}}{\partial \phi^i_{((\mu)(\nu))}}. \tag{2.29}$$

The $|\mu| = 0$ case obviously reproduces the Euler–Lagrange derivatives (2.27). These higher-order operators have also the fundamental property to absorb total derivatives, in the sense that, for any $k > 0$ and any local function f,

$$\frac{\delta(\partial_\nu f)}{\delta \phi^i_{(\mu)}} = \delta^{\bar\mu}{}_{\nu} \frac{\delta f}{\delta \phi^i_{(\mu')}}, \tag{2.30}$$

where the multi-index (μ'), of cardinal $k-1$, obeys $(\mu) = \bar\mu(\mu')$.

The boundary term $\mathbf{\Theta} \in \Omega^{(n-1,1)}$, a form of codegree one with respect to the horizontal derivative and of degree one with respect to the vertical derivative, collects the residue of all integration by parts needed to write the first term in Eq. (2.26) as $\mathbf{E}_i \delta\phi^i$. It is referred to as *presymplectic potential* for reasons that shall become clear soon. A straightforward computation shows that

$$\boldsymbol{\Theta} = \Theta^\mu (\mathrm{d}^{n-1}x)_\mu, \qquad \Theta^\mu = \frac{\partial \mathbf{L}}{\partial \phi^i_\mu}\delta\phi^i + \frac{\partial \mathbf{L}}{\partial \phi^i_{\mu\nu}}\partial_\nu(\delta\phi^i) - \partial_\nu\left(\frac{\partial \mathbf{L}}{\partial \phi^i_{\mu\nu}}\right)\delta\phi^i + \cdots \tag{2.31}$$

A compact expression of $\boldsymbol{\Theta}$ can be obtained using the so-called Anderson homotopy operator and is given by Eq. (2.47) below. As it arises as a boundary term in the variation of the Lagrangian (2.26), $\boldsymbol{\Theta}$ is determined up to the total derivative of a $(n-2,2)$-form $\mathbf{Y}$ on the jet bundle, *i.e.*

$$\boldsymbol{\Theta} \mapsto \boldsymbol{\Theta}' = \boldsymbol{\Theta} + \mathrm{d}\mathbf{Y}. \tag{2.32}$$

As undesirable as it might look like at first glance, this ambiguity in extracting the potential $\boldsymbol{\Theta}$ from the first-variation formula, coined as *Iyer–Wald ambiguity* [**?**], has many virtues to be exploited in physics, for regularity or renormalisation purposes. We shall come back on it in Section **??**. If the base manifold of interest has no boundary, the last term in Eq. (2.26) vanishes upon integration by virtue of Stokes' theorem. In the presence of a boundary, $\mathscr{I} = \partial\mathscr{M}$, the presymplectic potential acquires a major role as it controls the variational principle, in the sense that

$$\delta S[\phi] = \int_{\mathscr{M}} \mathbf{E}_i\delta\phi^i + \int_{\mathscr{I}} \boldsymbol{\Theta}. \tag{2.33}$$

The choice of sign in front of the boundary term corresponds to the choice of orientation of the normal to the boundary, chosen here as inwards pointing. Therefore, the variational problem (2.33), consisting of finding the extermal field configurations of $S[\phi]$, is not well-defined unless one provides *boundary conditions* to deal with the last term. Let us denote by $\hat{\mathscr{F}}$ the subset of fields in $\mathscr{F}$ that satisfy a given set of boundary conditions on $\mathscr{I}$.

Several configurations can arise. First, we call *conservative boundary conditions* any set of boundary conditions on $\mathscr{I}$ that allow the boundary term in Eq. (2.33) to be canceled, up to complementing the original action with a suitable boundary term. Mathematically, this means that $\boldsymbol{\Theta}$ is asymptotically δ-exact,

$$\exists \hat{\boldsymbol{\ell}} \in \Omega^{(n-1,0)} : \boldsymbol{\Theta}\big|_{\mathscr{I}} = \delta\hat{\boldsymbol{\ell}},\ \forall\phi^i \in \hat{\mathscr{F}},\ \forall\delta\phi^i \in \Omega^1(\hat{\mathscr{F}}). \tag{2.34}$$

The last condition is crucial for the variation $\delta\phi$ to preserve the boundary conditions. We can thus re-define the Lagrangian as $\mathbf{L}' = \mathbf{L} + \mathrm{d}\hat{\boldsymbol{\ell}}$ and compute the variation of the related action to obtain:

$$\delta S'[\phi] = \int_{\mathscr{M}} \mathbf{E}_i\delta\phi^i + \int_{\mathscr{I}} (\boldsymbol{\Theta} - \delta\hat{\boldsymbol{\ell}}) = \int_{\mathscr{M}} \mathbf{E}_i\delta\phi^i. \tag{2.35}$$

The first equality uses (2.11) and the fact that Euler–Lagrange are blind to boundary terms, while the second one, evaluated on $\hat{\mathscr{F}}$, follows from Eq. (2.34). Therefore, if the boundary conditions are obeyed *and* preserved by the variations, the modified action is stationary if the equations of motion are obeyed:

$$\boxed{\delta S'[\phi] \approx 0,\ \forall\delta\phi^i \in \Omega^1(\hat{\mathscr{F}}) \qquad \Rightarrow \qquad \mathbf{E}_i \approx 0.} \tag{2.36}$$

As we shall see in due time, conservative boundary conditions allow for charge conservation, hence their names. They are the most used in physics to model closed systems (in the thermodynamical sense) that have no interaction with the environment. We call *strongly conservative* choices of boundary conditions such that the integral Θ over the boundary $\mathscr{I}$ vanishes, which corresponds to the particular case of Eq. (2.34) for which $\hat{\boldsymbol{\ell}} = \mathbf{0}$. Examples are many: for instance, when $\mathscr{I}$ is repelled to spatial infinity and one demands that the fields decay sufficiently fast to be able to neglect the boundary terms.

Oppositely, *leaky boundary conditions* do not allow the exactness condition (2.34) to be true. Although it is always possible to remove the δ-exact part of the boundary value of Θ by choosing the boundary term in the action, we shall not be able to cancel the full boundary term. Leaky boundary conditions are best suited to model physical systems that interact with their environment, possibly leading to non-conservation of the charges, hence their name. The selection of the potential Θ is therefore not uniquely decided by mathematical requirements like in the conservative case, but is left to physical purpose. We shall come back on this important point later in these lectures. Assuming that the couple $S[\phi]$, including suitable boundary terms, and Θ are determined, the action is no longer stationary when the equations of motion are satisfied, but in fact obey the milder condition:

$$\boxed{\delta S[\phi] - \int_{\mathscr{I}} \Theta \approx 0,\ \forall \delta\phi^i \in \Omega^1(\hat{\mathscr{F}}) \qquad \Rightarrow \qquad \mathbf{E}_i \approx 0.} \tag{2.37}$$

The boundary term breaking the stationary of $S[\phi]$ has been referred to as *presymplectic flux* across the boundary $\mathscr{I}$. We shall also make this interpretation more precise in the following, but the physical interpretation of it is already clear. Indeed, if the system described by the action $S[\phi]$ is interacting with an environment $\mathscr{E}$ through the interface $\mathscr{I}$, the total action should be formed by the linear combination $S + S^{\text{int}}_{\mathscr{I}}$, where the second term is the action modeling interactions localised at the interface. Therefore, the required condition in Eq. (2.37) is nothing but the exigence that the total action is stationary when the fields are solutions of the equations of motion and the presymplectic flux provides a measure of the exchanged information through the interface upon variation $\phi^i \mapsto \phi^i + \delta\phi^i$. More developments on this paramount aspect can be found in [**?**, **?**].

To conclude, the above discussion should have convinced the reader that the sufficient and necessary amount of data required to define a physical theory does not reduce to the gift of the field content and the action. The latter must be supplemented by boundary conditions *and* the prescription of the boundary term in the action. Then, for a given theory in the sense we are advocating for, the extraction of Θ from the first-variation formula (2.26) is only ambiguous up to the addition of codegree two forms $\mathbf{Y}$ like in Eq. (2.32), and nothing else. Now that the stage has been set, let us embark into the discussion of symmetries and associated conservation laws.

2.5 Algebraic Poincaré lemma

At this point, let us mention a paramount fact. Because it takes the presence of the field fibration into account, the horizontal derivative has a richer cohomology structure than the usual exterior derivative on $\mathscr{M}$. For the latter, the Poincaré lemma states that, in a simply connected open subset of $\mathscr{M}$, the de Rham cohomology class $\mathsf{H}^k_{\mathscr{M}}$, *i.e.* the set of equivalence classes of closed k-forms modulo the exact forms, is empty for $0 < k \leq n$ and is $\mathbb{R}$ for $k = n$:

$$\mathsf{H}^k_{\mathscr{M}} = \begin{cases} \mathbb{R} & \text{if} \quad k = 0, \\ 0 & \text{if} \quad 0 < k \leq n. \end{cases} \tag{2.38}$$

That means that, locally, every closed k-form is exact, except for the trivial case $\mathrm{d}c = 0$ for any real constant 0-form c.

Proof. The proof holds in $\mathscr{M} \simeq \mathbb{R}^n$ up to the gift of suitable coordinate maps, and amounts to providing a linear operator $I^k : \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k-1}(\mathbb{R}^n)$ over a given star-shaped open set U of $\mathbb{R}^n$ such that $\{\mathrm{d}, I^k\} = \mathrm{id}$ if $0 < k \leq n$, referred to as *contracting homotopy operator*. Indeed, if such an operator exists, any p-form $\boldsymbol{\omega}$ can be divided into two pieces: $\boldsymbol{\omega} = \mathrm{d}(I^k\boldsymbol{\omega}) + I^k\mathrm{d}\boldsymbol{\omega}$. By linearity, a closed form therefore obeys $\boldsymbol{\omega} = D(I^k\boldsymbol{\omega})$ on U, *i.e.* is locally exact. The "primitive" of $\boldsymbol{\omega}$ is computed by $I^k\boldsymbol{\omega} \in \Omega^{k-1}(\mathbb{R}^n)$ up to the gift of an exact $(k-2)$-form. The $k = 0$ case is special because if $f \in \Omega^0(\mathbb{R}^n)$ is closed on U, it cannot be exact since the set of (-1)-form is empty by definition. Therefore, f is constant on U, as stated by the Poincaré lemma.

For completeness, we provide the construction of the contracting homotopy operator I^k, assuming that U contains the origin $x = 0$ for simplicity. Let $D : [0,1] \times U \longrightarrow U : (\tau, x) \mapsto D_\tau(x) = \tau\, x$ be the dilation operator on U and $\mathbf{P} = x^\mu \frac{\partial}{\partial x^\mu}$ the position vector. Then, one can define

$$I^k : \Omega^p(\mathbb{R}^n) \longrightarrow \Omega^{k-1}(\mathbb{R}^n) : \boldsymbol{\omega}(x) \mapsto I^k\boldsymbol{\omega}(x) := \int_0^1 \frac{\mathrm{d}\tau}{\tau} D_\tau^*(\iota_{\mathbf{P}}\boldsymbol{\omega}), \tag{2.39}$$

where the dilation operator replaces x^μ by τx^μ and $\mathrm{d}x^\mu$ by $\tau \mathrm{d}x^\mu$ in $\mathbf{P}$ and $\boldsymbol{\omega}$, and show that it suits for proving the Poincaré lemma. Differentiating this operator with respect to t yields

$$\frac{\mathrm{d}D_\tau(x)}{\mathrm{d}\tau} = \frac{1}{\tau} D_\tau(x)^\mu \frac{\partial}{\partial x^\mu} \quad \Rightarrow \quad \frac{\mathrm{d}D_\tau^*\boldsymbol{\omega}}{\mathrm{d}\tau} = D_\tau^*(\mathscr{L}_{\tau^{-1}\mathbf{P}}\boldsymbol{\omega}) = \frac{1}{\tau} D_\tau^*(\mathscr{L}_{\mathbf{P}}\boldsymbol{\omega}). \tag{2.40}$$

Integrating over t from 0 to 1 and using Cartan magic formula leads to

$$D_1^*\boldsymbol{\omega} - D_0^*\boldsymbol{\omega} = \mathrm{d}\left(\int_0^1 \frac{\mathrm{d}\tau}{\tau} D_\tau^*(\iota_{\mathbf{P}}\boldsymbol{\omega})\right) + \int_0^1 \frac{\mathrm{d}\tau}{\tau} D_\tau^*(\iota_{\mathbf{P}}\mathrm{d}\boldsymbol{\omega}). \tag{2.41}$$

Since D_1 is the identity operator, we conclude that $\boldsymbol{\omega} = D_0^*\boldsymbol{\omega} + \mathrm{d}(I^k\boldsymbol{\omega}) + I^k(\mathrm{d}\boldsymbol{\omega})$. If $k \geq 1$, the pullback of $\boldsymbol{\omega}$ by the constant map D_0^* vanishes, hence the identity $\{\mathrm{d}, I^k\}\boldsymbol{\omega} = \boldsymbol{\omega}$ is obeyed. If $k = 0$, $D_0^* f = f(0)$ and the homotopy formula instead reads as $f = f(0) + I^0(\mathrm{d}f)$. Assuming closedness, the last term disappears and we are left with the expected result, namely $f = f_0$ a constant. This concludes the proof. □

If the cohomological properties (2.38) were obeyed by the horizontal derivative (2.9), it would mean that every n-form on the jet bundle is locally exact. It would imply that every Lagrangian is a total derivative, hence any field theory is dynamically trivial. Fortunately, one can prove the following theorem [?] [?]

Theorem 2.1 (Algebraic Poincaré Lemma)**.** *The cohomology class $\mathsf{H}^p_{\mathcal{J}^\infty}$ for the horizontal derivative is given by*

$$\mathsf{H}^k_{\mathcal{J}^\infty} = \begin{cases} \mathbb{R} & \text{if} \quad k = 0, \\ 0 & \text{if} \quad 0 < k < n, \\ [\boldsymbol{\omega}^n] & \text{if} \quad k = n, \end{cases} \tag{2.42}$$

where $[\boldsymbol{\omega}^n]$ denotes equivalence classes of n-forms for the equivalence relation: $\boldsymbol{\omega}' \sim \boldsymbol{\omega}$ if and only if $\boldsymbol{\omega}' = \boldsymbol{\omega} + \mathrm{d}\boldsymbol{\alpha}$.

Algebraic Poincaré lemma can be equivalently phrased as follows: the horizontal cohomology class $\mathsf{H}^k_{\mathcal{J}^\infty}$ is empty for any $0 < k < n$, formed by the real constants if $k = 0$ and, most importantly, is given if $k = n$ by the equivalence classes of horizontal top forms that have the same Euler–Lagrange derivatives, since the latter identically annihilate total derivatives. This is the cohomological statement of the freedom of choosing boundary terms in the action principle without changing the local dynamics, *i.e.* the equations of motion.

Proof. To prove Theorem 2.1, one can proceed in the same way, *i.e.* finding a contracting homotopy operator $I^k_{\delta\phi}$ that obeys $[\mathrm{d}, I^k_{\delta\phi}] = \mathrm{id}$ while acting on (p,q)-forms defined over simply connected open sets of $\mathcal{J}^\infty$ for $0 < k < n$ and discuss the extremal cases $k = 0$ and $k = n$ separately. The defining property involves here a commutator rather than an anticommutator: this is rooted into the fact that d and δ anti-commute, see Eq. (2.17). The suitable operator is *Anderson's horizontal homotopy operator* [?] (see also [?])

$$I^k_{\delta\phi} : \Omega^{(k,\ell)} \longrightarrow \Omega^{(k-1,\ell+1)} : \boldsymbol{\omega} \mapsto I^k_{\delta\phi}\boldsymbol{\omega} := \sum_{(\mu)} \frac{|\mu|+1}{n-k+|\mu|+1}\partial_{(\mu)}\left(\delta\phi^i \frac{\delta}{\delta\phi^i_{(\mu)\nu}}\frac{\partial\boldsymbol{\omega}}{\partial \mathrm{d}x^\nu}\right) \tag{2.43}$$

where the last term involves the higher-order Lie–Euler operators (2.29). It can be checked that

$$\left[\delta, I^k_{\delta\phi}\right] = 0. \tag{2.44}$$

It is important to notice that unlike the usual contracting homotopy operator, Anderson's one increases the field-space degree by one unit, *i.e.* $\ell \mapsto \ell + 1$ in Eq. (2.43), which explains the presence of the unevaluated variation $\delta\phi$ in subscript position. For a path γ on field space, the contracting homotopy operator on the horizontal space can be defined as

$$I^k_H : \Omega^{(k,0)} \longrightarrow \Omega^{(k-1,0)} : I^k_H\boldsymbol{\omega} := \int_\gamma I^k_{\delta\phi}\boldsymbol{\omega} \tag{2.45}$$

since if $\boldsymbol{\omega}$ is truly horizontal, *i.e.* with a vertical trivial form degree, $I^k_{\delta\phi}\boldsymbol{\omega}$ is a field-space one-form. After a tedious, although straightforward, exercise of algebra, one can show that Anderson's homotopy operators obey:

$$\begin{cases} \delta\boldsymbol{\omega} = I^{k+1}_{\delta\phi}(\mathrm{d}\boldsymbol{\omega}) - \mathrm{d}\big(I^k_{\delta\phi}\boldsymbol{\omega}\big) & \text{if} \quad 0 \leq k < n, \\ \delta\boldsymbol{\omega} = \delta\phi^i \dfrac{\delta\boldsymbol{\omega}}{\delta\phi^i} - \mathrm{d}\big(I^n_{\delta\phi}\boldsymbol{\omega}\big) & \text{if} \quad k = n. \end{cases} \tag{2.46}$$

In fact, the proof of the second line above can be done by "integrating by parts" in the definition (2.8) of $\delta\boldsymbol{\omega}$ and collecting the residues into the boundary term that makes Anderson's operator appear. Applying the resulting expression to $\boldsymbol{\omega} = \mathbf{L}$ and comparing with the first-variation formula, Eq. (2.26), shows that the presymplectic potential reads as

$$\boxed{\boldsymbol{\Theta} = I^n_{\delta\phi}\mathbf{L}.} \tag{2.47}$$

The properties (2.46) are sufficient to readily prove the algebraic Poincaré lemma, which holds locally on $\mathcal{J}^\infty$, *i.e.*, locally on spacetime *and* field space. Indeed, integrating over a path γ in field space, these properties become

$$\begin{cases} \Delta\boldsymbol{\omega} = I^{k+1}_H(\mathrm{d}\boldsymbol{\omega}) + \mathrm{d}\big(I^k_H\boldsymbol{\omega}\big) & \text{if} \quad 0 \leq k < n, \\ \Delta\boldsymbol{\omega} = \displaystyle\int_\gamma \delta\phi^i \frac{\delta\boldsymbol{\omega}}{\delta\phi^i} + \mathrm{d}\big(I^n_H\boldsymbol{\omega}\big) & \text{if} \quad k = n, \end{cases} \tag{2.48}$$

where $\Delta\boldsymbol{\omega}$ is the difference between the values of the local multilinear form $\boldsymbol{\omega}$ evaluated at the extremal points of γ. The changes of sign results from the anticommutation between d and δ. In particular, the difference between an horizontal top-form and an exact horizontal top-form is linear in the Euler–Lagrange derivatives, hence insensitive to total derivatives: this gives rise to the equivalence classes in Theorem 2.1 for the extremal case $k = n$. If the path γ is taken as starting from the trivial configuration $\phi^i \equiv 0$, the proof of algebraic Poincaré lemma follows the same lines as the proof of standard Poincaré lemma. □

3 Symmetries and Noether theorems

3.1 Variational symmetries

In the parlance of jet-bundle formalism, we define a *variational symmetry* of a given theory as generated by a evolutionary vector field $\delta_Q\phi^i = Q^i$ with characteristic Q^i such that its prolongation onto the jet bundle leaves the Lagrangian invariant up to a boundary term $\mathbf{B}_Q \in \Omega^{(n-1,0)}$:

$$\delta_Q \mathbf{L} = \mathrm{d}\mathbf{B}_Q. \tag{3.1}$$

Because $[\delta_Q, \mathrm{d}] = 0$, if δ_Q generates a variational symmetry of $\mathbf{L}$, so does it for any equivalent Lagrangian $\mathbf{L}' = \mathbf{L}+\mathrm{d}\boldsymbol{\ell}$. Importantly, variational symmetries are symmetries of the equations of motion but the converse is not true in general.[1] Indeed, a direct algebraic computation shows that[2]

$$\frac{\delta}{\delta\phi^i}\left(\delta_Q\mathbf{L}\right) - \delta_Q\left(\frac{\delta\mathbf{L}}{\delta\phi^i}\right) = (-\partial)_{(\mu)}\left(\frac{\partial Q^j}{\partial\phi^i_{(\nu)}}\frac{\delta\mathbf{L}}{\delta\phi^j}\right). \tag{3.2}$$

If δ_Q generates a variational symmetry, the first term drops out, and we see that the transformation of the equations of motion is linear in the equations of motion and their derivatives.

Furthermore, we also define a *conserved current* $\mathbf{J}_Y \in \Omega^{(n-1,0)}$ as a codegree-one horizontal form that is closed when the equations of motion are satisfied, *i.e.* there exist local functions $Y^{i(\mu)}$ such that

$$\mathrm{d}\mathbf{J}_Y = \mathbf{Y}^i(E_i), \quad \mathbf{Y}^i := (\mathrm{d}^n x)\, Y^{i(\mu)}\partial_{(\mu)}. \tag{3.3}$$

Developing in the horizontal basis of multilinear form, this equation takes the more familiar form

$$\partial_\mu J^\mu_Y = \sum_{(\mu)} Y^{i(\mu)}\partial_{(\mu)}E_i \approx 0 \tag{3.4}$$

of a relativistic conservation law encoded into a vector density whose divergence vanishes on-shell. From their very definition, conserved currents naturally span equivalence classes $[\mathbf{J}]$ for the equivalence relation

$$\mathbf{J}' \in [\mathbf{J}_Y] \quad \Leftrightarrow \quad \mathbf{J}'_Y = \mathbf{J}_Y + \mathrm{d}\mathbf{K}_Y + \mathbf{L}^i_Y(E_i) + \delta^n{}_1 C_Y, \tag{3.5}$$

[1] However, non-variational symmetries, *i.e.*, symmetries of equations of motion which are *not* variational symmetries, play an important role in the context of integrable systems for instance, see *e.g.* [**?**].

[2] See, *e.g.*, Eq. (A.36) of [**?**].

where $\mathbf{K}_Y = K_Y^{\mu\nu}(\mathrm{d}^{n-2}x)_{\mu\nu}$ is a codegree-two form coined as *superpotential*, $\mathbf{L}_Y^i$ is a codegree-one form depending linearly on the equations of motion, $\mathbf{L}_Y^i(E_i) = L^{i\mu(\nu)}\partial_{(\nu)}E_i(\mathrm{d}^{n-1}x)_\mu$ and the constant C_Y spans the cohomology class of zero-forms in the $n = 1$ case. The existence of superpotential-ambiguity in defining conserved current is rooted to the cohomology of the horizontal derivative, emanating from $\mathrm{d}^2 = 0$, while the possibility to add terms that vanish on-shell derives from the conservation condition, which is meant to hold if and only if the equations of motion are obeyed. By definition, a conserved current is *trivial* if and only if it belongs to the equivalence class $[\mathbf{0}]$: in plain words, a conserved current is trivial if and only if it reduces on-shell to a boundary term, *i.e.* the total divergence of a superpotential. We can therefore state the following famous theorem:

Theorem 3.1 (First Noether theorem)**.** *To every variational symmetry of the action describing a given system is associated a (possibly trivial) conserved current, and reciprocally.*

Proof. Contracting the first-variation formula (2.26) with a variational symmetry generator δ_Q yields

$$\delta_Q\mathbf{L} = Q^i\mathbf{E}_i + \mathrm{d}(i_Q\Theta) = \mathrm{d}\mathbf{B}_Q \tag{3.6}$$

by virtue of (3.1). It then follows that $\mathbf{J}_Q := \mathbf{B}_Q - i_Q\Theta$ is a conserved current as it obeys $\mathrm{d}\mathbf{J}_Q = Q^i\mathbf{E}_i$. The codegree-one form $\mathbf{J}_Q \in \Omega^{(n-1,0)}$ is called a *Noether current* associated to the variational symmetry of characteristic Q^i. It constitutes a representative of the class $[\mathbf{J}_Q]$ of equivalent Noether currents related to δ_Q. Now applying several inverse Leibniz rules on Eq. (3.3), we obtain $\mathrm{d}\mathbf{J}'_Y = (-\partial)_{(\mu)}Y^{i(\mu)}\,\mathbf{E}_i$ for a given $\mathbf{J}'_Y$ collecting the resulting total derivative terms. If we posit $Q^i \equiv (-\partial)_{(\mu)}Y^{i(\mu)}$, we observe that $\mathbf{J}'_Y$ can be equated to the Noether current $\mathbf{J}_Q$ associated to the variational symmetry generated by δ_Q, up to d-exact terms. Indeed, we can use the contracted first-variation formula to show that $\delta_Q\mathbf{L} = \mathrm{d}\mathbf{B}_Q$ where $\mathbf{B}_Q \equiv \mathbf{J}_Q + i_Q\Theta$, again, up to d-exact terms and possible contributions vanishing on-shell. □

If $\mathbf{J}_Q$ is a Noether current, the associated Noether charge is defined as

$$H_Q := \int_\Sigma \mathbf{J}_Q = \int_\Sigma J_Q^\mu(\mathrm{d}^{n-1}x)_\mu \tag{3.7}$$

upon integration on a codimension one hypersurface $\Sigma \subset \mathscr{M}$. If Σ has no boundary or if boundary conditions ensures that the ambiguity $\mathbf{K} \in \Omega^{(n-2,0)}$ vanishes when approaching the boundary, we observe that the definition (3.7) does not depend on the particular representative of Noether current, using Stokes theorem. Cases when the contribution of $\mathbf{K}$ becomes critical is precisely the subject of these lectures. Considering two hypersurfaces Σ_1 and Σ_2, the conservation of the Noether charge H_Q on the account of the equations of motion while moving from Σ_1 to Σ_2 is a consequence of the conservation of the related current:

$$H_Q\big|_{\Sigma_2} - H_Q\big|_{\Sigma_1} = \int_{\Sigma_2}\mathbf{J}_Q - \int_{\Sigma_1}\mathbf{J}_Q = \int_{V_{12}}\mathrm{d}\mathbf{J}_Q \approx 0, \tag{3.8}$$

where V_{12} is the (codimension two) volume comprised between Σ_1 and Σ_2 and Stokes theorem is responsible for the second equality.

3.2 Gauge symmetries

Among the variational symmetries acting on the system, some of them can depend on arbitrary local functions $\lambda = (\lambda^\alpha) \subset \mathscr{C}^\infty(\mathscr{J}^\infty)$, where α is assumed to belong to a finite-dimensional index. In particular, each λ^α is an utterly free function of spacetime coordinates. One calls *gauge symmetry* such a transformation, and a theory admitting those as symmetries a *gauge theory* (with nuances that we shall point out later). Gauge symmetries are really special because the true parameters of the transformation are no longer the characteristics Q^i but the functions λ^α, coined as *gauge parameters*. We write infinitesimal gauge symmetry with parameters λ^α as the generator δ_λ of a variational symmetry with characteristics $R^i_\alpha = R^{i(\mu)}_\alpha \partial_{(\mu)}$, *i.e.*

$$\delta_\lambda \phi^i = R^i_\alpha[\lambda^\alpha] = R^{i(\mu)}_\alpha \partial_{(\mu)} \lambda^\alpha. \tag{3.9}$$

The formal way to implement the crucial difference brought by gauge transformations is to extend the jet bundle by adding new coordinates $(\lambda^\alpha, \lambda^\alpha_\mu, \lambda^\alpha_{\mu\nu}, \dots)$. This makes the machinery a bit heavy for the scope of these lectures, hence we shall refrain presenting this further extension in its full glory here. However, we refer the interested reader to the seminal references [**?**] for more details.

Replacing $Q^i \mapsto R^i_\alpha[\lambda^\alpha]$ in Eq. (3.6) and isolating the gauge parameters by means of several inverse Leibniz rules, we get

$$R^i_\alpha[\lambda^\alpha]\mathbf{E}_i = \mathrm{d}\mathbf{J}_\lambda = \lambda^\alpha R^{\dagger i}_\alpha(E_i)\,(\mathrm{d}^n x) + \mathrm{d}\mathbf{S}_\lambda(E_i), \tag{3.10}$$

where $\mathbf{J}_\lambda = \mathbf{B}_\lambda - i_{R[\lambda]}\mathbf{\Theta}$ is the usual representative of the Noether current related to δ_λ and $\mathbf{S}_\lambda$ is referred to as the *weakly vanishing Noether current*:

$$\mathbf{S}_\lambda = (-\partial)_{(\nu)}\left(R^{i\mu(\nu)(\nu')}_\alpha E_i\right)\partial_{(\nu')}\lambda^\alpha (\mathrm{d}^{n-1}x)_\mu. \tag{3.11}$$

It owes its name from the fact that, depending linearly on the equations of motion, it has to vanish on-shell, $\mathbf{S}_\lambda \approx \mathbf{0}$. If the gauge characteristics are exact to first-order, *i.e.* $R^i_\alpha[\cdot] = R^i_\alpha + R^{i\mu}_\alpha \partial_\mu$, the weakly vanishing Noether current simply reads as

$$\mathbf{S}_\lambda = \lambda^\alpha R^{i\mu}_\alpha \partial_\mu E_i. \tag{3.12}$$

The first term in the right-hand side of Eq. (3.10) involves the differential operators

$$R^{\dagger i}_\alpha(E_i) = (-\partial)_{(\mu)}\left(R^{i(\mu)}_\alpha E_i\right). \tag{3.13}$$

Taking a Euler–Lagrange derivative with respect to λ^α on the relation (3.10) yields

$$\boxed{R^{\dagger i}_\alpha(E_i) = 0.} \tag{3.14}$$

These relations have been coined as *Noether identities*. Although they involve linearly the equations of motion, they hold *off-shell*. There is one Noether identity for each gauge parameter λ^α. Therefore, the existence of a gauge symmetry implies that the equations of motion are not independent: there is some redundancy in the dynamical description of the theory. Moreover, the action of gauge symmetries also implies the presence of arbitrary functions of spacetime in the general solution of the equations of motion, since a given solution is mapped on another one by gauge transformation.[3] We have the following theorem:

Theorem 3.2 (Second Noether theorem)**.** *There is a bijective correspondence between Noether identities and gauge symmetries.*

On account of Noether identities (3.14), Eq. (3.10) indicates that the difference $\mathbf{J}_\lambda - \mathbf{S}_\lambda$ is identically closed. We can therefore write

$$\mathbf{J}_\lambda = \mathbf{S}_\lambda + \mathrm{d}\mathbf{K}_\lambda \tag{3.15}$$

for some arbitrary superpotential $\mathbf{K}_\lambda$. Therefore, Noether currents $\mathbf{J}_\lambda$ associated with a gauge symmetry are equivalent to the weakly vanishing Noether current $\mathbf{S}_\lambda$ that is conserved ($d\mathbf{S}_\lambda \approx \mathbf{0}$) and vanishes ($\mathbf{S}_\lambda \approx \mathbf{0}$) on shell. In particular, $\mathbf{J}_\lambda \approx d\mathbf{K}_\lambda$. A particularly striking rephrasing of this result is that Noether currents associated with gauge symmetries are trivial: it implies that the Noether charge (3.7) is undefined since it is given by the surface (*i.e.*, codimension-two) integral of an arbitrary superpotential,

$$H_\lambda = \int_\Sigma \mathbf{J}_\lambda \approx \int_{\partial\Sigma} \mathbf{K}_\lambda, \tag{3.16}$$

where $\partial\Sigma$ denotes the boundary of Σ. In other words, first Noether theorem is unable to provide a meaningful conserved current associated to a gauge symmetry by virtue of second Noether theorem. This does not invalidate Noether procedure, this just tells us that we have to use another method to determine the superpotential $\mathbf{K}_\lambda$ in Eq. (3.16). The explanation and motivation of this method is the main goal of these lectures.

Some important remarks are in order before moving on. First, there exists trivial gauge symmetries, which are therefore related to trivial Noether identities. Take for instance a theory for a single scalar field ϕ. The latter is not commonly recognised as a gauge theory, however there exists a trivial identity among its equations of motion. Indeed, if $E = \frac{\delta L}{\delta \phi}$, the differential operator $N := M^{\mu\nu}\partial_\mu E \partial_\nu$ is a *Noether operator*, *i.e.* $N(E) = 0$ off shell, if the matrix $M^{\mu\nu}$ is antisymmetric. The related Noether identity is *trivial* because the coefficients defining the Noether operator vanish on-shell, *i.e.* $N(\cdot) \approx 0$. More generically, one can show that

$$N^i = N^{i(\mu)}\partial_{(\mu)} \approx 0 \qquad \Rightarrow \qquad N^{i(\mu)} = M^{[i(\mu)j(\nu)]}\partial_{j(\nu)}E_i \tag{3.17}$$

for a given collection of local functions $M^{i(\mu)j(\nu)}$ packaged in a skew-symmetric matrix under the exchange $i(\mu) \leftrightarrow j(\nu)$. By virtue of Theorem 3.2, trivial Noether identities are related to trivial gauge symmetries, *i.e.* variational symmetries with characteristics $Q^i[f], f \in \mathrm{Loc}(\mathcal{J}^\infty)$ that vanish on-shell, $Q^i[f] \approx 0$. Therefore, they are trivial in the sense that they do not induce arbitrary functions in the solutions of equations of motion. The trivial gauge transformation related to the trivial Noether operators (3.17) are

$$Q^i[f] = M^{\dagger ij}\left(f, E_j\right), \quad M^{ij}(f,g) := M^{i(\mu)j(\nu)}\partial_{(\mu)}f\partial_{(\nu)}g \tag{3.18}$$

for any $f, g \in \mathrm{Loc}(\mathcal{J}^\infty)$. Trivial gauge transformations exist for any variational principle, since the fact that $M^{+ij}(f, E_j)$ is the characteristic of a variational symmetry is an algebraic statement that does not depend on the particular theory. Contracting the first-variation formula (2.26), we get $\delta_f \mathbf{L} = Q^i[f]\mathbf{E}_i + \mathrm{d}(i_{Q[f]}\boldsymbol{\Theta})$

[3]Indeed, gauge symmetries are symmetries of the equations of motion, *i.e.*, $\delta_\lambda \mathbf{E}_i = \mathbf{0}$, since they are in particular variational symmetries.

and the first term can be massaged as

$$Q^i[f]\mathbf{E}_i = fM^{ij}(E_i, E_j)(\mathrm{d}^n x) + \mathrm{d}\mathbf{S}_f(E_i) \tag{3.19}$$

after applying several inverse Leibniz rules. The first term identically vanishes by antisymmetry: it showcases in fact the Noether identities $N^i(E_i) = M^{ij}(E_i, E_j)$ induced by the trivial Noether operators (3.17). This establishes that $\delta_{Q[f]}$ generates a variational symmetry regardless the concrete expression of the action. Therefore, one is led to distinguish *proper gauge theories*, where there exists at least one non-trivial Noether identity, and *improper ones*, which encompasses all other local field theories.

We shall avoid dealing with these subtleties in the course of these lectures, by assuming that we focus on *irreducible gauge theories*. Such theories are gauge theories for which a given set $\mathcal{S} = \{R^{\dagger i}_\alpha\}$ of Noether operators obey the following axioms:

1. $\mathcal{S}$ is *non-trivial*: each $R^{\dagger i}_\alpha$, as a differential operator, does not vanish on-shell;
2. $\mathcal{S}$ is *irreducible*: if $Z^{\dagger\alpha} \circ R^{\dagger i}_\alpha \approx 0$ for a given differential operator Z^α, then the latter vanishes identically;
3. $\mathcal{S}$ is a *generating set*: for any Noether operator N^i, *i.e.* $N^i(E_i) = 0$, there exists a differential operators Z^α such that $N^i \approx Z^{\dagger\alpha} \circ R^{\dagger i}_\alpha$. From our above discussion, this means that there also exist differential operators $M^{ij}(\cdot,\cdot)$ as defined in Eq. (3.18) such that

$$N^i(E_i) = 0 \quad \Leftrightarrow \quad N^i = Z^\dagger \circ R^{\dagger i}_\alpha + M^{ij}(\cdot, E_j). \tag{3.20}$$

   The set of Noether identities associated with a generating set is complete, in the sense that any Noether operator can be expressed in terms of (derivatives of) the Noether operators related to the generating set, modulo a trivial Noether identity given by a antisymmetric bilinear combination of the equations of motion.

The task of finding generating sets among gauge transformations is generally hard: it is in fact equivalent to finding all the non-trivial gauge symmetries of the theory. A generating set of gauge symmetries is given by (3.9) where the operators $R^{\dagger i}_\alpha$ form a generating set of Noether identities. Then, any gauge symmetry $Q^i[f]$ depending on a local function f can be decomposed as

$$Q^i[f] = R^i_\alpha \circ Z^\alpha[f] + M^{\dagger ij}(f, E_j), \tag{3.21}$$

where

$$M^{\dagger ij}(f, E_j) = (-\partial)_{(\mu)} \left(f M^{[i(\mu)j(\nu)]} \partial_{(\nu)} E_j \right), \tag{3.22}$$

as a direct consequence of Eq. (3.20). In the examples discussed in these lectures, one can prove, with a bit of time but without any major difficulty, that the involved symmetries form a generating set. Therefore, we shall always assume that the characteristics denoted as R^i_α form a generating set of the system's gauge symmetries.

3.3 Global symmetries

Let us quickly come back on the fundamental Theorem 3.1. In a gauge theory, one defines *global symmetries* as variational symmetries that are not gauge symmetry, *i.e.* whose characteristics do *not* depend on arbitrary local functions. From our discussion above, a global symmetry is then naturally defined as an equivalence class of characteristics that differ by a gauge transformation, which one can always split into a non-trivial one (picked in a generating set of gauge symmetries) and a trivial one. In formulas,

$$Q'^i \in [Q^i] \quad \Leftrightarrow \quad Q'^i = Q^i + R^i_\alpha[\lambda^\alpha] + M^{\dagger ij}(E_j), \tag{3.23}$$

where

$$M^{\dagger ij}(E_j) := M^{\dagger ij}(1, E_j) = (-\partial)_{(\mu)} \left(M^{[i(\mu)j(\nu)]} \partial_{(\nu)} E_j \right) \tag{3.24}$$

to ease the notation. Although first Noether theorem is unable to yield something useful for gauge symmetries, it still provides meaningful invariants for global symmetries. Therefore, there exists a refined statement of Theorem 3.1, proven by Barnich, Brandt and Henneaux in [**?**] by cohomological methods, and which can be presented as follows:

Theorem 3.3 (First Noether theorem for gauge theories)**.** *Consider a Lagrangian theory of fields admitting variational symmetries, some of which may be gauge symmetries. To every global symmetry* $[Q]$ *corresponds one and only one equivalence classes of Noether currents* $[\mathbf{J}_Q]$ *through the correspondence*

$$[Q^i] \mapsto [\mathbf{B}_Q - i_Q \Theta], \quad [\mathbf{J}] \mapsto [Q^i] = \left[\sum_{(\mu)} (-1)^{|\mu|} \partial_{(\mu)} Y^{i(\mu)} \right] \tag{3.25}$$

where the local functions $Y^{i(\mu)}$ *are related to* $\mathbf{J}$ *through Eq.* (3.3)*.*

3.4 Reducibility parameters

By virtue of second Noether theorem, conserved charges associated with gauge symmetries have to emerge from superpotentials, *i.e.* codegree-two forms $\mathbf{K}_\lambda$ in Eq. (3.16). For this reason, one refers to such Noether charges as *surface charges*, since they are integrated over codimension-two submanifolds, *i.e.* surfaces in the usual four-dimensional case. From now on, our task is to provide tools to derive a suitable $\mathbf{K}_\lambda$ to compensate the failure of first and second Noether theorems to provide a representative for the superpotential.

Barnich, Brandt and Henneaux have shown in [**?**] a generalisation of first Noether theorem that establishes a correspondence between (equivalence classes of) conserved codegree-two forms and an important subclass of gauge parameters, which are referred to as *reducibility parameters*. Among the gauge parameters (λ^α) of a generating set $R^i_\alpha[\lambda^\alpha]$ of gauge symmetries, one says that $\bar\lambda^\alpha$ are reductibility parameters if there exists a differential operator $M^{\dagger ij}_{\bar\lambda}$, with the same properties and local expression as (3.24), such that

$$R^i_\alpha[\bar\lambda^\alpha] = M^{\dagger ij}_{\bar\lambda}(E_j) \approx 0. \tag{3.26}$$

It must be stressed that the above is a non-trivial constraint. Indeed, we have assumed to work with irreducible gauge theories, for which R^i_α form a non-trivial and irreducible generating set, so in particular R^i_α

does *not* vanish on-shell as a differential operator. This does not of course prevent R^i_α to have a non-trivial kernel when the equations of motion are obeyed. The latter is precisely spanned by the reducibility parameters. Therefore, the corresponding gauge transformation leave all solutions of the equations of motion invariant. In other words, they do not map solutions onto solutions but preserve each solution pointwise in solution space. A trivial way to solve the constraint (3.26) is to have parameters that vanish on-shell, $\bar\lambda \approx 0$. The latter are referred to as *trivial reducibility parameters*, which induces the following equivalence relation among reducibility parameters:

$$\bar\lambda \sim \bar\lambda' \quad \Leftrightarrow \quad \bar\lambda' \approx \bar\lambda, \tag{3.27}$$

i.e. if they differ by a trivial reducibility parameter. Moreover, the cancellation of $R^i_\alpha[\lambda^\alpha]$ can occur without invoking the equations of motions, *i.e.*

$$R^i_\alpha[\bar\lambda^\alpha] = 0. \tag{3.28}$$

Again, this does not contradict any of our previous hypothesis regarding the irreducibility of the generating set, and merely states that R^i_α may have a non-trivial kernel. Parameters solving Eq. (3.28) are referred to as *exact reducibility parameters*.

Furthermore, and in the same spirit as before, one can define a *conserved codegree-two form* $\mathbf{K}_J \in \Omega^{(n-2,0)}$ as obeying $\mathrm{d}\mathbf{K}_J \approx \mathbf{0}$, *i.e.*

$$\mathrm{d}\mathbf{K}_J = \mathbf{J}^i(E_i) = J^{i\mu}(E_i)(\mathrm{d}^{n-1}x)_\mu, \quad J^{i\mu} := J^{i\mu(\nu)}\partial_{(\nu)} \tag{3.29}$$

for a given collection of local functions $J^{i\mu(\nu)}$. By definition, a conserved codegree-two form is an ambiguous piece of data, fixed up to the gift of an exact codegree-two form $\mathrm{d}\mathbf{L}$ and a weakly vanishing codegree-two form $\mathbf{T}^i(E_i) \approx \mathbf{0}$. We therefore have the following equivalence classes:

$$\mathbf{K}' \in [\mathbf{K}] \quad \Leftrightarrow \quad \mathbf{K}' = \mathbf{K} + \mathrm{d}\mathbf{L} + \mathbf{T}^i(E_i) + \delta^n{}_2 C. \tag{3.30}$$

The last term is again due to the first class of cohomology of the horizontal derivative d, which is $\mathbb{R}$ for zero forms. Barnich, Brandt and Henneaux's result can then be termed as follows.

Theorem 3.4 (Generalised Noether theorem)**.** *There is a bijective correspondence between the set of equivalence classes of reducibility parameters associated with a given gauge theory and the set equivalence classes of conserved codegree-two forms.*

Proof. As for Theorem 3.3, the complete and rigorous proof of this result requires cohomological methods whose exposition goes far beyond the scope of these lectures. We offer here a more concrete check of it by showing how to derive the existence of a conserved codegree-two form from reducibility parameters and reciprocally.

First, we contract the reducibility identity, Eq. (3.26), off shell, with the equations of motion to get

$$\mathrm{d}\mathbf{S}_{\bar\lambda}(E_i) = R^i[\bar\lambda]\mathbf{E}_i = M^{\dagger ij}_{\bar\lambda}(E_j)\mathbf{E}_i \tag{3.31}$$

by virtue of Noether's second theorem. Integrating by parts in the right-hand side, we deduce that there exists a weakly vanishing codegree-one form $\mathbf{B}^{ij}_{\bar\lambda}$, non-necessarily skew-symmetric on $(i \leftrightarrow j)$, such that

$$M^{\dagger ij}_{\bar\lambda}(E_j)\mathbf{E}_i = \mathrm{d}\mathbf{B}^{ij}_{\bar\lambda}(E_i, E_j) + M^{ij}_{\bar\lambda}(E_i, E_j)(\mathrm{d}^n x), \tag{3.32}$$

and the last term vanishes because $M^{(ij)} = 0$. Coupling the two previous equations, we deduce that $\mathbf{J}_{\bar\lambda} := \mathbf{S}_{\bar\lambda}(E_i) + \mathbf{B}^{ij}(E_i, E_j)$ is closed off shell, *i.e.* $\mathrm{d}\mathbf{J}_{\bar\lambda} = \mathbf{0}$. Therefore, for any reducibility parameter $\bar\lambda$, there exists a codegree-two form $\mathbf{K}_{\bar\lambda}$ such that $\mathbf{J}_{\bar\lambda} = \mathrm{d}\mathbf{K}_{\bar\lambda}$. The latter is conserved as $\mathbf{J}_{\bar\lambda} \approx \mathbf{0}$ by construction.

Second, suppose that we have a particular conserved codegree-two form $\mathbf{K}_J$, which, by definition, obeys Eq. (3.29). Acting with the horizontal derivative on this equation yields $\partial_\mu J^{i\mu}(E_i) = \partial_\mu \partial_\nu K^{\mu\nu}_J = 0$. Since this equality holds off-shell, the differential operator $\partial_\mu \circ J^{i\mu}$ is a Noether operator. From the assumtion that the characteristics $R^i_\alpha[\lambda^\alpha]$ define a generating set of gauge symmetries, there must exist differential operators $Z^\alpha(\cdot)$ and $M^{ij}(\cdot,\cdot)$, with the same notations as before, such that

$$\partial_\mu \circ J^{i\mu} = Z^{\dagger\alpha} \circ R^{\dagger i}_\alpha - M^{ij}(\cdot, E_j) \tag{3.33}$$

from Eq. (3.20). Acting on an arbitrary local function f, and integrating by parts in the operators, we find

$$-J^{i\mu} \circ \partial_\mu f = R^i_\alpha \circ Z^\alpha[f] - M^{\dagger ij}(f, E_j). \tag{3.34}$$

Particularising for $f = 1$ yields

$$R^i_\alpha[\bar\lambda^\alpha] = M^{\dagger ij}(E_j), \quad \bar\lambda^\alpha := Z^\alpha[1], \tag{3.35}$$

which is nothing but Eq. (3.26). Hence, there exists a reducibility parameter accounting for the conservation of the initial codegree-two current, $\mathbf{K}_J$, which concludes the proof. □

For exact reducibility parameters, it is possible to extract an explicit expression for a representative $\mathbf{K}_{\bar\lambda}$ out of the weakly vanishing Noether current. Indeed, since such gauge parameters annihilate the characteristics (3.28), we derive from Eqs. (3.10) and (3.14) that the weakly vanishing Noether current is closed off-shell, $\mathrm{d}\mathbf{S}_{\bar\lambda} = \mathbf{0}$. By virtue of algebraic Poincaré lemma (Theorem 2.1), it is therefore locally exact and we can "integrate" it by means of Anderson's homotopy operator (2.43). Recalling that the latter sends (k, ℓ)-forms on $(k-1, \ell+1)$-forms, we posit

$$\mathbf{k}_{\bar\lambda} := -I^{n-1}_{\delta\phi}\mathbf{S}_{\bar\lambda} \in \Omega^{(n-2,1)}, \tag{3.36}$$

up to an exact codegree-two form, of course. Using the properties (2.46), we indeed have $\mathrm{d}\mathbf{k}_{\bar\lambda} = \delta\mathbf{S}_{\bar\lambda} - I^n_{\delta\phi}(\mathrm{d}\mathbf{S}_{\bar\lambda}) = \delta\mathbf{S}_{\bar\lambda}$, since the weakly vanishing Noether current is closed. Consider now two solutions ϕ^i_1 and ϕ^i_2 of the equations of motion and a given path γ on the solution space linking them. We can obtain a $(n-2, 0)$-form by integrating over this path, so we further posit

$$\mathbf{K}_{\bar\lambda} := \int_\gamma \mathbf{k}_{\bar\lambda} = -\int_\gamma I^{n-1}_{\delta\phi}\mathbf{S}_{\bar\lambda} \in \Omega^{(n-2,0)}. \tag{3.37}$$

The integral is of course well-defined, *i.e.* only depends on the end points of the path γ and not on the choice of path itself, if and only if the codegree-two form (3.36) is δ-exact, *i.e.* obeys the integrability condition $\delta\mathbf{k}_{\bar\lambda} = \mathbf{0}$ since we have assumed that the topology of the field space is trivial. Finally, as a sanity check, we can observe that the codegree-two form (3.37) is utterly conserved,

$$\mathrm{d}\mathbf{K}_{\bar{\lambda}} = -\int_\gamma \delta \mathbf{S}_{\bar{\lambda}} = S_{\bar{\lambda}}\big|_{\phi_1} - S_{\bar{\lambda}}\big|_{\phi_2} \approx 0, \tag{3.38}$$

since both ϕ_1 and ϕ_2 are solutions of the equations of motion, for which the weakly vanishing Noether current cancels. For the following, it is worth keeping in mind that the crucial hypothesis here comes from the fact that the gauge parameters are reducibility parameters. Generalised Noether theorem affirms that these are the only gauge parameters for which a conserved quantity can be extracted. We shall see that if the reducibility property holds in a weaker sense, only asymptotically for instance, the construction still works but the related codegree-two currents are no longer obliged to be conserved.

3.5 Examples

We examplify here all the described results above by means of two well-known gauge theories: Maxwell electrodynamics and Einstein gravity. The latter allows us to already show the limits of the construction of conserved codegree-two currents from exact reducibility parameters and leads to the very important notion of asymptotic symmetries.

Maxwell electrodynamics

Let us consider a theory $S[\mathbf{A}]$ for an Abelian gauge field $\mathbf{A} = A_\mu \mathrm{d}x^\mu$ over a given n-dimensional spacetime $(\mathscr{M}, \mathbf{g})$ covered by coordinates (x^μ). The associated curvature $\mathbf{F} = \mathrm{d}\mathbf{A} = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ choosing ∇ as the Levi-Civita connection related to $\mathbf{g}$. In the presence of a matter current $\mathbf{j}$, Maxwell's Lagrangian for this theory reads

$$\mathbf{L} = \sqrt{-g}\left(-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu\right)(\mathrm{d}^n x) \tag{3.39}$$

for a real positive constant e. The jet coordinates are here $\phi^i_{(\mu)} \longrightarrow A_{\mu,(\nu)}$ where the comma separates the index of the gauge field from the derivative indices, and we shall keep the convenient notation $F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}$ in what follows. From the first-variation formula, Eq. (2.26), we deduce the equations of motion and the presymplectic potential related to (3.39)

$$\mathbf{E}^\nu = \frac{1}{e^2}\sqrt{-g}\left(\nabla_\mu F^{\mu\nu} - e^2 j^\nu\right)(\mathrm{d}^n x), \quad \mathbf{\Theta} = -\sqrt{-g}\,\frac{1}{e^2} F^{\mu\nu}\delta A_\nu (\mathrm{d}^{n-1}x)_\mu \tag{3.40}$$

that obey $\delta \mathbf{L} = \mathbf{E}^\nu \delta A_\nu - \mathrm{d}\mathbf{\Theta}$. As a side exercise, one can prove that acting with Anderson homotopy operator as in Eq. (2.47) reproduces the right result.

In the absence of external matter current, $\mathbf{j} = \mathbf{0}$, the theory is invariant under the Abelian gauge transformation $\delta_\lambda A_\mu = \partial_\mu \lambda$. In our general notations, the characteristic is $R_\mu[\lambda] = \partial_\mu \lambda$, that is the simplest possible differential operator. Since $\delta_\lambda F_{\mu\nu} = 0$, there is no related boundary term, $\mathbf{B}_\lambda = \mathbf{0}$ and the canonical representative $\mathbf{J}_\lambda$ of the Noether current is simply given by

$$\mathbf{J}_\lambda = \mathbf{B}_\lambda - i_{R[\lambda]}\mathbf{\Theta} = \sqrt{-g}\,\frac{1}{e^2} F^{\mu\nu}\partial_\nu \lambda (\mathrm{d}^{n-1}x)_\mu. \tag{3.41}$$

The only independent component of $\mathrm{d}\mathbf{J}_\lambda$ is evaluated on-shell to

$$\partial_\mu J^\mu_\lambda = \sqrt{-g}\frac{1}{e^2}\nabla_\mu\left(F^{\mu\nu}\partial_\nu\lambda\right) \approx \sqrt{-g}\, j^\mu\partial_\mu\lambda. \tag{3.42}$$

The Noether current is then conserved on-shell when there is no matter current. Furthermore, it is trivial because ∂_ν can be straightforwardly integrated by parts on-shell, resulting in $\mathbf{J}_\lambda = \mathrm{d}\mathbf{K}_\lambda$ where

$$\mathbf{K}_\lambda = \sqrt{-g}\frac{1}{e^2}\lambda F^{\mu\nu}(\mathrm{d}^{n-2}x)_{\mu\nu} = \frac{\lambda}{e^2}\star\mathbf{F} \tag{3.43}$$

To be more precise, integrating by parts in the conservation law of $\mathbf{J}_\lambda$ yields

$$R_\mu[\lambda]E^\mu = -\frac{\sqrt{-g}}{e^2}\nabla_\mu(\lambda\nabla_\nu F^{\mu\nu}) + \frac{\sqrt{-g}}{e^2}\lambda\nabla_\mu\nabla_\nu F^{\mu\nu}. \tag{3.44}$$

The last term yields the Noether identity $\nabla_\mu\nabla_\nu F^{\mu\nu} = 0$, which holds off-shell as the differential Bianchi identity for the gauge curvature $\mathbf{F}$. The first term provides the weakly vanishing Noether current

$$\mathbf{S}_\lambda = -\frac{\sqrt{-g}}{e^2}\lambda\nabla_\nu F^{\mu\nu}(\mathrm{d}^{n-1}x)_\mu. \tag{3.45}$$

We have now all the quantities to explicitly check the generalised Noether theorem. First, let us notice that reducibility parameters in Maxwell theory are pretty simple as they are exact and reduce to constant gauge parameters, $\bar\lambda \in \mathbb{R}$. Therefore, the related surface charge can be obtained by integration over the boundary $\partial\Sigma$ of a spacelike given volume Σ in $\mathscr{M}$:

$$\delta H_{\bar\lambda} = \int_{\partial\Sigma}\mathbf{K}_{\bar\lambda} = \frac{\bar\lambda}{e^2}\int_{\partial\Sigma}\star\mathbf{F}. \tag{3.46}$$

Particularising to $\bar\lambda = 1$ yields the electric charge in Σ computed from the flux of electric field across its boundary $\partial\Sigma$. We can also recover this famous result by means of the homotopy operator acting on the weakly vanishing Noether current. For this computation, the following identities are useful:

$$\begin{gathered}(\mathrm{d}^{n-2}x)_{\mu\nu} = \frac{1}{2}\frac{\partial}{\partial \mathrm{d}x^\nu}(\mathrm{d}^{n-1}x)_\mu, \qquad \frac{\partial\mathbf{S}_{\bar\lambda}}{\partial A_\alpha} = \mathbf{0}, \qquad \frac{\partial\mathbf{S}_{\bar\lambda}}{\partial A_{\alpha,\mu}} = \mathbf{0},\\ \frac{\partial\mathbf{S}_{\bar\lambda}}{\partial A_{\alpha,\mu\nu}} = -\frac{\sqrt{-g}}{e^2}\bar\lambda\left(g^{\alpha(\mu}g^{\nu)\beta} - g^{\alpha\beta}g^{\mu\nu}\right)(\mathrm{d}^{n-1}x)_\beta.\end{gathered} \tag{3.47}$$

Then, the only first three terms in the sum defining the Anderson homotopy operator contribute to the formula in Eq. (3.36). Since there is no gravity, hence the geometry is a background field, the variation utterly crosses the covariant derivatives, $\delta\nabla = 0$, and we end up with

$$\mathbf{k}_{\bar\lambda} = -I^{n-1}_{\delta\phi}\mathbf{S}_{\bar\lambda} = \frac{\bar\lambda}{e^2}\sqrt{-g}\,\delta F^{\mu\nu}(\mathrm{d}^{n-2}x)_{\mu\nu} = \frac{\bar\lambda}{e^2}\delta(\star\mathbf{F}). \tag{3.48}$$

This local expression is readily integrable on field space are reproduces correctly the expected result (3.43) upon integrating over a path connecting the target solution to a reference solution whose electric charge is assumed to vanish as a choice of off-set.

To be continued