

Quantum bounds and fluctuation-dissipation relations

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"Strange Metals, SYK Models and Beyond"

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Outline

- Quantum bounds and Planckian scales
- Fluctuation-dissipation theorem: from frequency to time
- OTOCs as two-point function: bound on chaos via FDT
- Blurring on a Planckian scale

Quantum bounds

The most famous bound: the uncertainty principle

$$\Delta x \Delta p \geq \frac{1}{2} \hbar$$

Most distinctive feature of how quantum mechanics differs from the classical world

More recently new bounds on physical time scales

Towards a bound on chaos

Lyapunov exponent

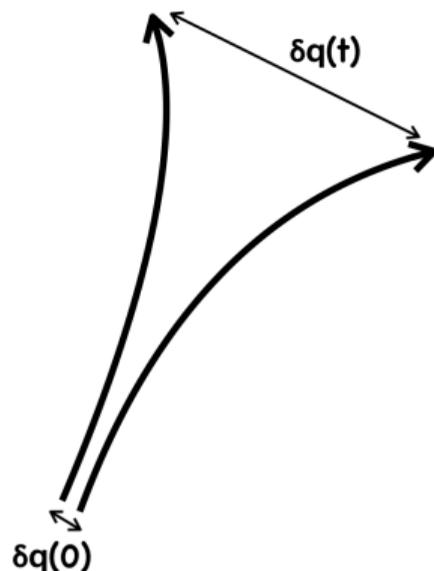
Classical

$$\left| \frac{\delta q(t)}{\delta q(0)} \right|^2 = |\{q(t), p(0)\}|^2 \simeq e^{\lambda t}$$

Quantum

$$\left\langle \left(i[q(t), p(0)] \right)^2 \right\rangle \simeq \hbar^2 e^{\lambda t}$$

Larkin and Ovchinnikov (1969)



NB: Focus on operators/observables, not on wavefunction

Out-of-Time-Order Correlations

$$\begin{aligned} -\langle [A(t), B(0)]^2 \rangle &= \langle A(t)B(0)B(0)A(t) \rangle + \langle B(0)A(t)A(t)B(0) \rangle \\ &\quad - \langle A(t)B(0)A(t)B(0) \rangle - \langle B(0)A(t)B(0)A(t) \rangle \end{aligned}$$

4-point Out-of-Time-Order Correlation (OTOC)

$$\langle A(t)B(0)A(t)B(0) \rangle$$

$$\langle \bullet \rangle = \frac{1}{Z} \text{Tr} [e^{-\beta H} \bullet]$$

Bound on chaos

Regularized OTOC

$$\frac{1}{Z} \text{Tr} [e^{-\beta H/4} A(t) e^{-\beta H/4} B(0) e^{-\beta H/4} A(t) e^{-\beta H/4} B(0)] \simeq a - \epsilon e^{\lambda t}$$

for $t_d \ll t \ll t_{Ehr}$

$$\lambda \leq \frac{2\pi T}{\hbar}$$

Maldacena, Shenker and Stanford (2016)

Sachdev-Ye-Kitaev (SYK) model

$$H_{SYK} = - \sum_{ijkl} J_{ijkl} \chi_i \chi_j \chi_k \chi_l$$

$$\{\chi_i, \chi_j\} = 2\delta_{ij} \quad J_{ijk} \text{ Gaussian i.i.d. variables}$$

Sachdev, Ye (1993)

Kitaev (2015)

- ▶ Asymptotically saturates the bound

Planckian time scales and transport coefficients

$$\tau \propto \tau_P = \frac{\hbar}{T}$$

- ▶ Same dependence on T for very different systems
- ▶ No details of the system \rightarrow Universality
- ▶ Shortest possible time scale in many-body systems?
- ▶ Conjectured bounds on transport coefficients (e.g. viscosity)

Chowdhury, Georges, Parcollet, Sachdev (2021)
Hartnoll, Mackenzie (2021)

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Correlations and responses

$$S(t) = \frac{1}{Z} \text{Tr} [e^{-\beta H} A(t) A(0)] = C(t) - i\hbar R(t)$$

Symmetrized correlation function

$$C(t) = \frac{1}{Z} \text{Tr} \left[e^{-\beta H} \frac{1}{2} \{A(t), A(0)\} \right]$$

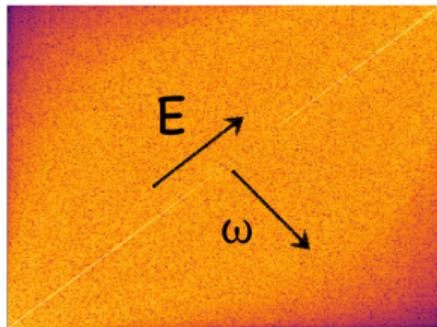
Response function (Kubo formula)

$$\chi(t) = \theta(t) 2R(t) = \theta(t) \frac{i}{Z\hbar} \text{Tr} [e^{-\beta H} [A(t), A(0)]]$$

An intermediate regulated function

$$F(t) = \frac{1}{Z} \text{Tr} \left[e^{-\frac{1}{2}\beta H} A(t) e^{-\frac{1}{2}\beta H} A(0) \right]$$

Meaning in ETH



$$A_{\alpha\beta} = \mathcal{A}(E)\delta_{\alpha\beta} + e^{-S(E)/2} f(E, \omega) R_{\alpha\beta}$$

$$E = (\epsilon_{\alpha} + \epsilon_{\beta})/2 \quad \omega = \epsilon_{\alpha} - \epsilon_{\beta}$$

M. Srednicki (1999)

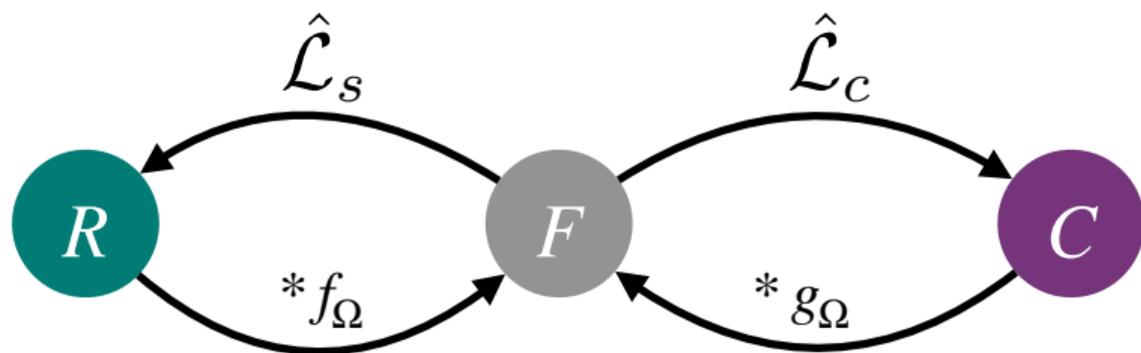
FDT in the frequency domain

$$\hbar \operatorname{Im} R(\omega) = \tanh\left(\frac{\hbar\beta\omega}{2}\right) C(\omega)$$

$$C(\omega) = \cosh\left(\frac{\hbar\beta\omega}{2}\right) F(\omega)$$

$$\hbar \operatorname{Im} R(\omega) = \sinh\left(\frac{\hbar\beta\omega}{2}\right) F(\omega)$$

FDT in the time domain domain



Going from one function to the other via convolutions or differential operators

FDT in the time domain domain (1)

$$C(t) = \cos\left(\frac{\hbar\beta}{2} \frac{d}{dt}\right) F(t)$$

$$R(t) = -\frac{1}{\hbar} \sin\left(\frac{\hbar\beta}{2} \frac{d}{dt}\right) F(t)$$

Classically $\hbar \rightarrow 0$

$$C(t) = F(t) \quad \chi(t) = \theta(t) 2R(t) = -\theta(t) \beta \partial_t C(t) \quad (1)$$

FDT in the time domain domain (2)

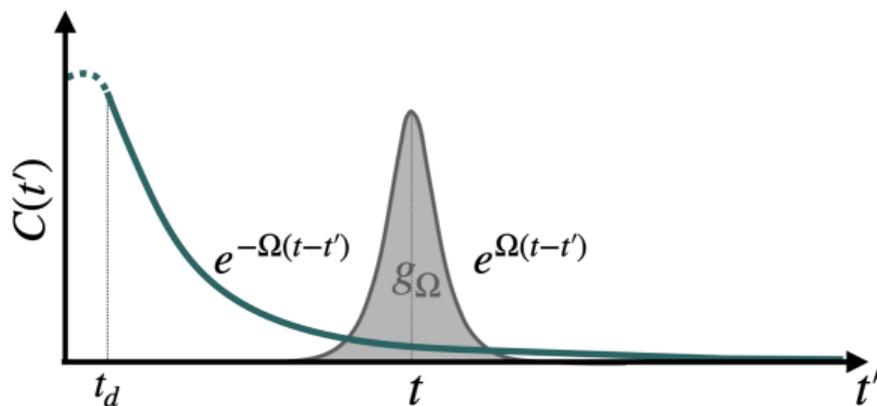
$$F(t) = \int dt' g_{\Omega}(t-t')C(t')$$

Blurring function

$$g_{\Omega}(t) = FT \left[\frac{1}{\cosh \hbar \beta \omega / 2} \right] = \frac{\Omega}{\pi} \frac{1}{\cosh \Omega t}$$

Planckian scale $\Omega = \frac{\pi T}{\hbar} = \frac{1}{\tau_{\Omega}}$

Blurring functions



- ▶ Smooths out the details on a scale $\simeq \hbar/T$
- ▶ Suppression of high frequency components $F(\omega) = \frac{C(\omega)}{\cosh \beta\omega/2}$
- ▶ $g_\Omega(t) \rightarrow \delta(t)$ for $\hbar \rightarrow 0$

t-FDT with (integrated) response

$$\begin{aligned} F(t) &= \int_{-\infty}^{\infty} dt' f(t-t') R(t') \\ &= \frac{\psi(\infty)}{\beta} + \int_0^{\infty} [\tilde{f}_{2\Omega}(t-t') + \tilde{f}_{2\Omega}(t+t')] \psi(t') \end{aligned}$$

$$\psi(t) = \int_0^t \chi(t') = 2 \int_0^t R(t')$$

$$f(t) = FT \left[\frac{1}{\sinh \hbar \beta \omega / 2} \right] = \frac{1}{\beta} \tanh \Omega t$$

Blurring function $\tilde{f}_{2\Omega}(t) = \frac{\Omega}{(\cosh \hbar \Omega t)^2}$

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OTOCs as two point functions (1)

$$\begin{aligned} & \frac{1}{Z} \text{Tr} \left[e^{-\beta H/2} A(t) A(0) e^{-\beta H/2} A(t) A(0) \right] \\ &= \frac{1}{Z} \sum_{ij} e^{-\frac{\beta}{2}(E_i + E_j)} \langle ij | A(t) \otimes A(t) A(0) \otimes A(0) | ji \rangle \\ &= \frac{1}{Z} \text{Tr} \left[e^{-\beta_2 \mathbb{H}} \mathbb{A}(t) \mathbb{A}(0) \mathbb{P} \right] \propto S_{\mathbb{A}, \mathbb{A}\mathbb{P}}(t) \end{aligned}$$

$$\mathbb{H} = H \otimes I + I \otimes H \quad \mathbb{A} = A \otimes A \quad \mathbb{P} |ij\rangle = |ji\rangle \quad \beta_2 = \beta/2$$

OTOCs as two point functions (2)

$$F_{\mathbb{A},\mathbb{A}\mathbb{P}}(t) \propto \text{Tr} \left[e^{-\frac{\beta_2}{2}\mathbb{H}} \mathbb{A}(t) e^{-\frac{\beta_2}{2}\mathbb{H}} \mathbb{A}(0)\mathbb{P} \right] \rightarrow MSS$$

$$C_{\mathbb{A},\mathbb{A}\mathbb{P}}(t) \propto \text{Tr} \left[e^{-\beta_2\mathbb{H}} \left\{ \mathbb{A}(t), \mathbb{A}(0)\mathbb{P} \right\} \right]$$

$$R_{\mathbb{A},\mathbb{A}\mathbb{P}}(t) \propto \text{Tr} \left[e^{-\beta_2\mathbb{H}} \left[\mathbb{A}(t), \mathbb{A}(0)\mathbb{P} \right] \right]$$

Behaviour of C and F

$$F_{A,AP}(t) \simeq a - \epsilon e^{\lambda t} \quad \text{for } t_d \ll t \ll t_{Ehr}$$

$$C_{A,AP}(t) \simeq b - \delta e^{\lambda t} \quad \text{for } t_d \ll t \ll t_{Ehr}$$

$$0 < \delta \ll 1 \quad 0 < \epsilon \ll 1$$

Exponential behavior from **positive defined** squared commutator

Bound on chaos from FDT

$$\text{t-FDT} \quad C(t) = \cos\left(\frac{\hbar\beta}{2} \frac{d}{dt}\right) F(t)$$

$$\text{If } F(t) = \epsilon e^{\lambda t} \quad \text{then} \quad C(t) = \cos\left(\frac{\hbar\beta\lambda}{2}\right) F(t)$$

Sign argument

$$\cos\left(\frac{\hbar\beta_2\lambda}{2}\right) \geq 0 \quad \text{for } \lambda \leq \frac{2\pi T}{\hbar}$$

Tsuji, Shitara, Ueda (2018)

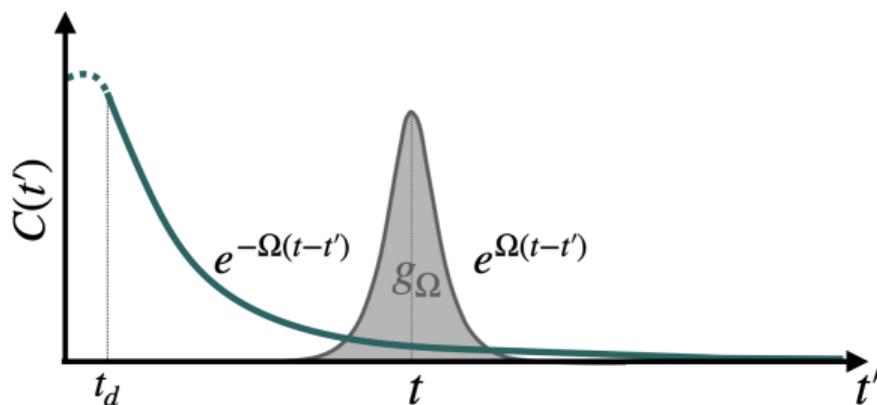
Pappalardi, Foini, Kurchan (2021)

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FDT as a blurring

$$F(t) = \int dt' g_{\Omega}(t-t') C(t') \quad g_{\Omega}(t) = \frac{\Omega}{\pi} \frac{1}{\cosh \Omega t}$$



Suppose

$$C(t) \propto e^{-at} \quad t \gg t_d$$

- ▶ If $a < \Omega$ integrand peaked at $t' \simeq t$
- ▶ If $a > \Omega$ dominated by short times

Bound on the decay of R

$$C(t) \simeq Ae^{-at} \quad t \gg t_d$$

$$F(t) = \frac{A}{\cos \beta a/2} e^{-at} + c_\Omega e^{-\Omega t} + \dots \propto e^{-t/\tau}$$

$$c_\Omega = \frac{2\Omega}{\pi} \int_{-\infty}^{\infty} dt e^{-\Omega t} C(t)$$

$$R(t) = A \tan \beta a/2 e^{-at} + c_\Omega e^{-\Omega t} + \dots \propto e^{-t/\tau}$$

Therefore, if $c_\Omega \neq 0$ the rate of F and R is bounded:

$$\frac{1}{\tau} \leq \Omega = \frac{\pi T}{\hbar}$$

Bound on the decay of C

If $c_\Omega = 0$ one can repeat the same starting from $R(t)$ which decays exponentially

$$F(t) \simeq C(t) \simeq e^{-t/\tau}$$

$$r_{2\Omega} = \frac{4\Omega}{\pi} \int_{-\infty}^{\infty} dt e^{-2\Omega t} R(t)$$

If $r_{2\Omega} \neq 0$ the rate of F and C is bounded:

$$\frac{1}{\tau} \leq 2\Omega = \frac{2\pi T}{\hbar}$$

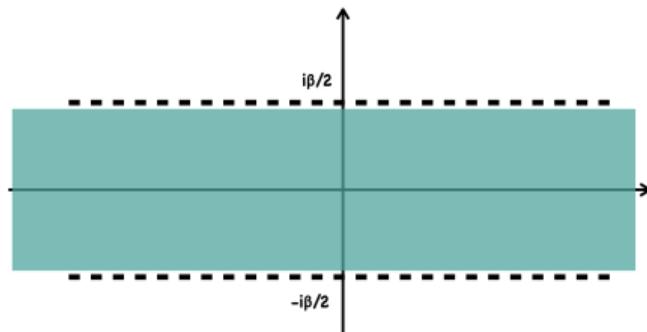
No proof of bound

$$c_{\Omega} = \frac{2\Omega}{\pi} \int_{-\infty}^{\infty} dt e^{-\Omega t} C(t) = 0$$

$$r_{2\Omega} = \frac{4\Omega}{\pi} \int_{-\infty}^{\infty} dt e^{-2\Omega t} R(t) = 0$$

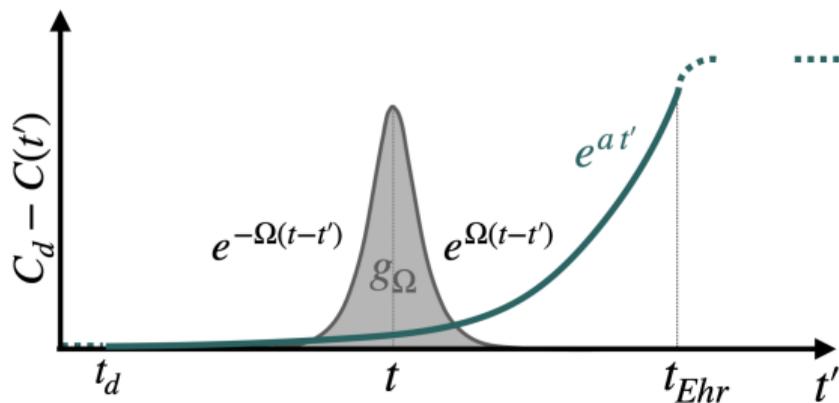
Sum rules \rightarrow Relation between short and long times

Analyticity properties of $F(t)$ in the strip



Blurring for exponentially increasing functions

$$C(t) = C_d - Ae^{at} \quad \text{for } t_d \ll t \ll t_{Ehr}$$



- ▶ If $a < \Omega$ integrand peaked at $t' \simeq t$
- ▶ If $a > \Omega$ dominated by large times

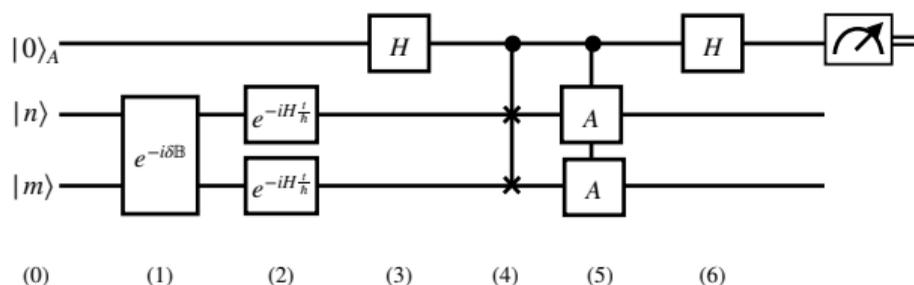
Ω with $\beta_2 \rightarrow$ bound on chaos

Conclusions

- Bound on chaos as a consequence of FDT
- Quantum FDT induces blurring on a Planckian scale

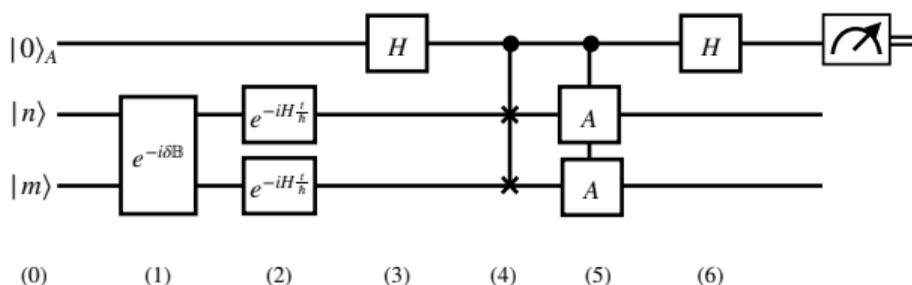
Conclusions

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- Response function in double space as a possible probe of OTOC?



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Thank you!