IS THE SCATTERING AMPLITUDE ANALYTIC IN A FIELD THEORY WITH A COMPACT SPATIAL COORDINATE?

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MOTIVATION

- My motivation: Question of Andre about analyticity of amplitude in a QFT with compactified spatial coordinate.

- **INTRODUCTION**
- **A SCALAR FIELD IN $D = 5$ DIMENSIONS.**
- **COMPACTIFICATION OF $D = 5$ THEORY TO $R^{3,1} \otimes S^1$**
- **ANALYTICITY PROPERTIES OF ELASTIC SCATTERING AMPLITUDE**
- **SUMMARY AND CONCLUSIONS**

- Let us recapitulate the known rigorous results for $D = 4$ theories derived from axiomatic field theory.
  The Froissart bound
  $$\sigma_t \leq \frac{4\pi}{t_0} ln^2 \frac{s}{s_0}$$
  t_0 is a parameter derived from first principle ($t_0 = 4m_\pi^2$) for most hadronic processes. s_0 is energy scale to make argument of *log* dimensionless and cannot be determined from axiomatic field theoretic frame work. The bound is arrived at from the following ingredients which can be derived from axiomatic field theory.

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- 1. Analyticity of scattering amplitude, $F(s, t)$, in the cut s-plane. $|F(s, t)| \leq s^N, N \in Z$, it is polynomially bounded (EGM) and it satisfies dispersion relation for t inside Lehmann-Martin ellipse.
  2. Crossing symmetry.
  3. Convergence of partial wave amplitude inside Lehmann-Martin ellipse.
  4. Unitarity. The partial wave amplitudes satisfy positivity condition

- $$0 \leq |f_l(s)|^2 \leq Im\ f_l(s) \leq 1$$

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- In order to derive statements (1) to (4) above one adopts following steps in the frameworks of general field theories, say LSZ.
- The scattering amplitude, $F(s,t)$, is the boundary value of an analytic function such that

  $$F(s,t) = lim_{\epsilon \to 0} F(s + i\epsilon, t)$$

  with a right hand cut starting from the threshold, s_{thr}, (say $4m^2$) and a left hand cut starting from $u = u_{thr}$. Partial wave expansion:

  $$F(s,t) = \frac{k}{\sqrt{s}} \sum_{l=0}^{\infty} (2l+1) f_l(s) Pl(cos\theta)$$

  The partial wave expansion converges inside the Lehmann-Martin ellipse in the $cos\theta$ plane which is larger than the usual domain i.e. $-1 \leq cos\theta \leq +1$. Thus the dispersion relations in s can be proved for fixed t, lying in the Lehmann-Martin ellipse.

- Khuri [Ann. Phys. **242**, 471(1995)] studied scattering in a quantum mechanical model where a spatial dimension is compactified on a circle, S^1, of radius R; $\frac{1}{R} << 1$. He derived expression for the scattering amplitude in the frame work of perturbation theory. The spatial geometry is $R^3 \otimes S^1$. The potential is $V(r, \Phi)$. $\mathbf{r} \in \mathbf{R^3}$, $r = |\mathbf{r}|$ and Φ has period 2π. $V(r, \Phi)$ is such that as $r \to \infty$ $V(r, \Phi) \to 0$. He showed that in the perturbative frame work the forward scattering amplitude violates analyticity properties for a class of potentials in certain situations. However, a model without S^1 compactification, with same potential (in $d = 3$) has good analyticity properties known from 1957.

- The scattering amplitude depends on three variables - the momentum of the particle, k, the scattering angle θ, and an integer n which appears due to the periodicity of the Φ-coordinate. Thus forward scattering amplitude is denoted by $T_{nn}(K)$, where $K^2 = k^2 + \frac{n^2}{R^2}$. The starting point is the Schrödinger equation

$$\left[\nabla^2 + \frac{1}{R^2}\frac{\partial^2}{\partial \mathbf{\Phi}^2} + K^2 - V(r, \Phi)\right]\Psi(\mathbf{r}, \Phi) = 0$$

The free plane wave solutions are

$$\Psi_0(\mathbf{x}, \Phi) = \frac{1}{(2\pi)^2} e^{i\mathbf{k}.\mathbf{x}} e^{in\Phi}$$

and $n \in \mathbf{Z}$. The total energy is

$$\mathbf{K}^2 = k^2 + \frac{n^2}{R^2}$$

He extracted the scattering amplitude adopting the standard Greens function method.

- (SKIP IT) The free Green's function in this case has the following form

$$G_0(\mathbf{K}; \mathbf{x}, \Phi : \mathbf{x}', \Phi') = -\frac{1}{(2\pi)^4} \sum_{n=-\infty}^{n=+\infty} \int d^3p \frac{e^{i\mathbf{p}.(\mathbf{x}-\mathbf{x}')} e^{in(\Phi-\Phi')}}{[p^2 + \frac{n^2}{R^2} - \mathbf{K}^2 - i\epsilon]}$$

It satisfies the free Schrödinger equation

$$G_0(\mathbf{K}; \mathbf{x}-\mathbf{x}'; \Phi-\Phi') = -\frac{1}{(8\pi^2)} \sum_{n=-\infty}^{n=+\infty} \frac{e^{i\sqrt{K^2-(n^2/R^2)}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} e^{in(\Phi-\Phi')}$$

Khuri's prescription: $\sqrt{K^2 - n^2/R^2}$ is defined in such a way that when $n^2/R^2 > K^2$

$$i\sqrt{K^2 - n^2/R^2} \rightarrow -\sqrt{n^2/R^2 - K^2}, \quad n^2 > K^2R^2$$

Expansion for $G_0(\mathbf{K}; \mathbf{x}-\mathbf{x}'; \Phi-\Phi')$ is damped for large enough $|n|$. The Green's function, G_0, satisfies the properties satisfied by those in usual potential scattering for fixed k^2.

- He started from an integral equation as is prescribed in potential scattering theory. Then extracted the scattering amplitude. He explicitly showed that for $n = 1$ the forward amplitude does not have analyticity properties as was the case with a $d = 3$ uncompactified theory.

- He started from an integral equation as is prescribed in potential scattering theory. Then extracted the scattering amplitude. He explicitly showed that for $n = 1$ the forward amplitude does not have analyticity properties as was the case with a $d = 3$ uncompactified theory.
- Andre (Martin) brought this paper to my attention. He was concerned. If the analyticity of a scattering amplitude breaks down in a relativistic local field theory then it has serious consequences. What will be fate of all rigorous results derived from axiomatic field theory and which have been tested experimentally?

- Analyticity and causality are deeply connected in QFT. In special theory of relativity, c is the limiting velocity - no signal travels faster then light. In case of potential scattering there is only Galilean invariance. Thus violation of analyticity in potential scattering is not such a matter of concern as in QFT.

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- I systematically studied this problem for a field theory in the LSZ formulation.

Analyticity Property of Forward Amplitude in a Compactified Field Theory

- How do we proceed?
- Consider a $D = 5$ flat spacetime, $R^{4,1}$, with a neutral, scalar massive field theory with mass m_0.
- Compactify one spatial dimension on S^1. The geometry is $R^{3,1} \otimes S^1$.
  The Spectrum: A massive scalar field of the original theory (mass m_0) and tower of KK states.
  Goal: To derive analyticity properties of the amplitude without appealing to any specific model.
- Assumptions: In the resulting compactified theory with above spectrum, all particles are stable, there are no bound states, the vacuum is unique.

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- Assumptions: In the resulting compactified theory with above spectrum, all particles are stable, there are no bound states, the vacuum is unique.
- We work in the LSZ formulation. Start with the five dimensional uncompactified theory; in the 5-D theory all variables are denoted with a **hat**. The postulates are:

- **A1.** The states of the system are represented in a Hilbert space, $\hat{\mathcal{H}}$. All the physical observables are self-adjoint operators in the Hilbert space, $\hat{\mathcal{H}}$.
  A2. The theory is invariant under inhomogeneous Lorentz transformations.
  A3. The energy-momentum of the states are defined. It follows from the requirements of Lorentz invariance that we can construct a representation of the orthochronous Lorentz group. The representation corresponds to unitary operators, $\hat{U}(\hat{a}, \hat{\Lambda})$, and the theory is invariant under these transformations. Thus there are hermitian operators corresponding to spacetime translations, denoted as $\hat{P}_{\hat{\mu}}$, with $\hat{\mu} = 0, 1, 2, 3, 4$. $[\hat{P}_{\hat{\mu}}, \hat{P}_{\hat{\nu}}] = 0$ If translation operators are chosen to be diagonal we have basis vectors span the Hilbert space

$$\hat{P}_{\hat{\mu}} |\hat{p}, \hat{\alpha} >= \hat{p}_{\hat{\mu}} |\hat{p}, \hat{\alpha} >$$

  The vacuum is Lorentz invariant. Another important postulate is microcausality.

- **A4.**. The microcausality: for two bosonic local operators $\mathcal{O}(x)$ *and* $\mathcal{O}(x')$

$$\left[\mathcal{O}(\hat{x}), \mathcal{O}(\hat{x}')\right] = 0, \qquad \textit{for} \ \ (\hat{x} - \hat{x}')^2 < 0$$

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- The asymptotic fields: define $\hat{\phi}(\hat{x})^{in,out}$ which satisfy free field equations. We may construct complete set of states from $\hat{\phi}^{in}$ or $\hat{\phi}^{out}$. $\hat{\phi}(\hat{x})$ is the interacting field; $\hat{\phi}(\hat{x})^{in,out}$ are defined with suitable limiting procedure from $\hat{\phi}(\hat{x})$. The vacuum is unique. Single particle states created by $\hat{\phi}(\hat{x})^{in}$ and $\hat{\phi}(\hat{x})^{out}$ are the same.

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- R-products

$$R\ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n) = \qquad (-1)^n \sum_P \theta(\hat{x}_0 - \hat{x}_{10})...\theta(\hat{x}_{n-10} - \hat{x}_{n0})$$
$$[[...[\hat{\phi}(\hat{x}), \hat{\phi}_{i_1}(\hat{x}_{i_1})], \hat{\phi}_{i_2}(\hat{x}_{i_2})]..], \hat{\phi}_{i_n}(\hat{x}_{i_n})]$$

$R\ \hat{\phi}(\hat{x}) = \hat{\phi}(\hat{x})$; the field is kept where it is. R-product is Lorentz invariant. The VEV of R-product is translationally invariant; consequently, $R(\hat{x}, ..\hat{x}_n)$ depends on difference of coordinates: $\hat{\xi}_1 = \hat{x} - \hat{x}_1$, $\hat{\xi}_2 = \hat{x}_1 - \hat{x}_2$....

- $R^{4,1} \rightarrow \; R^{3,1} \otimes S^1$
  Decompose the 5-dimensional spacetime coordinates as:
  $\hat{x}^{\hat{\mu}} = (x^\mu, y), \;\; \mu = 0,1,2,3, \;\; y \in S^1$. Periodicity of y: $y + 2\pi R = y$, R is radius of compactification. Consider, $\hat{\phi}(\hat{x})^{in}$ which satisfies free field equation: $[\Box_5 + m_0^2]\hat{\phi}^{in,out}(\hat{x}) = 0$. We expand the field

$$\hat{\phi}^{in,out}(\hat{x}) = \hat{\phi}^{in,out}(x,y) = \phi_0^{in,out}(x) + \sum_{n=-\infty}^{+n=\infty} \phi_n^{in,out}(x) e^{\frac{iny}{R}}$$

$\phi_0^{in,out}(x)$ has no y-dependence, it is called *zero mode*. For $n \neq 0$

$$[\Box - \frac{\partial}{\partial y^2} + m_n^2]\phi_n^{in,out}(x,y) = 0$$

where $\phi_n^{in,out}(x,y) = \phi_n^{in,out} e^{\frac{iny}{R}}$ and $n = 0$ term is
$\phi_0^{in,out}(x) = \phi^{in,out}(x)$ from now on. Here $m_n^2 = m_o^2 + \frac{n^2}{R^2}$.
Momentum associated along y-direction is quantized: $q_n = \frac{n}{R}$; it is additive conserved quantum number.

- Let us look at Källen-Lehmann spectral reprsentation for the 5-dimensional theory

$$< 0|[\hat{\phi}(\hat{x}), \hat{\phi}(\hat{y})]|0 >= \quad \sum_{\hat{\alpha}} \Big(< 0|\hat{\phi}(0)\hat{\alpha} > e^{-i\hat{p}_{\hat{\alpha}}.(\hat{x}-\hat{y})} \\ \times < \hat{\alpha}|\hat{\phi}(0)|0 > -(\hat{x} \leftrightarrow \hat{y}) \Big)$$

If we expand $\hat{\phi}(\hat{x})$ in fourier modes as we have done for $\hat{\phi}(x, y)^{in}$ earlier then we arrive at

$$< 0|[\hat{\phi}(x, y), \hat{\phi}(x', y')]|0 >= \quad < 0|[\phi_0(x) + \sum_{-\infty}^{+\infty} \phi_n(x, y),\ \phi_0(x') + \\ \sum_{-\infty}^{+\infty} \phi_l(x', y')]|0 >$$

The VEV of a commutator of two $\hat{\phi}$ fields in the (KL) representation decompose as sums of several VEV's. Vacuum has $q_n = 0$ thus terms like $< 0|[\phi_n, \phi_{-n}]|0 >$ are admissible

$$< 0|[\phi_0(x), \phi_0(x')]|0 >, \quad < 0|[\phi_n(x), \phi_{-n}(x')]|0 >, ...$$

- The interacting field satisfies equations of motion with a source current, $\hat{j}(\hat{x})$ and it can be expanded as

$$\hat{j}(x,y) = j(x) + \sum_{n=-\infty}^{n=+\infty} J_n(x) e^{iny/R}$$

$\phi(x)$ and $\phi_n(x)$ interpolate to corresponding *in* and *out* fields. $\phi^{in,out}$ and each of the fields $\phi_n^{in,out}(x)$ create their Fock spaces. For example the single particle (say 'in') states are:

$a^{\dagger,in}\mathbf{k})|0>=|\mathbf{k},k_0,in>, k_0>0;\ A^{\dagger,in}(\mathbf{p},q_n|0>=|\mathbf{p},p_0;q_n,in>, p_0>0$

Each sector contains a complete set of states is designated with a conserved charge $q_n=\frac{n}{R}$. Thus $<\mathbf{p}',q_n'|\mathbf{p},q_n>=\delta^3(\mathbf{p}'-\mathbf{p})\delta n',n$. Thus $\hat{\mathcal{H}}$ decomposes as

$$\hat{\mathcal{H}} = \sum \oplus \mathcal{H}_n$$

- Definitions and conventions

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- Field and four momenta associated with $n = 0$ charge are respectively denoted as $\phi(x)$ and k. Fields carrying nonzero charges and four momenta are: $\chi(x)$ and p. The elastic scattering between particles (with charge conservation) are of following types:
  (i) $\phi + \phi' \to \phi + \phi'$
  (ii) $\phi + \chi(n) \to \phi' + \chi'(n)$
  (iii)$\chi(m) + \chi(n) \to \chi'(m) + \chi'(n)$. We shall consider scattering of particles with equal charge reaction (iii) without any loss of generality; with this choice (i) and (iii) describe equal mass scattering whereas (ii) is unequal mass scattering. We don't deal with (i) and (ii) in this talk. Details are in arXiv 1810.11275

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- The Mandelstam variables are:

$$s = (\tilde{p}_a + \tilde{p}_b)^2, \quad t = (\tilde{p}_a - \tilde{p}_d)^2, \quad u = (\tilde{p}_a - \tilde{p}_c)^2$$

$\mathbf{M}_a^2, \quad \mathbf{M}_b^2, \quad \mathbf{M}_c^2, \quad \mathbf{M}_d^2$, are two or more particle states carrying same quantum number as a, b, c, d.
$(\mathbf{M}_{ab}, \mathbf{M}_{cd}), \quad (\mathbf{M}_{ac}, \mathbf{M}_{bd}), \quad (\mathbf{M}_{ad}, \mathbf{M}_{bc})$ two or more particle states having quantum numbers of $(ab, cd), (ac, bd), (ad, bc)$ respectively.

- **Elastic Scattering of $\mathbf{n(p_a) + n(p_b) \to n(p_c) + n(p_d)}$**

- **Elastic Scattering of $\mathbf{n(p_a) + n(p_b) \to n(p_c) + n(p_d)}$**
- We can proceed by adopting LSZ reduction technique. The fields are denoted by $\chi_a, \chi_b, \chi_c, \chi_d$ with respective momenta p_a, p_b, p_c, p_d. Following standard prescription

$$
\begin{aligned}
< p_d, p_c \ out | p_b, p_a \ in >= \quad & < p_d, p_c \ in | p_b, p_a \ in > \\
& - \frac{1}{(2\pi)^3} \int d^4x \int d^4x' e^{-i(p_a.x - p_c.x')} \\
& < p_d | \theta(x'_0 - x_0)[J_c(x'), J_a(x)] | p_b >
\end{aligned}
$$

$J_a(x)$ is source current for $\chi_a(x)$ and similarly for $J_c(x')$. Invoke unitarity,

$$
F(s,t) = i \int d^4x e^{i(p_a + p_c).\frac{x}{2}} \theta(x_0) < p_d | [J_a(x/2), J_c(-x'/2)] | p_b >
$$

We evaluate the imaginary part of this amplitude, $F(s,t,)$

-

$$
\begin{aligned}
Im\ F(s,t) = \quad & \frac{1}{2i}(F - F^*) \\
& = \frac{1}{2}\int d^4x e^{i(p_a+p_c).\frac{x}{2}} < p_d|[J_a(x/2), J_c(-x/2)]|p_b >
\end{aligned}
$$

We use the fact that F^* is invariant under interchange $p_b \to p_d$ and also $p_d \to p_b$; $\theta(x_0) + \theta(-x_0) = 1$. Open up the commutator of the two currents; introduce a complete set of states $\sum_{\mathcal{N}} |\mathcal{N} >< \mathcal{N}| = 1$. implement translation operations in each of the (expanded) matrix elements to express arguments of each current as $J_a(0)$ and $J_c(0)$ finally integrate over d^4x to get the δ-functions. Then

$$
\begin{aligned}
F(p_d, p_c; p_b, p_a) - F^*(p_b, p_a; p_c, p_d) = \quad & 2\pi i \sum_{\mathcal{N}} \Big[\delta(p_d + p_c - p_n) \\
& F(p_d, p_c; n) F^*(p_a, p_b; n) \\
& -\delta(p_a - p_c - p_n) \\
& F(p_d, -p_a; n) F^*(p_b, -p_c; n) \Big]
\end{aligned}
$$

Generalized unitarity relation. Forward case: implies optical theorem.

- Look at the first term: δ function implies $p_a + p_b = p_n = p_c + p_d$. This is s-channel process, $p_n^2 = \mathcal{M}_n^2 = s$.
  Look at second term: $p_b + (-p_c) = p_n = p_d + (-pa)$: $p_n^2 = \mathcal{M}_n^2 = (p_b - p_c)^2$. Go to a Lorentz frame $p_b = (m_b, \mathbf{0})$, then

$$\mathcal{M}_n^2 = 2m_b(m_b - p_c^0), \quad p_c^0 > 0$$

  Note: $m_a = m_c$, $p_c^0 = \sqrt{m_c^2 + \mathbf{p}_c^2}$; $\mathcal{M}_n^2 < 0$. $\mathcal{M}_n$ is intermediate physical state carrying n charge. Thus above condition cannot be satisfied. The 2nd term does not contribute to s-channel process. Instead look at cross channel process:

$$p_b + (-p_c) \to p_d + (-p_a); \quad -p_a^0 > 0, \ and \ -p_c^0 > 0$$

  p_b and p_c are incoming (hence the negative sign for p_c) and p_d and p_a are outgoing. However, the first term does not contribute. Here is hint of *crossing symmetry* (it is not a proof - can be proved ?). We are not interested to prove crossing symmetry here! The δ-functions guarantee energy momentum conservation. Generalized Unitarity implies there is a cut off for KK towers *as intermediate states* so long as s is finite, s could be very large.

- *Forward Scattering Amplitude: The Analyticity Property*

- *Forward Scattering Amplitude: The Analyticity Property*
- In this case the process is $p_c = p_a$ and $p_d = p_b$. We have forward scattering of equal mass particles, $m_n^2 = m_0^2 + \frac{n^2}{R^2}$. Starting from

$$F(p_b, p_a; p_b, p_a) = \int d^4x e^{ip_a.x} (\Box_x + m_n^2)^2 < p_b | R\chi_a(x)\chi_a(0) | p_b >$$

We arrive at

$$F(p_b, p_a; p_b, p_a) = \int d^4x e^{ip_a.x} < p_b | RJ_a(x)J_a(0) | p_b >$$

We go to the rest frame of particle 'b': $p_b = (m_b, \mathbf{0})$ and define $\omega = \frac{p_a.p_b}{m_n}$. In this frame (adopted by Symanzik)

$$F(p_b, p_a : p_b, p_a) = i \int_0^\infty \int d^3\mathbf{x} e^{ip_a^0 x^0 - i\sqrt{(p_a^0)^2 - m_n^2}\hat{\mathbf{e}}.\mathbf{x}} \tilde{f}(\mathbf{x}, x_0)$$

$\hat{\mathbf{e}}$ is the unit vector along $\mathbf{p}_a$. We can identify $\tilde{f}(\mathbf{x}, x_0)$; and from microcausality, we conclude $\tilde{f}(\mathbf{x}, x_0) = 0$, unless $x_0 > |\mathbf{x}|$.

- After the angular integration

$$F(p_b, p_a; p_b, p_a) = \int_0^\infty \mathcal{F}(\omega, r) dr$$

with

$$\mathcal{F}(\omega, r) = \quad 4\pi i \frac{sin\sqrt{\omega^2 - m_n^2} r e^{i\omega r}}{\sqrt{\omega^2 - m_n^2} r} \times \int_r^\infty dt e^{i\omega(r-t)} < p_b|[J_a(x), J_a(0)]|p_b >$$

Technicalities - SKIP: $\mathcal{F}(\omega, r)$ is analytic function of ω for $Im\ \omega \geq 0$ (upper half plane). (i) No branch point at $\omega = \pm m_n$ since $\frac{sin\sqrt{\omega^2 - m_n^2} r}{r\sqrt{\omega^2 - m_n^2}}$ even in $r\sqrt{(\omega^2 - m_n^2}$.
(ii) For, $\omega < m_n$ problem in behavior of $sin\sqrt{\omega^2 - m_n^2} r$? The presence of $e^{i\omega r}$ takes care. (iii) Assume, F is well behaved in s - no subtractions. To write dispersion relation for F, we have to interchange integration over r and ω. Write a dispersion relation for $\mathcal{F}(\omega, r)$ (assume it vanishes for large ω), then

- $$\mathcal{F}(\omega, r) = \frac{1}{\pi}\int_{\infty}^{+\infty} \frac{\mathrm{Im}\ \mathcal{F}(\omega', r)}{\omega' - \omega - i\epsilon} d\omega'$$

  Note that $\mathrm{Im}\ \mathcal{F}$ has the property: $\mathrm{Im}\ \mathcal{F}(r, \omega) = -\mathrm{Im}\ \mathcal{F}(-\omega, r)$. The integral is

  $$\mathcal{F}(\omega, r) = \frac{1}{\pi}\int_{0}^{+\infty} \mathrm{Im}\ \mathcal{F}(\omega', r)\left[\frac{1}{\omega' - \omega - i\epsilon} + \frac{1}{\omega' - \omega + i\epsilon}\right] d\omega'$$

  Now $\mathrm{Im}\ F$ is expressed as

  $$\mathrm{Im}\ F(p_b, p_a; p_b, p_a) = \frac{1}{2}\int d^4x e^{p_a.x} < p_b|[J_a(x), J_a(0)]|p_b > \qquad (1)$$

  We can open up the commutator, insert complete set of states, use translation operation and carry out the angular integration to get

  $$\begin{aligned}\mathrm{Im}\ F(p_b, p_a; p_b, p_a) = \quad & \frac{1}{2}(2\pi)^4 \sum_n | < p_b|J_a(0)|p_b > |^2\Big] \\ & \times[\delta^4(p_b + p_a - p_n) - \delta^4(p_b - p_a + p_n)\Big]\end{aligned}$$

- The expression for $\mathcal{F}(\omega, r)$ is

$$\mathrm{Im}\ F(p_b, p_a; p_b, p_a) = \frac{1}{2}\int dr 4\pi r^2 \frac{sin\sqrt{\omega^2 - m_n^2}r}{\sqrt{\omega^2 - m_n^2}r} \times \int_{-\infty}^{+\infty} e^{i\omega t} < p_b|[J_a(x), J_a(0)]|p_b > dt$$

While writing dispersion integral for $F(p_a, p_b, p_c, p_d)$ the issue of interchanging t and ω integral comes up. Symanzik has resolved this in his (1957) paper on *forward dispersion relation for* πN *scattering*. Here is a simple problem of scattering of equal mass bosons. Thus the dispersion relation written above for forward scattering amplitude holds $F(\omega)$. Moreover, Bogoliubov's approach leads to same conclusion.

- The expression for $\mathcal{F}(\omega, r)$ is

$$
\begin{aligned}
\mathrm{Im}\ F(p_b, p_a; p_b, p_a) = \quad & \frac{1}{2}\int dr 4\pi r^2 \frac{sin\sqrt{\omega^2 - m_n^2}r}{\sqrt{\omega^2 - m_n^2}r} \times \\
& \int_{-\infty}^{+\infty} e^{i\omega t} < p_b|[J_a(x), J_a(0)]|p_b > dt
\end{aligned}
$$

While writing dispersion integral for $F(p_a, p_b, p_c, p_d)$ the issue of interchanging t and ω integral comes up. Symanzik has resolved this in his (1957) paper on *forward dispersion relation for πN scattering*. Here is a simple problem of scattering of equal mass bosons. Thus the dispersion relation written above for forward scattering amplitude holds $F(\omega)$. Moreover, Bogoliubov's approach leads to same conclusion.

- Thus the forward amplitude satisfies dispersion relation. We have assumed good behavior for large ω. We discuss subtractions later. Conclusion of this section: Analyticity is not violated. This is different from the conclusion of Khuri who studied analyticity of amplitude perturbatively in potential scattering.

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- The elastic scatterings amplitudes for (i) $(n = 0) + (n = 0)$ and (ii) $(n = 0) + (n \neq 0)$ satisfy analyticity properties. In fact all the known results of analyticity in LSZ frame work can be derived (not proved here).
- We systematically studies forward elastic scattering amplitude of $(n) + (n)$ for $n \neq 0$. We showed that the forward scattering amplitude satisfies dispersion relations.

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- How about fixed-t dispersion relation? It is possible but requires more work (hard?). We need to prove J-L-D Theorem and then existence of Lehmann Ellipses and then the crossing symmetry (may be possible). Thus, domain of convergence of partial wave expansions will be established. One need to prove Martin's theorem before establishing analog of Jin-Martin theorem! Then only TWO subtractions !! It requires a lot of work $* * **$.

THANK YOU