Entanglement, Fluctuations Quantum Quenches

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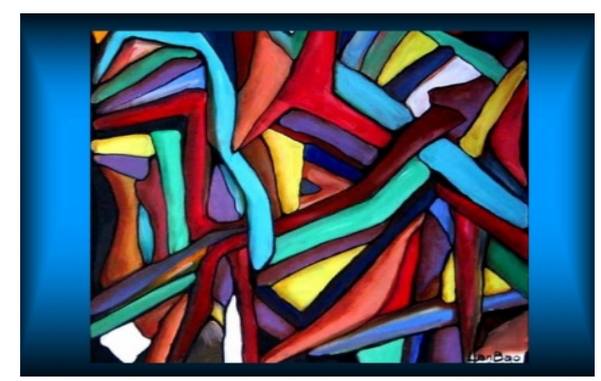
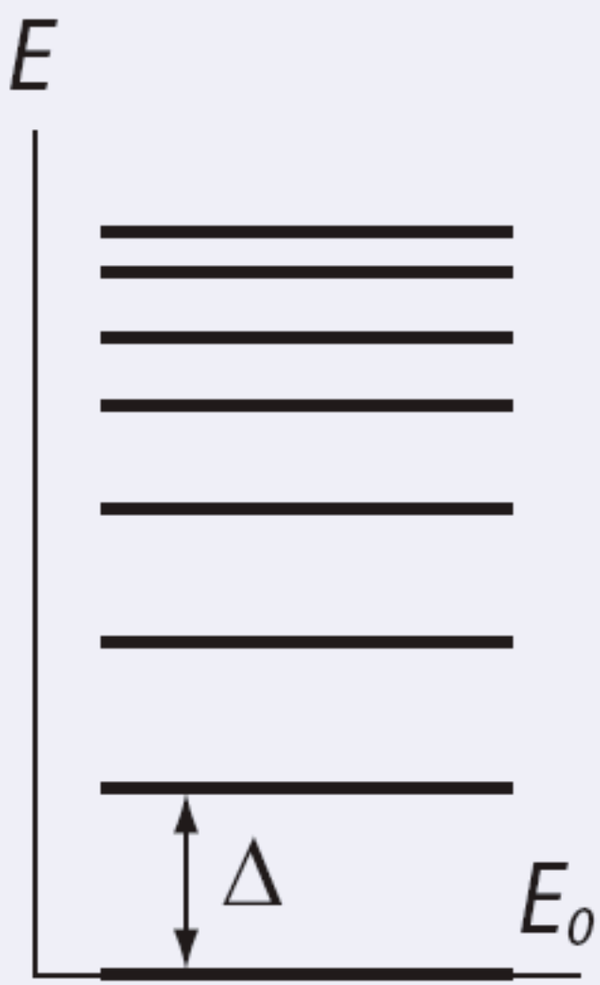


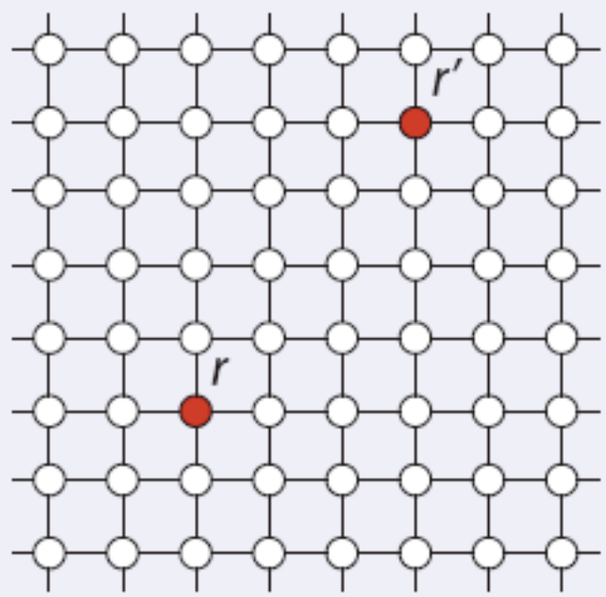
"Traditional" Many-Body Physics

Ground state and excited state energies



Gap, spectrum

Correlation functions

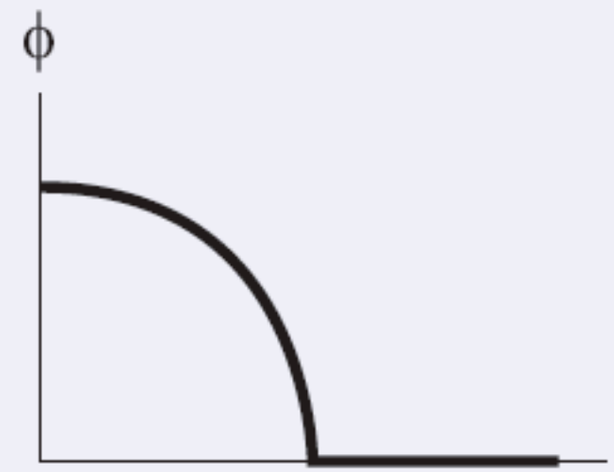


$\langle \hat{\psi}^\dagger(r,t)\hat{\psi}(r',t')\rangle$

$\langle \hat{\rho}(r,t)\hat{\rho}(r',t')\rangle$

Long-range order, response to perturbations

Symmetry-breaking order parameters



$\langle \phi \rangle \neq 0$

Non-Traditional

Entanglement & (number particle/spin) Fluctuations

Francis Song (PhD) Yale University

Stephan Rachel Yale University

Nicolas Laflorencie LPS Orsay, Paris
(→ Toulouse)

H. Francis Song, Stephan Rachel, Christian Flindt, Israel Klich, Nicolas Laflorencie, KLH,
arXiv:1109.1001, 30 pages+25 pages Supplementary Material
Phys. Rev. B **85**, 035409 (2012) [27 pages] (Editors' Suggestion) + long Suppl. Material

Also with I. Klich (Virginia) and C. Flindt (Geneva)

From Sadi Carnot to …

Thermodynamics
2nd principle, **1824**
Carnot cycle

Irreversibility and Entropia (1864)

Sadi Carnot (1796-**1832**)
died at 36 years old
Ecole Polytechnique

Entropy: « energy » & « tropos »

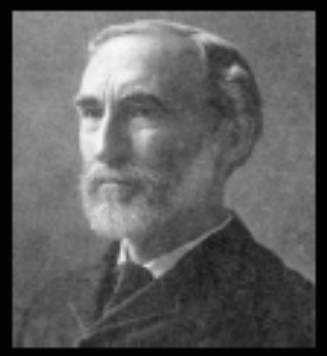
The word entropy was introduced by ***Clausius***, 1864

<u>Thermodynamics Foundations:</u>

Change of entropy is obtained by adding the small portions of heat received by the system divided by T during the heat process

<u>Measure of uncertainty</u> by ***Boltzmann & Gibbs:***

$$S = -k_B \int \rho(X) \, |\log \rho(X)| \, dX$$



Here, $\rho(X)\, dX$ is the *ensemble* density

Operator & Quantum mechanical perspective by ***von Neumann***, 1927

$$S = - \mathrm{Tr}\, \rho \log \rho$$

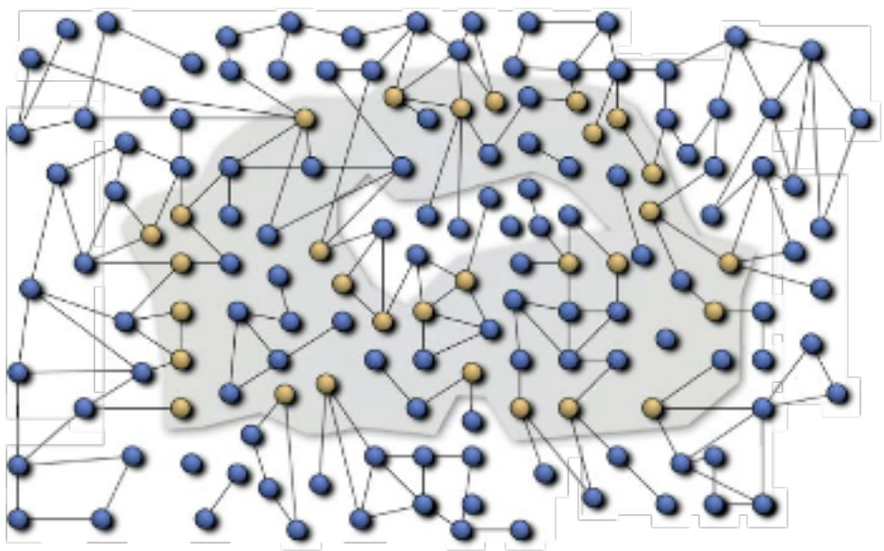
Entanglement Entropy

quantum mechanics:
entropy > 0 without an objective lack of information

non-degenerate
pure ground state

$$\rho_0 = |\psi\rangle\langle\psi|$$

$$\Rightarrow S(\rho_0) = 0$$



(Eisert *et al.*, RMP, 2010)

shaded region A
remainder B

$$\rho_A = \mathrm{tr}_B(\rho)$$

$$\Rightarrow S(\rho_A) \neq 0$$

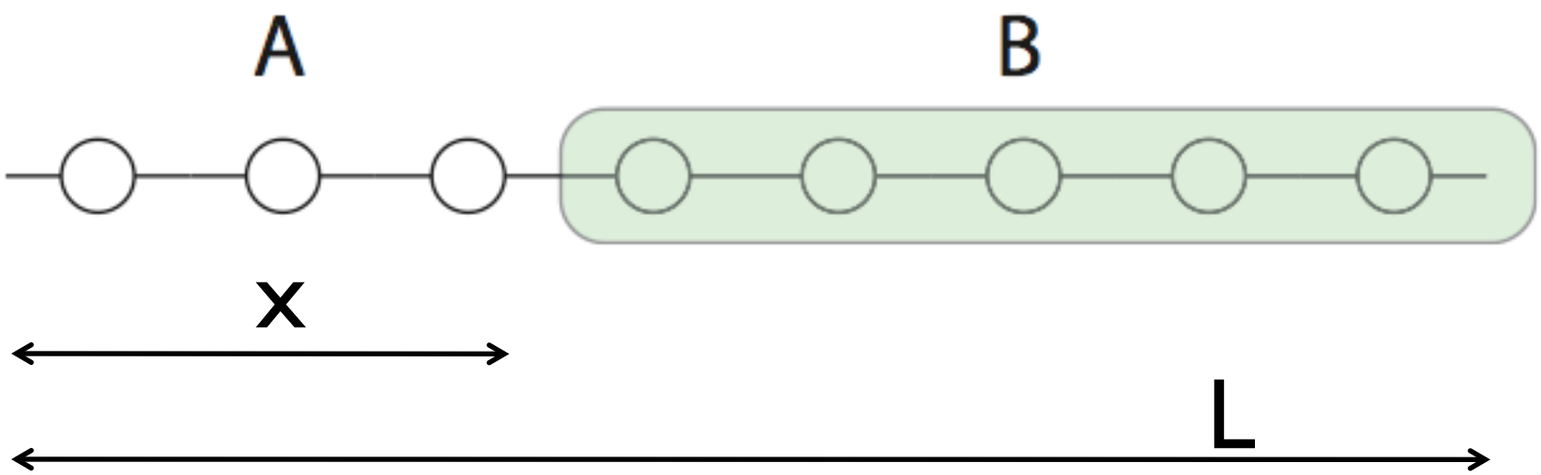
Entanglement
Entropy

von Neumann entropy

$$S(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$$

L. Amico et al. Rev Mod Phys (2008)

entanglement entropy in one dimension



Given a pure state $|\psi\rangle$, the reduced density matrix is

$$\hat{\rho}_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|.$$

Entanglement entropy (von Neumann entropy):

$$\mathcal{S}_A = -\mathrm{Tr}\ \hat{\rho}_A \log \hat{\rho}_A.$$

Definition is extremely general.

entanglement entropy

Conformal field theory (CFT) (Calabrese & Cardy):

$$\mathcal{S}(x) = \frac{c}{3} \ln \frac{x}{a}, \qquad c = \text{central charge.}$$

- Finite-size:

$$\mathcal{S}(x, L) = \frac{c}{3} \ln \left(\frac{L}{\pi} \sin \frac{\pi x}{L} \right) + s_1.$$

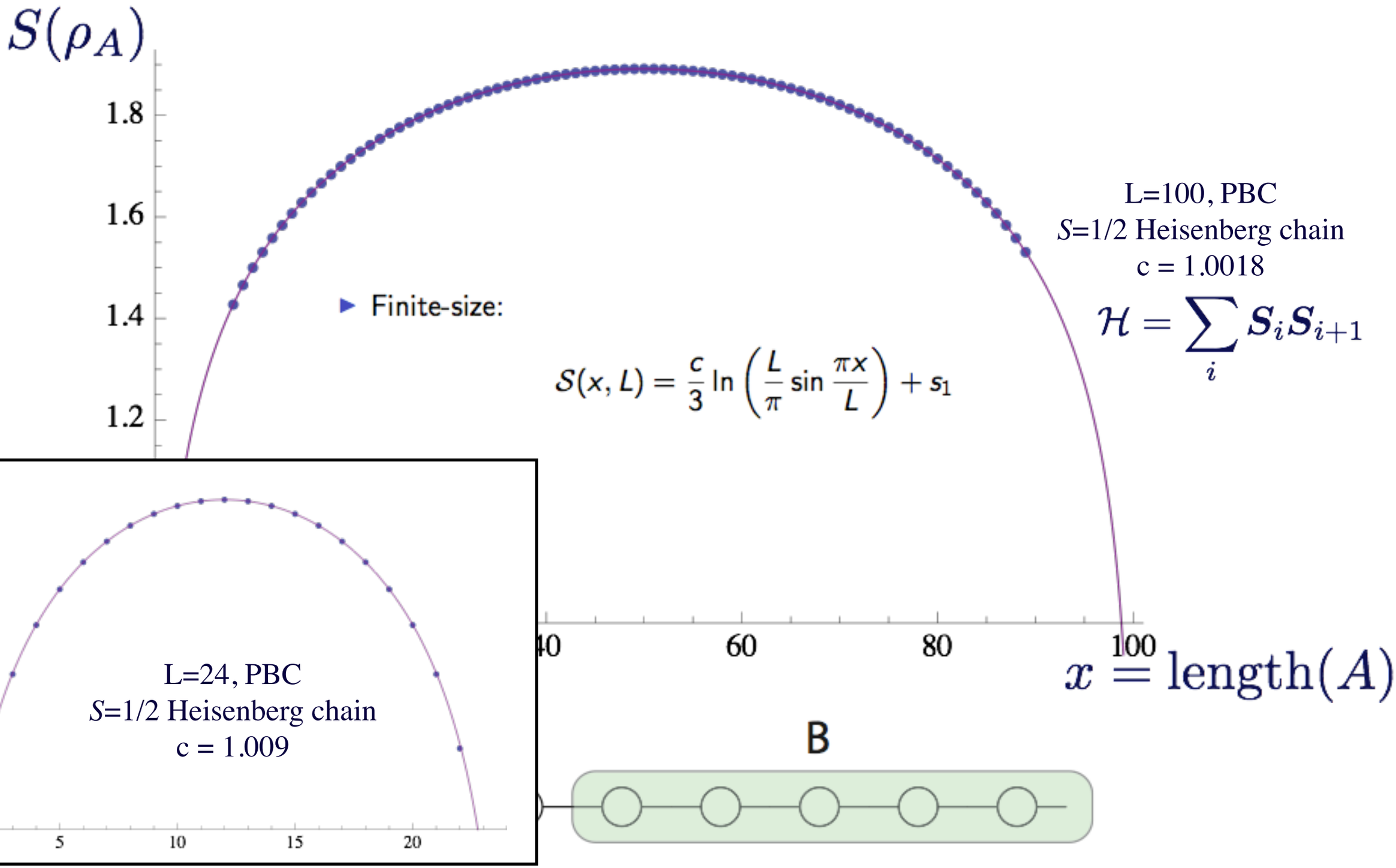
- Similar expression for finite temperature, free boundaries.

Are there observables that behave similarly?
Yes. Suggestions: QPC with free fermion (Klich & Levitov), fractional quantum Hall (Hsu et al.) wires.

$$S = -\lim_{n \to 1} \frac{\partial}{\partial n} \mathrm{Tr} \rho_A^n$$

CFT results for entropy: see also Holzhey, Larsen and Wilczek, 1994

entanglement entropy



DMRG results allow to extract c: Stephan Rachel and Francis Song

entanglement entropy

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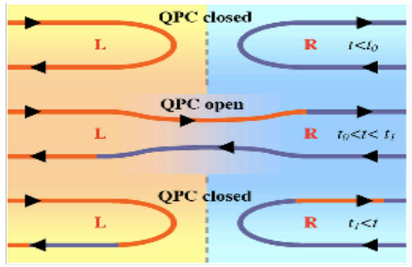


QPC, free fermions: Current fluctuations are Gaussian with a log. Variance:
$C_2=(1/\pi^2)\ln|t_1-t_0|$ & $S\sim 1/3.\ln|t_1-t_0|$

$C_2 = F$

Definitions

$$\hat{N}_A \to \hat{S}_z = \sum_{i \in A} \hat{S}_z^i$$

Number (spin) fluctuations:

$$\mathcal{F}_A = \langle (\hat{N}_A - \langle \hat{N}_A \rangle)^2 \rangle.$$

- Symmetric (cf. entanglement entropy):

$$\mathcal{F}_A = \mathcal{F}_B.$$

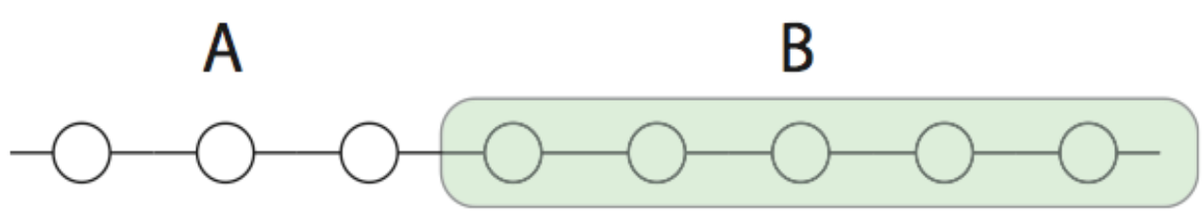
- Zero for a product state

$$\mathcal{F}_A = 0 \text{ if } |\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle.$$

- Luttinger liquids: (see lecture by C. Pepin and book by T. Giamarchi)

$$\pi^2 \mathcal{F}(x) = \frac{1}{\pi^2} \langle [\phi(x) - \phi(0)]^2 \rangle \sim K \ln \frac{x}{a}.$$

K=1 free fermions (Klich-Levitov)
K=ν for fractional QH edges
(Fradkin et al)



H. Francis Song, Stephan Rachel & Karyn Le Hur, PRB 2010

Number Fluctuations

Important properties of $\mathcal{F} = C_2 = \langle \hat{N}_A^2 \rangle - \langle \hat{N}_A \rangle^2$ with total number conservation

1. With one exception*, zero if and only if $|\Psi\rangle$ is a product state

$$|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle.$$

2. Symmetry between subsystems

$$\mathcal{F}_A = \mathcal{F}_B \equiv \mathcal{F}.$$

This suggests the importance of the boundary.

3. Subadditivity

$$\mathcal{F}_A + \mathcal{F}_B \geq \mathcal{F}_{A \cup B} = 0.$$

* One exception

The fluctuations are zero if particle number is **separately conserved** in the subsystems (cf. entropy for partitioning in momentum space).

Conserved U(1) current **(fixed total number of particles or spin)**
U(1) currents are described by a massless free boson, Critical theory

CFT + U(1)

- Generating function:

$$M_A(\lambda) = \langle e^{i\lambda(\hat{N}_A - \langle \hat{N}_A \rangle)} \rangle = \left(\frac{x}{a}\right)^{-g\lambda^2/(2\pi^2)}.$$

$$\pi^2 \mathcal{F}_A = -\pi^2 M_A''(0) = g \ln \frac{x}{a}.$$

- g fixed heuristically by comparing to finite-T:

$$\pi^2 \mathcal{F}(x, \beta) = g \ln \left(\frac{\beta}{\pi a} \sinh \frac{\pi x}{\beta} \right).$$

Compare to $\mathcal{F}(x \gg \beta, \beta) \sim \kappa x/\beta$:

$$g = \pi v \kappa.$$

High T: sub-system A in grand canonical ensemble in equilibrium with a thermal bath

Relation between Entanglement Entropy and Fluctuations

$$\frac{\mathcal{S}(x)}{\pi^2 \mathcal{F}(x)} \sim \frac{c}{3\pi v \kappa}, \qquad x \gg a.$$

Simple interpretation for previous results.

$\pi \, v \, \kappa \; = K$ for Luttinger theories

Also Valid for gapless spin chains…

Sub-leading corrections & boundaries?

For spin systems, the compressibility turns into the uniform magnetic susceptibility

XX Model – Periodic Boundary Conditions

- Hamiltonian:

$$H_{\mathrm{XX}} = \sum_i (\hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y).$$

  Free fermions, and therefore solvable.

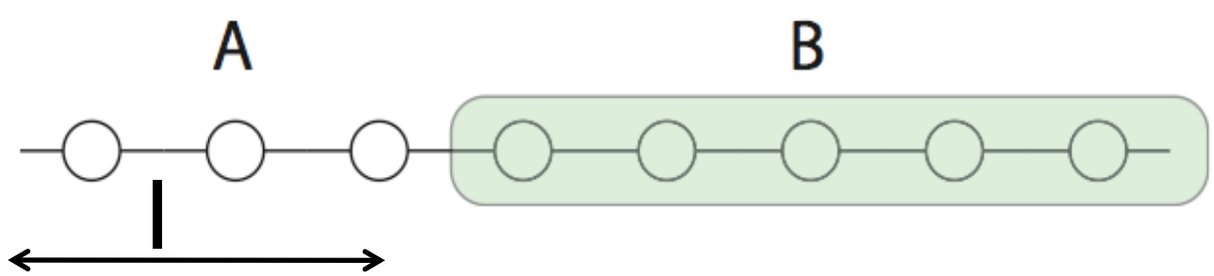
- Entanglement entropy (Jin & Korepin, 2004):

$$\mathcal{S}(\ell, L) = \frac{c}{3} \log_2 \ell + s_1, \qquad s_1 \simeq 1.047.$$

- Spin fluctuations:

$$\pi^2 \mathcal{F}(\ell, L) = \ln \ell + f_1 + O(\ell^{-2}), \qquad f_1 = 1 + \gamma + \ln 2.$$

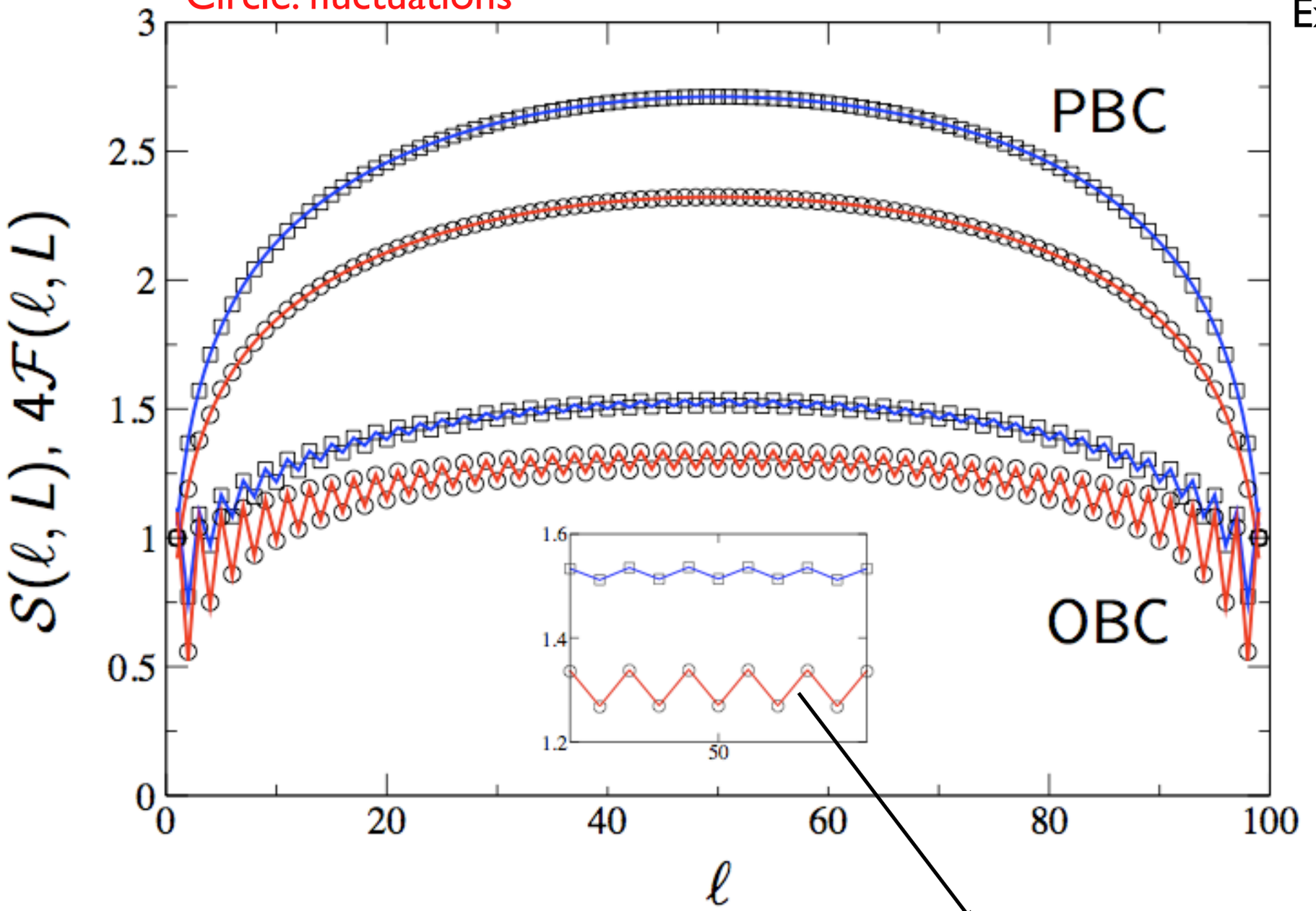
  γ is Euler's constant.



XX Model

Exact diagonalization
L=100
+
Exact Results

Square: entropy
Circle: fluctuations



Exact result

XX Model – Open Boundary Conditions

- Entanglement entropy

$$\mathcal{S}_{\mathrm{OBC}}(\ell, L) = \frac{1}{2} S_{\mathrm{PBC}}(2\ell, L) + a_1 \frac{1}{(2\ell)} - a_2 \frac{(-1)^\ell}{(2\ell)}.$$

(Laflorencie, Sorensen, Chang & Affleck, PRL 2006)

- Spin fluctuations:

$$\mathcal{F}_{\mathrm{OBC}}(\ell, L) = \frac{1}{2} \mathcal{F}_{\mathrm{PBC}}(2\ell, L) + \frac{1}{2\pi^2} \frac{1}{(2\ell)} - \left[\frac{\ln(2\ell) + \gamma + \ln 2}{\pi^2} \right] \frac{(-1)^\ell}{(2\ell)}$$

Friedel like contribution

Important NOTE:

Bose Gas: $S = \ln F^{1/2}$ Klich, Refael & Silva PRA 2006

Linear Relation between S and F works if total number of particles (or S_z) is conserved

It does not work if A and B don't exchange particles:

S. Furukawa and Y.B. Kim arXiv:1009.3016 (PRB)

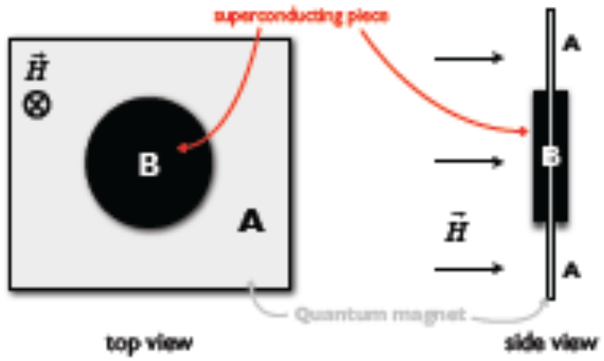
Relation perhaps does not hold in more complex geometries

J. Cardy, arXiv:1012.5116 (PRL 2011)

Example: more than one cut

At least this is encouraging

Entanglement Entropies
Fluctuation Theorem(s)



Bipartite Fluctuations as a Probe of Many-Body Entanglement
H. Francis Song, Stephan Rachel, Christian Flindt, Israel Klich, Nicolas Laflorencie, KLH,
arXiv:1109.1001, 30 pages+25 pages Supplementary Material
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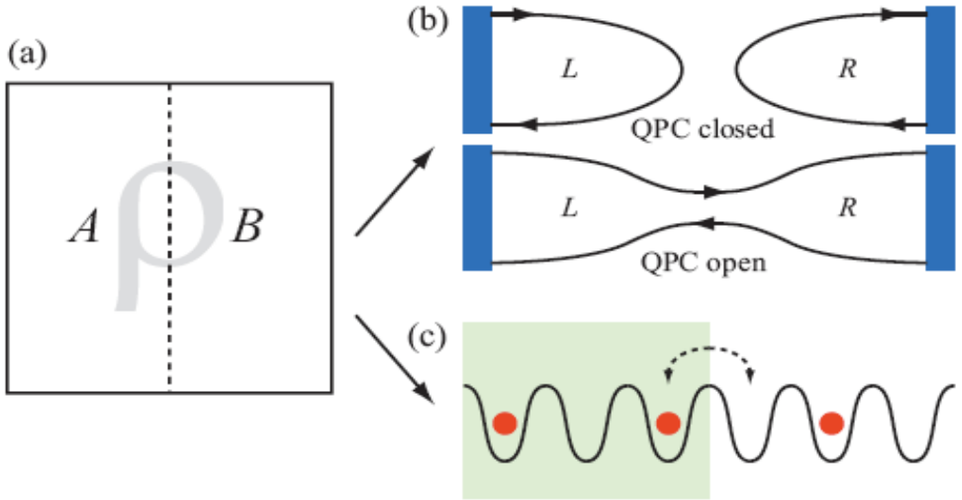
Application of mathematical combinatorics: XVIII



Can We Measure Entanglement Entropy? (Cont.)

Our claim

"Easier" way, for systems of non-interacting fermions: measure particle-number fluctuations

$$C_n = (-i\partial_\lambda)^n \ln \chi(\lambda)|_{\lambda=0}, \qquad \chi(\lambda) = \langle e^{i\lambda \hat{N}_A} \rangle.$$

$\chi(\lambda)$ is the cumulant generating, or characteristic, function.

Applications: Free fermions in any dimension, spin-1/2 XX chain $\equiv$ hard-core bosons in an optical lattice, integer quantum Hall effect, topological insulators, ...

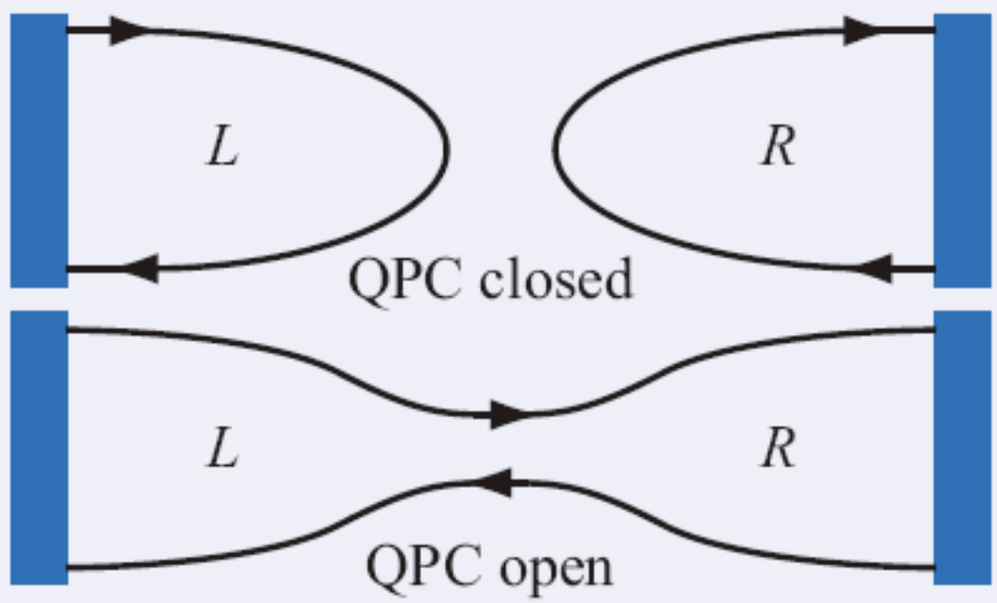
Further claim

Number fluctuations are interesting to study in their own right, e.g., Full Counting Statistics (FCS) in Quantum Point Contacts (QPC) and applications to detecting quantum phase transitions.

Number Fluctuations

All Cumulants for the QPC accessible…

There is already an incredibly active field in this direction: FCS (Levitov & Lesovik '93) in mesoscopic transport.



In a QPC, $\chi(\lambda, t) = \sum_n P_n(t) e^{i\lambda n}$ where $P_n(t)$ is the probability that n charges were transferred during the span $[0, t]$, say from L ("source") to R ("drain").

Entanglement Entropy and Fluctuations

Entanglement entropy of free fermions

$$\mathcal{S} = \lim_{K\to\infty} \sum_{n=1}^{K+1} \alpha_n(K) C_n,$$

where

$$\alpha_n(K) = \begin{cases} 2\sum_{k=n-1}^{K} \frac{S_1(k,n-1)}{k!k} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

Here $S_1(n,m)$ are **unsigned Stirling numbers of the first kind**. Practically, K is the number of available cumulants and should be taken to be even.

Order of limit is important

$$\lim_{K\to\infty} \left[\sum_{n=1}^{K+1} \alpha_n(K) C_n \right] \neq \sum_{n=1}^{\infty} \left[\lim_{K\to\infty} \alpha_n(K) \right] C_n.$$

The RHS is usually not convergent.

Klich-Levitov Series

Properties of the Series

Only even cumulants contribute

The formula works for pure states, and only even cumulants are symmetric between A and B.

What happens when we bring the limit inside?

For even n,

$$\alpha_n(\infty) = 2\zeta(n) = \frac{(2\pi)^n |B_n|}{n!},$$

where $\zeta(n)$ is the Riemann zeta function and B_n are Bernoulli numbers. So

$$\alpha_n(K) C_n \sim 2C_n \text{ for large } n,$$

but

$$\lim_{n\to\infty} |C_n| = \infty$$

Here "correct" means "convergent, plus the approximation is an increasingly sharper **lower bound** to the exact entanglement entropy."

Basic idea

$$\hat{H} = \sum_{i,j} \hat{a}_i^\dagger H_{ij} \hat{a}_j,$$

$$M_{ij} = \langle a_j^\dagger a_i \rangle$$

Expand logarithms to get

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{\mathrm{Tr}[M(1-M)^n + M^n(1-M)]}{n},$$

and notice that the factorial cumulants $F_n = \partial_\lambda^n \ln \chi(-i \ln \lambda)|_{\lambda=1}$ are given by

$$F_k = (-1)^{k-1}(k-1)![\mathrm{Tr}(M^k) - q], \qquad k \geq 1$$

to get

$$\mathcal{S} = \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n-1}}{n} \left[\frac{F_n}{(n-1)!} + \frac{F_{n+1}}{n!} \right] + \sum_{k=0}^{n} \binom{n}{k} \frac{F_{k+1}}{k!n} \right\}$$

Relation to ordinary cumulants

$$F_n = \sum_{k=1}^{n} (-1)^{n-k} S_1(n,k) C_k, \qquad n \geq 1$$

Other uses

Factorial cumulants were recently shown to reveal the effect of interactions on the full counting statistics in electron transport through a quantum dot.

$$\mathcal{S} = \lim_{K\to\infty} \sum_{n=1}^{K+1} \alpha_n(K) C_n$$

with **cutoff-dependent** coefficients

$$\alpha_n(K) = \begin{cases} 2\sum_{k=n-1}^{K} \frac{S_1(k,n-1)}{k!k} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

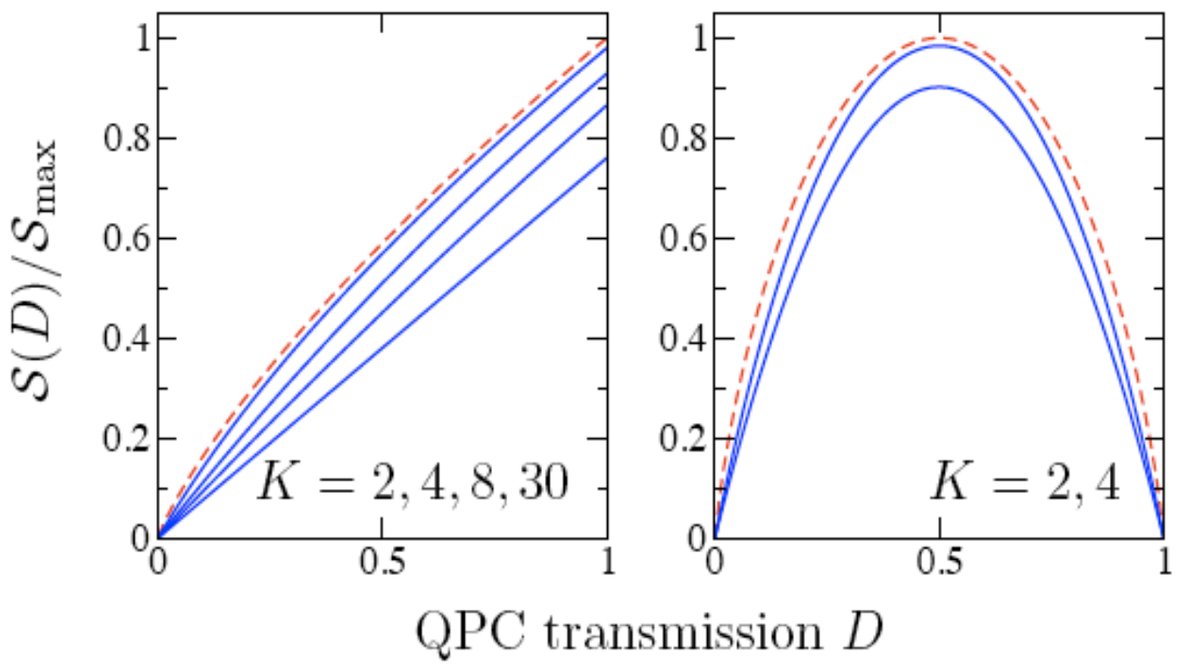
Dependence on cutoff K

The cutoff K corresponds to the partial sums of the matrix series with terms

$$T_n = \frac{M(1-M)^n + M^n(1-M)}{n}.$$

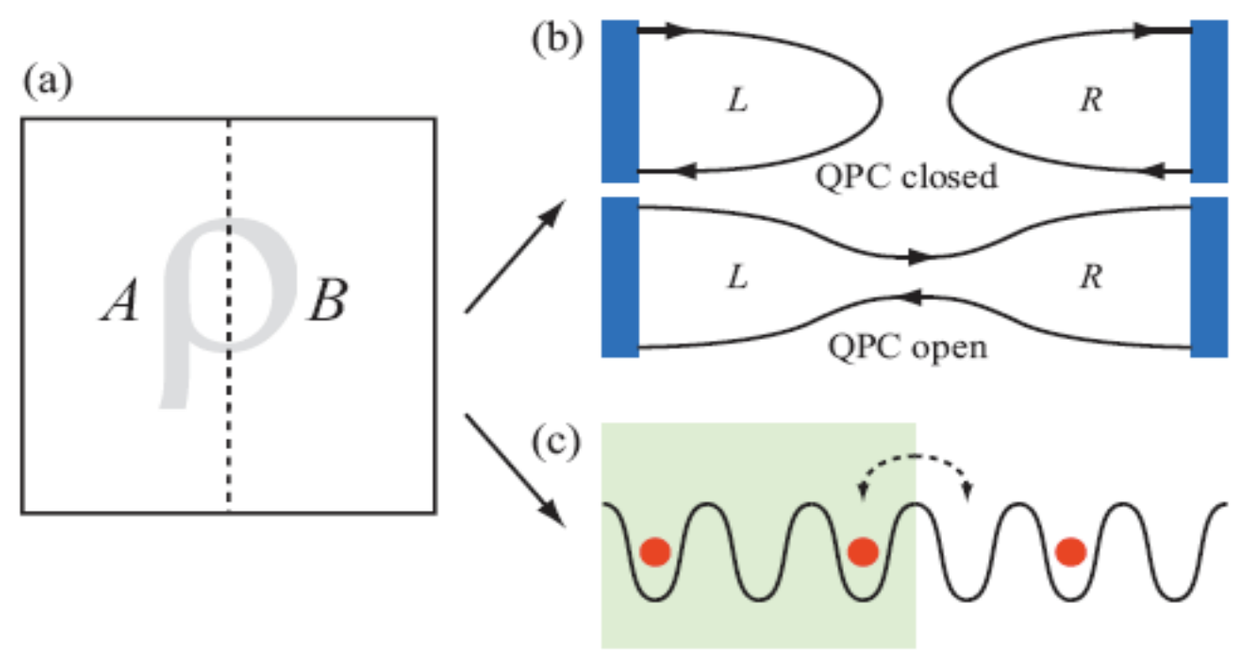
- Since the spectral radius (maximum absolute value of the eigenvalue) of M, $\rho(M)$, is less than 1, we know the partial sums of the matrix series converge.
- Each term is positive, so increasing K always produces a better **lower bound** to $\mathcal{S}$.

Check of Our New Formula: QPC...



If a DC voltage bias V is applied, the non-equilibrium generating function at zero temperature (the quantum shot noise regime) is found to be the binomial distribution $\log\chi(\lambda) = (eVt/h)\log(1 - D + De^{i\lambda})$, for which $\mathcal{S} = -(eVt/h)[D\log D + (1-D)\log(1-D)]$ [9]. This is also the electron-hole entanglement predicted for a biased tunnel junction [20], and indeed in both cases the Rényi entropies for small D are $\mathcal{S}_n \simeq [n/(n-1)]N$, where $N = (eVt/h)D$ is the number of transferred particles [21].

Higher-order cumulants become accessible in mesoscopic systems (B. Reulet-D. Prober; C. Glattli et al.)



Simple QPC case, 1e: **"partitioning"**

Now, the distribution $P(m)$ for the number m of transferred excitations between spin A and spin B is

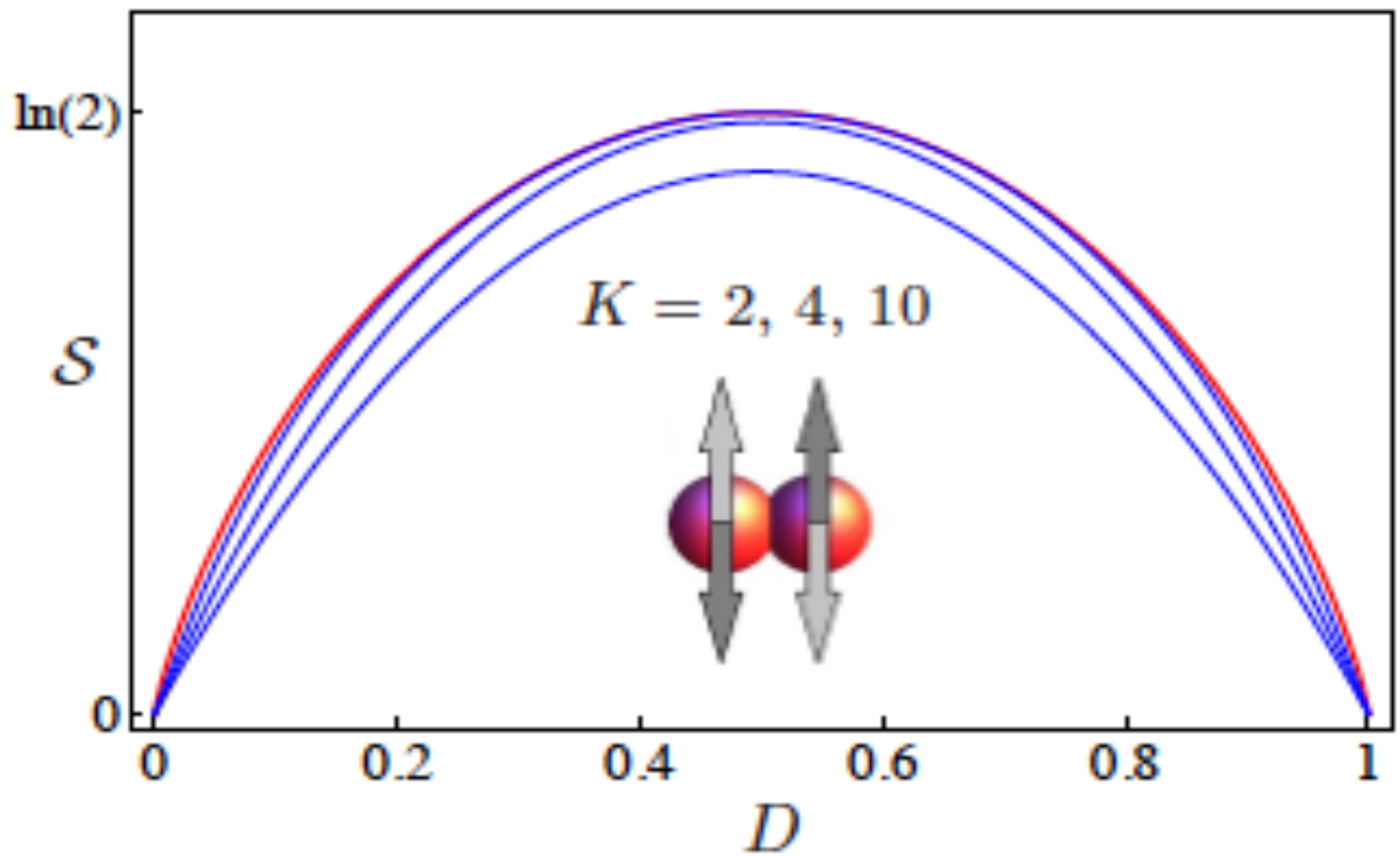
$$P(m) = \begin{cases} 1 - D & \text{for } m = 0, \\ D & \text{for } m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding generating function is

$$\chi(\lambda) = \sum_m P(m) e^{i\lambda m} = 1 + (e^{i\lambda} - 1)D,$$

$$\mathcal{S} = -[D \ln D + (1 - D) \ln(1 - D)]$$

How Many cumulants do we need: K=10, but…


ln(2)
$\mathcal{S}$
0
$K = 2, 4, 10$
0
0.2
0.4
0.6
0.8
1
D

QPC: many free particles

Generating function still accessible (Levitov, Lee & Lesovik, 1996)

$$\ln \chi(\lambda) = \frac{(i\lambda_*)^2}{2\pi^2} G(t)$$

with

$$G(t) = \ln \left\{ \frac{\tau_\beta}{\tau_c} \sinh \frac{t}{\tau_\beta} \right\}$$

D=1
agrees with CFT

determining the time dependence. Here $\tau_\beta = h/(\pi k_B T)$ is the time scale associated with the temperature T and τ_c is an ultraviolet short-time cutoff. Additionally, the counting field λ enters via the relation

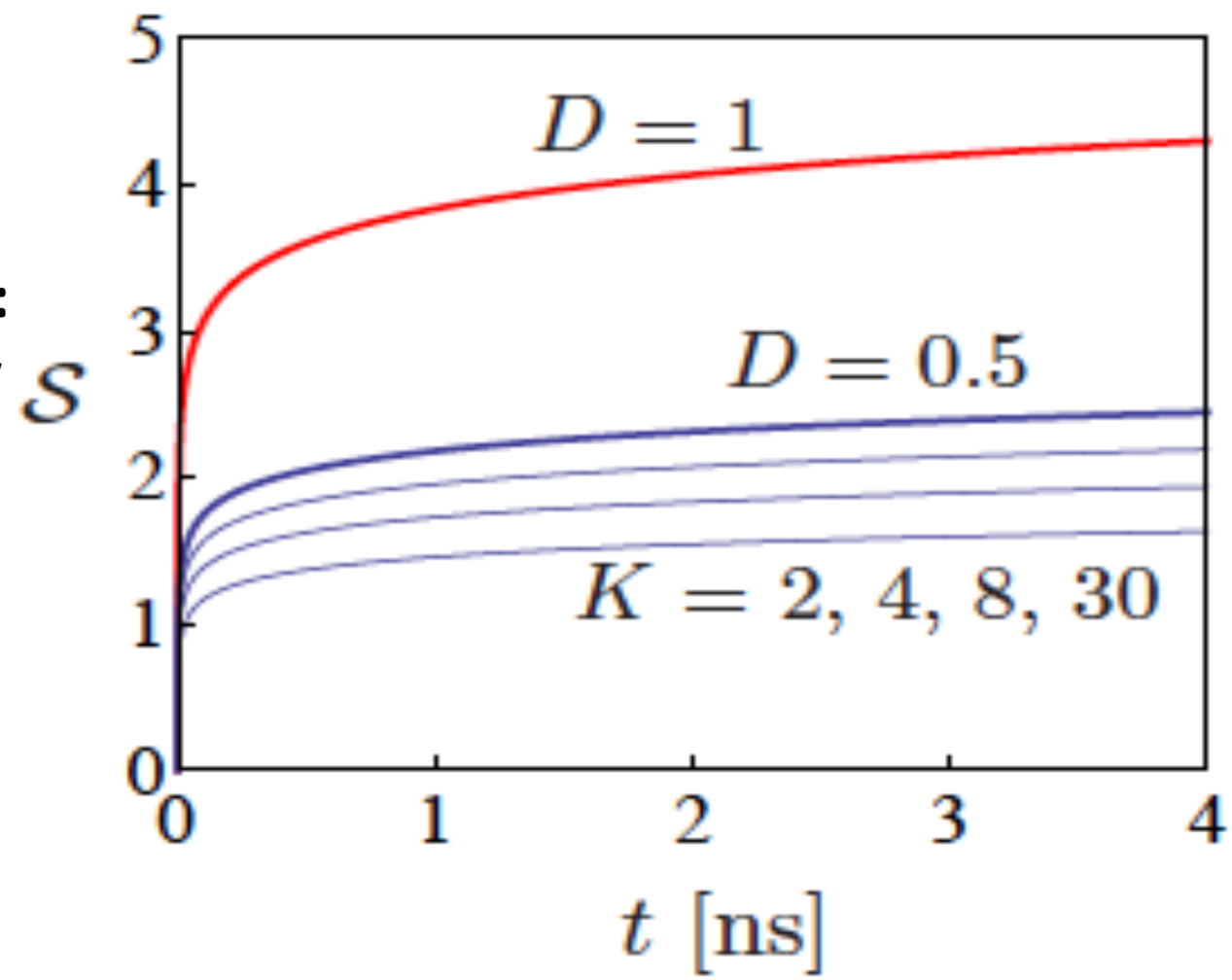
$$\sin \frac{\lambda_*}{2} = \sqrt{D} \sin \frac{\lambda}{2}$$

involving the QPC transmission D.

QPC I: T=0 and Away From Perfect Transparency

Generating Function known: Levitov, Lee and Lesovik (1996)
We are then able to compute all the cumulants and measure S…

GigaHertz limit: Experimentally feasible… (?)



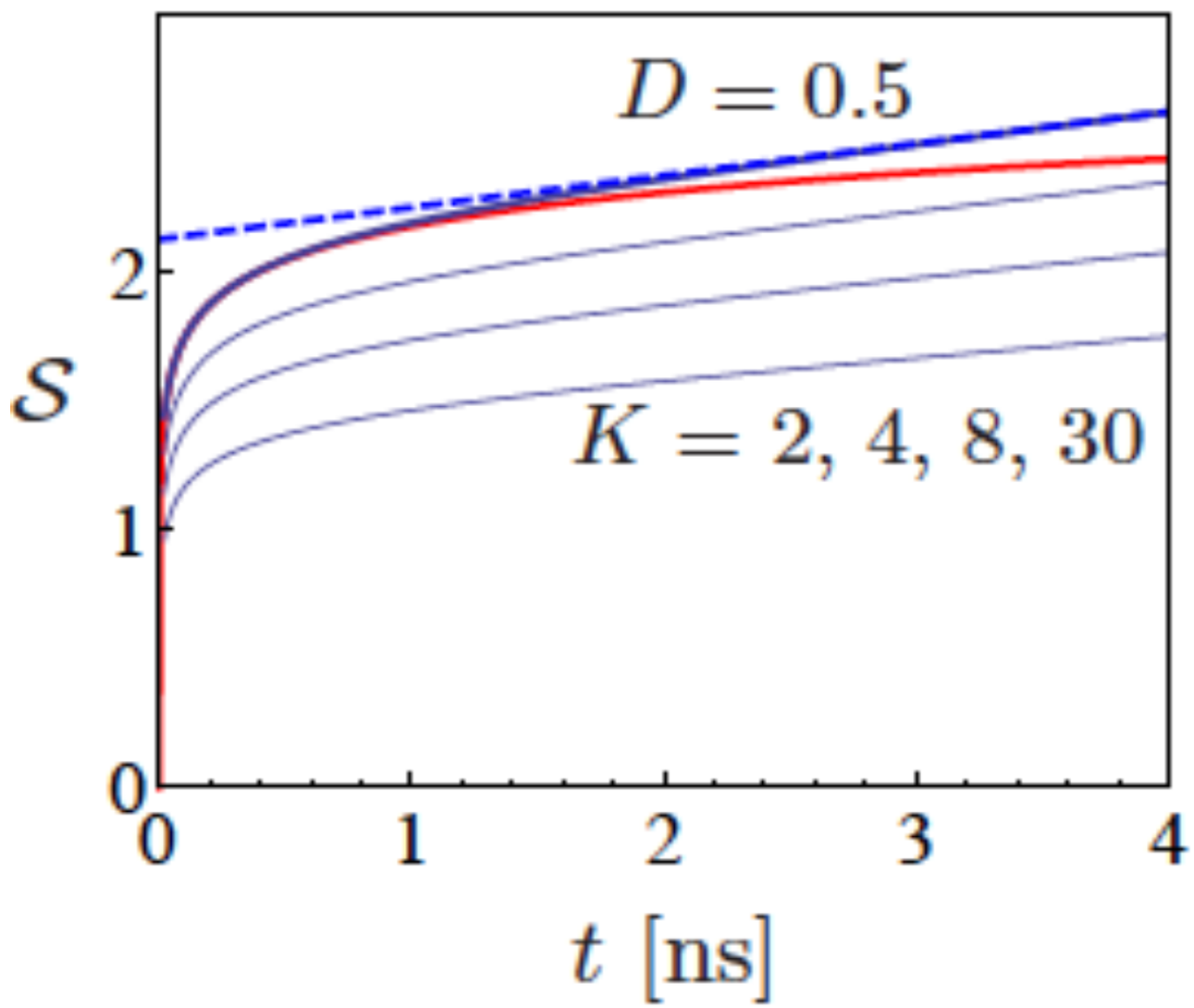
Gaussian equilibrium Quantum noise

D=1: Results from CFT (Klich-Levitov): entropy grows logarithmically with time
D=0.5: Higher cumulants matter, but the entropy maintains its logarithmic growth
With K=2: $S=5/(2\pi)^2 * \ln(t/\tau)$**: Lower bound on the full entanglement entropy**

QPC II: Effect of Thermal Fluctuations

Temperature produces classical thermal fluctuations which eventually cause the entropy to become linear in time (extensive function)

In principle, zero-temperature behavior still accessible



Blue: high temperature Limit

Red: zero temperature Limit

T=10mK

τ_β=1.5ns

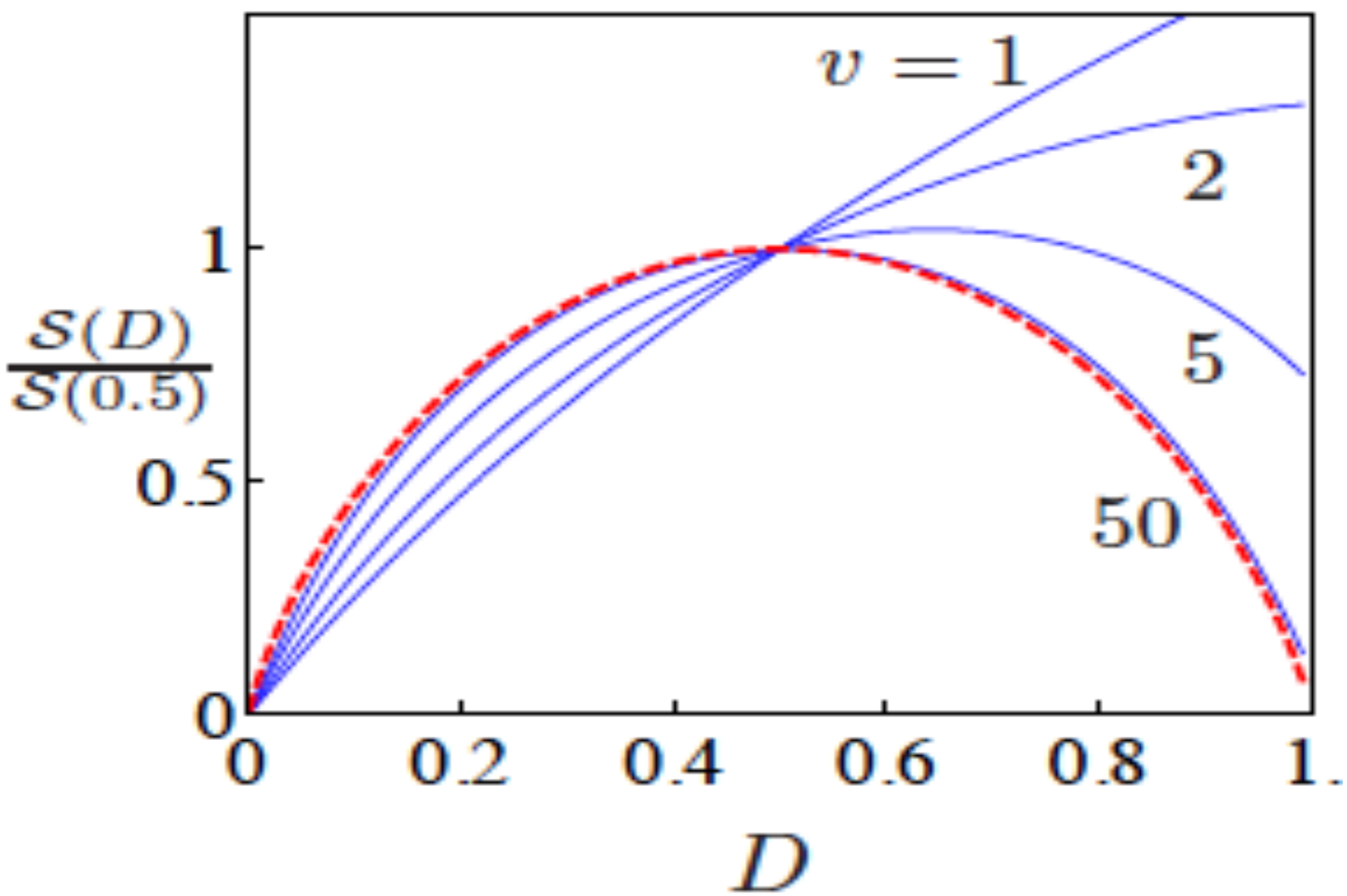
QPC III: Voltage Effects (X-function still accessible)

Large Voltage: shot noise, **classical** binomial charge transfer (Beenakker)

$$\mathcal{S} = -\frac{eVt}{h}\left[D\ln D + (1-D)\ln(1-D)\right]$$

(see also Y. Nazarov, 2011)

$v = eV/2k_BT$



K=10

Time larger than $\tau_\beta \sim 1/k_B T$

Concluding Remarks:

Fluctuation number: useful bi-partite object (QMC)

Brings New Information on short-range fluctuations and "potentially related" to entanglement-scaling

Useful Tool to detect QPTs: see our PRL 2012

Entanglement Entropy and Full Counting Statistics: Free fermions

Measure of EE and Renyi entropies in Interacting systems: quantum switch
D. Abanin & E. Demler, PRL 2012
See also J. Cardy, PRL