Information Scrambling at Late Time

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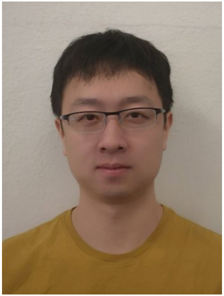
References & Collaborators

1. Y. Gu, A. Kitaev, and PZ, “A two-way approach to out-of-time-order correlators”, J. High Energ. Phys. **2021**, 94 (2021).
2. PZ and Y. Gu, “Operator Size Distribution in Large-N QM”, to appear.



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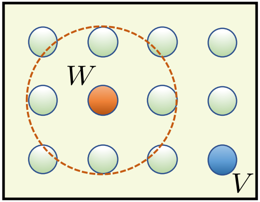
Outline

Introduction: SYK and Scramblons

From the Heisenberg Picture:

$$W(t) = e^{iHt}W(0)e^{-iHt}$$
$$= W(0) + it[H, W(0)] - \frac{t^2}{2}[H,[H,W(0)]]...$$

Operators become more and more complicated!



How $W(t)$ overlaps with V defines the out-of-time-order correlators

$$\langle |[W(t),V]|^2\rangle_\beta = \langle V^\dagger W^\dagger(t)W(t)V\rangle_\beta + \langle W^\dagger(t)V^\dagger V W(t)\rangle_\beta - \underbrace{\langle W^\dagger(t)V^\dagger W(t)V\rangle_\beta}_{\text{OTOC}} - \langle V^\dagger W^\dagger(t) V W(t)\rangle_\beta .$$

Information Scrambling $\leftrightarrow$ Decay of OTOC

Roberts, Stanford, and Streicher, 2018.

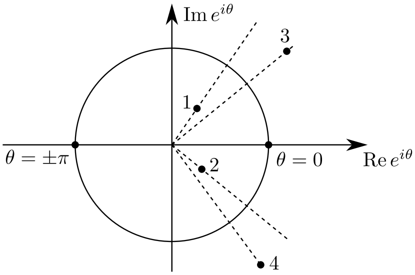
OTOC in Large-N Quantum Mechanics

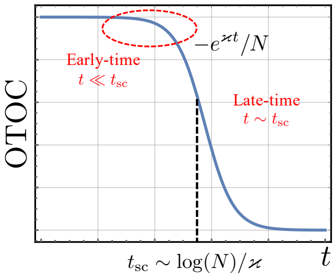
More generally, OTOC reads $(\theta_i = \tau_i + it_i)$

$$\begin{aligned}\text{OTOC}(\{\theta_i\}) &= \langle \mathcal{T}_\tau X_1(\theta_1)X_2(\theta_2)X_3(\theta_3)X_4(\theta_4)\rangle \\ &= (-1)^{X_2X_3}\,\langle X_1(\theta_1)X_3(\theta_3)X_2(\theta_2)X_4(\theta_4)\rangle\end{aligned}$$

with

$$\tau_1 \geq \tau_3 \geq \tau_2 \geq \tau_4, \qquad t_1 \approx t_2 \approx \frac{t}{2} \gg t_3 \approx t_4 \approx -\frac{t}{2}$$





Shenker and Stanford, 2014.

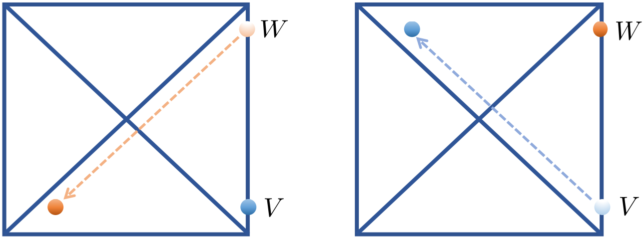
Holography: OTOC as a Bulk Scattering

Consider $\tau_i = 0$, $X_1^\dagger = X_2 = W$ and $X_3^\dagger = X_4 = V$,

$$\text{OTOC}(t) = \langle W^\dagger(t/2)V^\dagger(-t/2)W(t/2)V(-t/2)\rangle = \langle \text{in}|\text{out}\rangle,$$

with

$$|\text{in}\rangle = V(-t/2)W(t/2)|\beta\rangle \qquad |\text{out}\rangle = W(t/2)V(-t/2)|\beta\rangle$$



OTOC $= \mathcal{S}$ matrix of the bulk Scattering:

$$\text{OTOC} = \left\langle \mathcal{S}^\dagger \right\rangle = \left\langle e^{-i\delta} \right\rangle = \left\langle e^{-i\# G_N p_W p_V e^{2\pi t/\beta}} \right\rangle$$

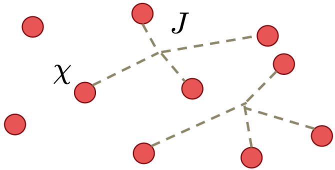
The Sachdev-Ye-Kitaev Model

The SYK$_q$ model ($q = 2n$)

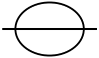
$$H_{\mathrm{SYK}} = \sum_{1 \le i_1 < i_2 < \ldots < i_q \le N} J_{i_1 i_2 \ldots i_q} \chi_{i_1} \chi_{i_2} \cdots \chi_{i_q},$$

The random variables $J_{i_1 i_2 \ldots i_q}$ satisfy

$$\overline{J_{i_1 i_2 \ldots i_q}} = 0, \qquad \overline{J^2_{i_1 i_2 \ldots i_q}} = \frac{(q-1)! J^2}{N^{q-1}} = \frac{(q-1)! 2^{q-1} \mathcal{J}^2}{q N^{q-1}}.$$



Solvable under large-N expansion:

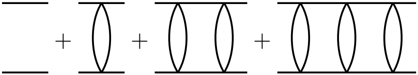
$\Sigma(\tau) =$  $= J^2 G^{q-1}(\tau).$

Free energy is determined $F = S[G, \Sigma]$.

Sachdev and Ye, 1993; Kitaev, 2015.

Early-time OTOC in SYK-like Models

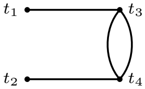
Early-time OTOC_c is a sum of ladders $(\log N \gg t\varkappa \gg 1)$:

$\text{OTOC}_c =$  $+ \ldots,$

This gives a self-consistent equation:

$$\int dt_5\, dt_6\, K^{\text{R}}(t_1, t_2; t_5, t_6)\text{OTOC}_c(t_5, t_6, t_3, t_4) \approx \text{OTOC}_c(t_1, t_2, t_3, t_4).$$

with SYK-like retarded kernel

$K^{\text{R}}(t_1, t_2; t_3, t_4) =$ 

.

Or in short

$$K^{\text{R}} \circ \text{OTOC}_c = \text{OTOC}_c.$$

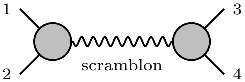
Maldacena and Stanford, 2016; Gu and Kitaev, 2019.

Scramblons

K^{R} is time-translational invariant. The eigenfunctions can be labeled by a center-of-mass 'frequency' α:

$$F_\alpha(t_1,t_2) = e^{-\alpha\frac{t_1+t_2}{2}}\Upsilon^{\mathrm{R}}_\alpha(t_{12}), \qquad K^{\mathrm{R}}\circ F_\alpha = k_{\mathrm{R}}(\alpha)F_\alpha.$$

The quantum Lyapunov exponent satisfies $k_{\mathrm{R}}(-\varkappa)=1$:

$\mathrm{OTOC}_c =$ 

$= \left(-\frac{e^{\varkappa t}}{Nc}\right)\Upsilon^{\mathrm{R}}(t_{12})\Upsilon^{\mathrm{A}}(t_{34}).$

Adding back disconnected part ($\log N \gg t\varkappa \gg 1$):

$\mathrm{OTOC} = \;)\quad(\; +$  $+\,O(N^{-2})$

$$= G(t_{12})G(t_{34}) + \left(-\frac{e^{\varkappa t}}{Nc}\right)\Upsilon^{\mathrm{R}}(t_{12})\Upsilon^{\mathrm{A}}(t_{34}) + O(N^{-2}).$$

Maldacena and Stanford, 2016; Gu and Kitaev, 2019.

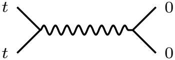
Information Scrambling at Late Time

Large-N Expansion Revisit

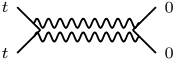
Scramblons are collective modes with effective actions

$$S \sim N(\phi\, G_0^{-1}\phi + \lambda_3\phi^3 + \lambda_4\phi^4 + \ldots) + \sum_i \phi\chi_i\chi_i + \ldots$$

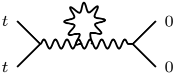
Each scramblon propagator $\sim e^{\varkappa t}/N$, each scramblon vertex $\sim N$.

① 

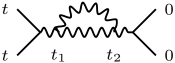
$\sim \quad e^{\varkappa t}N^{-1}$

② 

$\sim \quad (e^{\varkappa t}N^{-1})^2$

③ 

$\sim \quad e^{\varkappa t}N^{-2}$

④ 

$\sim \quad e^{\varkappa t}e^{\varkappa t_{12}}N^{-2}$

1. Early-time regime $e^{\varkappa t} \sim O(1)$:

   Only ① $\sim N^{-1}$ contributes, leads to

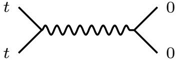
$$\mathrm{OTOC} = \;)\qquad(\; +$$  $$+\,O(N^{-2}).$$

Large-N Expansion Revisit

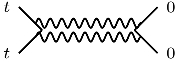
Scramblons are collective modes with effective actions

$$S \sim N(\phi\, G_0^{-1}\phi + \lambda_3\phi^3 + \lambda_4\phi^4 + ...) + \sum_i \phi\chi_i\chi_i + ...$$

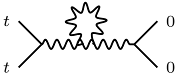
Each scramblon propagator $\sim e^{\varkappa t}/N$, each scramblon vertex $\sim N$.

① 

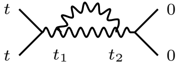
$\sim \quad e^{\varkappa t}N^{-1}$

② 

$\sim \quad (e^{\varkappa t}N^{-1})^2$

③ 

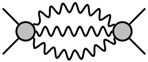
$\sim \quad e^{\varkappa t}N^{-2}$

④ 

$\sim \quad e^{\varkappa t}e^{\varkappa t_{12}}N^{-2}$

2. Late-time regime $e^{\varkappa t}/N \sim O(1)$:

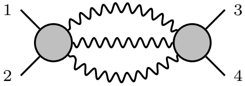
①, ② $\sim N^0$ contribute, ④ contributes if $t_1 \approx t$ and $t_2 \approx 0$.

$$\text{OTOC} = \sum_{\#\text{ of scramblons}}$$ 

Late Time OTOC & Scramblon Diagrams

This gives

$$\text{OTOC}(\{\theta_i\}) =$$

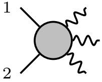


$$= \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} \Upsilon^{\text{R},m}(\theta_{12}) \, \Upsilon^{\text{A},m}(\theta_{34}).$$

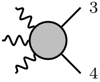
Here the propagator of scramblons

$$-\lambda = -\frac{e^{\varkappa t}}{Nc} = -\frac{e^{i\varkappa(\pi-\theta_1-\theta_2+\theta_3+\theta_4)/2}}{Nc}$$

Scattering amplitudes in the future/past



$$= \Upsilon^{\text{R},m}(\theta_{12}),$$



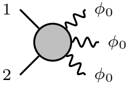
$$= \Upsilon^{\text{A},m}(\theta_{34}).$$

In particular, $\Upsilon^{\text{R/A},0}(\theta) = G(\theta)$ and $\Upsilon^{\text{R/A},1}(\theta) = \Upsilon^{\text{R/A}}(\theta)$.

Coherent State of Scramblons

The trick to determine $\Upsilon^{\mathrm{R},m}(\theta_{12})$ is to condense the scramblons. Then the two-point function of fermions is given by

$$f^{\mathrm{R}}(\phi_0,\theta_{12}) = \langle \chi(\theta_1)\chi(\theta_2)\rangle_{\phi_0} =$$



$$= \sum_m \frac{(\phi_0)^m}{m!}\Upsilon^{\mathrm{R},m}(\theta_{12}).$$

After obtaining $f^{\mathrm{R}}(\phi_0,\theta)$, we can define $h^{\mathrm{R}}(p,\theta)$ through

$$f^{\mathrm{R}}(\phi_0,\theta) = \int_0^\infty dp\ h^{\mathrm{R}}(p,\theta)e^{\phi_0 p}, \qquad \Upsilon^{\mathrm{R},m}(\theta) = \int_0^\infty dp\ h^{\mathrm{R}}(p,\theta)p^m.$$

This gives

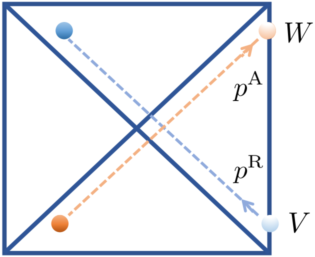
$$\begin{aligned}\mathrm{OTOC}(\{\theta_i\}) &= \sum_{m=0}^{\infty}\frac{(-\lambda)^m}{m!}\,\Upsilon^{\mathrm{R},m}(\theta_{12})\,\Upsilon^{\mathrm{A},m}(\theta_{34})\\ &= \int_0^\infty dp^{\mathrm{A}}\int_0^\infty dp^{\mathrm{R}}\ h^{\mathrm{R}}(p^{\mathrm{A}},\theta_{12})h^{\mathrm{A}}(p^{\mathrm{R}},\theta_{34})\exp(-\lambda p^{\mathrm{R}}p^{\mathrm{A}})\end{aligned}$$

Emergent Bulk Scattering

$$\mathrm{OTOC}(\{\theta_i\}) = \int_0^\infty dp^{\mathrm{R}} \int_0^\infty dp^{\mathrm{A}}\ h^{\mathrm{R}}(p^{\mathrm{A}}, \theta_{12}) h^{\mathrm{A}}(p^{\mathrm{R}}, \theta_{34}) \exp(-\lambda p^{\mathrm{R}} p^{\mathrm{A}})$$

This takes the form of a bulk scattering! No assumption of holography or maximal chaos!

1. $h^{\mathrm{R/A}}(p, \theta_{12}) \quad \rightarrow \quad$ Probability of having a particle with null momentum p.
2. $-\lambda p^{\mathrm{R}} p^{\mathrm{A}} \quad \rightarrow \quad$ the phase shift $-i\delta$. In particular, for zero imaginary time separation: $\lambda = |\lambda| e^{i\varkappa\pi/2}$.
3. maximal chaos $\quad \rightarrow \quad$ elastic scattering!



Shenker and Stanford, 2015; Stanford, Yang, and Yao, 2021.

Extract the Scramblon Data

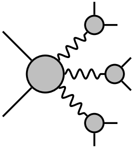
To condense ϕ, we should apply an operator $e^{za_\phi^\dagger}$. Scramblon can be excited by a pair of fermion operators $\chi\chi$:

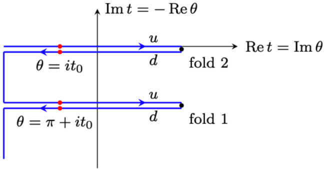
$$a_\phi^\dagger \leftrightarrow N\chi\chi, \qquad \text{or} \qquad e^{za_\phi^\dagger} \leftrightarrow e^{zN\chi\chi}$$

We choose

$$I_{\text{ pert}} = \underbrace{\frac{c}{2\cos(\varkappa\pi/2)\Upsilon^{\text{A}}(\pi)}}_{u} z e^{\varkappa t_0} \sum_{j=1}^{N} \Big(\chi_j^u(\pi + it_0) - \chi_j^d(\pi + it_0)\Big)\Big(\chi_j^d(it_0) - \chi_j^u(it_0)\Big).$$

Traditional $1/N$ ($t_0 \to -\infty$ and $t_1, t_2 \sim O(1)$):

$G(t_1, t_2) =$  $= \sum_m \frac{(-ze^{\varkappa t})^m}{m!} \Upsilon^{\text{R},m}(\theta_{12})$



We find $\phi_0 = -ze^{\varkappa t}$, $t = \frac{t_1+t_2}{2}$.

Example: Large-q SYK Model

$G(t_1,t_2)$ can also be solved using the Schwinger-Dyson equation:

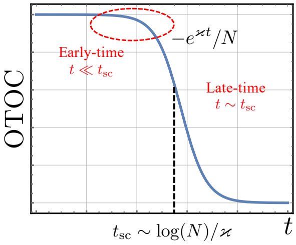
$$G(t_1,t_2)=\int_{-\infty}^{+\infty} dtdt'\ G^{\mathrm{R}}(t_1,t)\left(J^2G(t,t')^{q-1}-zue^{\varkappa t_0}\delta(t-t_0)\delta(t'-t_0)\right)G^{\mathrm{A}}(t',t_2)$$

In the large-q limit, this gives

$$f^{\mathrm{R}}(z_{\mathrm{R}};t_1,t_2)=\frac{1}{2}\left(\frac{\cos\frac{\pi v}{2}}{\cosh\frac{vt_{12}}{2}-\phi_0}\right)^{2\Delta},\qquad \phi_0=-ze^{\frac{v(t_1+t_2)}{2}},$$

which leads to

$$\mathrm{OTOC}(\{t_i\})=\frac{(\cos\frac{\pi v}{2})^{4\Delta}}{4\lambda^{2\Delta}}U\left(2\Delta,1,\frac{\cosh\frac{vt_{12}}{2}\cosh\frac{vt_{34}}{2}}{\lambda}\right).$$



Short-time $\lambda \ll 1$, reduce to previous results.

Long-time $\lambda \gg 1$, decays as $te^{-2\Delta\varkappa t}$.

Eberlein, Kasper, Sachdev, and Steinberg, 2017.

Late-time Operator Size Distribution

Operator Size Distribution at Late Time

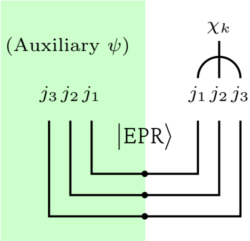
The Heisenberg evolution makes operators complicated:

$$\chi_1(t) = \sum_n \sum_{\{i_n\}} 2^{n/2} c_{i_1,i_2\ldots i_n}(t) \underbrace{\chi_{i_1}\chi_{i_2}\ldots\chi_{i_n}}_{\text{length}=n}$$

In the late-time limit, we introduce $n = sN$, with $s \in [0,1]$.

$$\mathcal{P}(s,t) \equiv 2N \sum_{\{i_n\}} |c_{i_1,i_2\ldots i_n}(t)|^2.$$

$\mathcal{P}(s,t)$ can be computed using the trick:



Prepare an EPR state with

$$c_j = \frac{1}{\sqrt{2}}(\chi_j + i\psi_j), \quad \underbrace{c_j^\dagger c_j}_{n_j} |\text{EPR}\rangle = 0.$$

This leads to

$$\mathcal{P}(s,t) = 2\langle \text{EPR}|\chi_1(t)\delta(s - \sum_j n_j/N)\chi_1(t)|\text{EPR}\rangle$$

Qi and Streicher, 2018.

Operator Size Distribution at Late Time

Generating function method

$$\mathcal{S}(\nu,t)=\int ds\ \mathcal{P}(s,t)e^{-s\nu}=2\langle\chi_1(\theta_1)\chi_1(\theta_2)e^{-\nu\left(\frac{1}{2}-\frac{1}{N}\sum_j\chi_j(\theta_3)\chi_j(\theta_4)\right)}\rangle$$

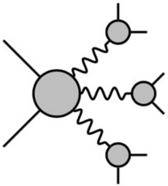
with $\theta_1=it+\epsilon$, $\theta_2=it-\epsilon$, $\theta_3=0$, and $\theta_4=-2\epsilon$.

A comparison:

$$\langle\chi_i\chi_i e^{-z\sum_j\chi_j\chi_j}\rangle$$

Each insertion $\sim ze^{\varkappa t}$

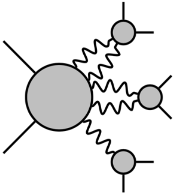
Vanishes at $\varkappa t\sim-\log z$, early-time



$$\langle\chi_i\chi_i e^{-z\sum_j\chi_j\chi_j/N}\rangle$$

Each insertion $\sim ze^{\varkappa t}/N$

Vanishes at $\varkappa t\sim\log N$, late-time



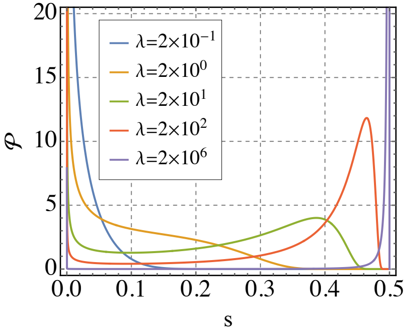
Operator Size Distribution at Late Time

Summing up scramblon diagrams gives

$$\mathcal{S}(\nu,t) = 2\sum_m \frac{\nu^m}{m!}\sum_{\{n_i>0\}}\frac{1}{n_1!...n_m!}\left[-\frac{e^{\varkappa t}}{Nc}\right]^{\sum_i n_i}\Upsilon^{\mathrm{R},\sum_i n_i}(0)\Upsilon^{\mathrm{A},n_1}(0)...\Upsilon^{\mathrm{A},n_m}(0)$$

$$= 2\int_0^\infty dp_{\mathrm{A}}\ h^{\mathrm{R}}(p_{\mathrm{A}},0)e^{-\nu\left[\frac{1}{2}-f^{\mathrm{A}}\left(\frac{e^{\varkappa t}p_{\mathrm{A}}}{cN},0\right)\right]}$$

We identify operator size $s \leftrightarrow \frac{1}{2} - f^{\mathrm{A}}\left(\frac{e^{\varkappa t}p_{\mathrm{A}}}{cN},0\right)$!



1. Short-time
$f^{\mathrm{A}} \to \frac{1}{2} - \#\frac{e^{\varkappa t}p_{\mathrm{A}}}{N}, \quad \mathcal{S} \sim f^{\mathrm{R}}(\#\frac{e^{\varkappa t}}{N},0)$
match Qi-Streicher result.

Size $\sim$ Momentum

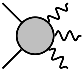
2. Long-time
$f^{\mathrm{A}} \to 0, \quad \mathcal{P} \sim \delta(s-1/2)$
maximally scrambled operator!

Size $\neq$ Momentum

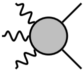
Summary

Scramblon diagrams for late-time information scrambling

 $= -\dfrac{e^{\varkappa t}}{Nc}$,

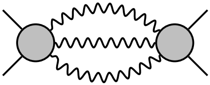


$= \Upsilon^{\mathrm{R},m}(\theta_{12})$,

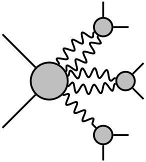


$= \Upsilon^{\mathrm{A},m}(\theta_{34})$.

Out-of-time-order correlator



Operator size distribution



Results are closely related to gravity counterparts.

Y. Gu, A. Kitaev, and PZ, "Wormhole Teleportation in Large-N QM", to appear.