Non-quasiparticle approaches to transport

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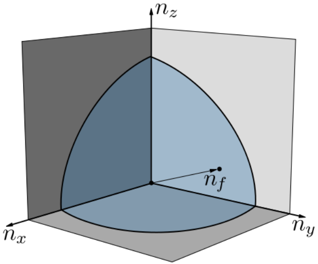
Goal of the lectures

- The goal of these lectures is to give a short introduction to non-quasiparticle approaches to transport at strong coupling, e.g. hydrodynamics, memory matrices and AdS/CFT.
- After setting up the stage, I will mostly focus on momentum relaxation in metallic phases.
- At the worksop, I will talk about spontaneous symmetry breaking.

General references

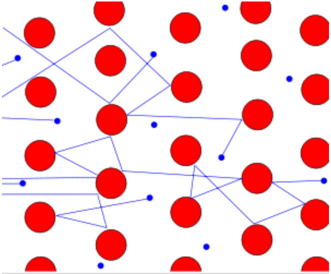
- *Lectures on hydrodynamics*, Pavel Kovtun, [arXiv:1205.5040].
- *Holographic quantum matter*, Sean Hartnoll, Andrew Lucas and Subir Sachdev, [arXiv:1612.07324].
- *Hydrodynamic fluctuations, broken symmetry and correlation functions*, D. Forster, 1975.

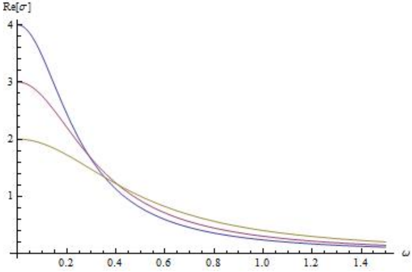
Transport with long-lived quasiparticles



- Transport in a weakly-coupled metallic phase is accounted for by tracking the dynamics of the weakly-interacting quasiparticles.
- Infinite number of quasi-conserved quantities $\tau_{qp} \gg \hbar/(k_B T)$ (or $\tau_{el} \ll \tau_{inel}$).
- Kinetic Boltzmann equation: captures the dynamics of $n_{\delta k}$, the qp density at wavector $\delta k = k - k_F$. Difficulty: solving the collision integral but this is a technical obstacle, not a conceptual one.

Transport with long-lived quasiparticles





For transport, this typically means that the ac conductivity

$$\sigma(\omega, k=0) \sim \frac{\omega_p^2}{\Gamma - i\omega}, \quad \Gamma = \frac{1}{\tau_{qp}}$$

There is a sharp Drude-like peak at $\omega = 0$ and

$$\sigma_{dc} = \lim_{\omega \to 0} \sigma(\omega) = \frac{ne^2 \tau_{qp}}{m} \gg \frac{1}{T}$$

This can be taken as an operational definition of a good metal.

Transport without long-lived quasiparticles

- What about cases without long-lived quasiparticles $\tau_{qp} \sim 1/T$?

- Specifically, I will focus here on cases with an emerging long lived collective mode: momentum.

- Hydrodynamics: relaxation towards equilibrium $\tau \gg \tau_{th} \sim 1/T$. Expansion in small gradients which encapsulates the assumption that $\tau/\tau_{th} \gg 1$ or equivalently $\xi/\ell_{mfp} \gg 1$.

- The memory matrix formalism does not assume small gradients: 'disorder' can vary importantly on microscopic scales. However it is only practically useful if there is only a small number of long-lived operators.

- AdS/CFT gives results consistent with both previous approaches, and allows to describe the crossover from weak to strong breaking.

1. Hydrodynamics

2. AdS/CFT

Relativistic hydrodynamics

- The starting point is conservation equations for the stress-energy tensor and current, in the presence of an applied electric field $E_i \sim F_{0i}$:

$$\nabla_\mu T^{\mu\nu} = j_\mu F^{\mu\nu} \,, \qquad \nabla_\mu j^\mu = 0$$

- We give a constitutive relation to currents order by order in gradients

$$j^\mu = \rho u^\mu - T\sigma_o \left(\nabla^\mu (\mu/T) - E^i \delta^\mu_i\right) + O(\nabla^2) \,,$$

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} - 2\eta\sigma^{\mu\nu} + O(\nabla^2)$$

$$\sigma_{\mu\nu} = P^\alpha_\mu P^\beta_\nu \nabla_{(\alpha} u_{\beta)} - \frac{1}{d} g_{\mu\nu} \nabla \cdot u$$

Two transport coefficients at $\sim O(\nabla)$: η, σ_o.

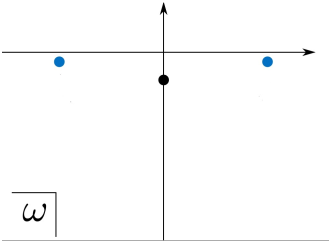
- Finally, solve for linearized fluctuations around equilibrium

$$\mu(t,x) = \bar{\mu} + \delta\mu(t,x) \,, \quad T(t,x) = \bar{T} + \delta T(t,x) \,, \ldots$$

in terms of E_i, using the relations between vevs and sources

$$\pi^i = \chi_{PP} v^i = (\epsilon + p) v^i \,, \quad \begin{pmatrix} \delta\rho \\ \delta s \end{pmatrix} = \begin{pmatrix} \chi_{\rho\rho} & \chi_{\rho s} \\ \chi_{\rho s} & \chi_{ss} \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta T \end{pmatrix}$$

Spectrum of modes



- There are three longitudinal modes: two acoustic and a diffusive mode

$$\omega_{\pm} = \pm c_s k - i\gamma_s(\eta, \sigma_o)k^2 \,, \qquad \omega_{inc} = -iD_{inc}(\sigma_o)k^2$$

- The sound modes are carried by momentum and pressure fluctuations

$$G^R_{\pi\pi},\, G^R_{\delta p \delta p} \sim \frac{1}{\omega^2 - c_s^2 k^2 - 2i\gamma_s k^2}$$

- The diffusive mode is carried by a combination of charge and entropy

$$G^R_{\delta\rho_{inc}\delta\rho_{inc}} \sim \frac{1}{\omega + iD_{inc}k^2} \,, \quad \delta\rho_{inc} = s\delta\rho - \rho\delta s$$

Thermoelectric conductivities

- Generalized Ohm's law

$$\begin{pmatrix} j \\ j_q \end{pmatrix} = \begin{pmatrix} \sigma & T\alpha \\ T\alpha & T\bar{\kappa} \end{pmatrix} \begin{pmatrix} E \\ -\partial\delta T/T \end{pmatrix}$$

$$\sigma(\omega) = \sigma_o + \frac{\rho^2}{(\epsilon+p)}\left(\frac{i}{\omega} + \pi\delta(\omega)\right)$$

$$\alpha(\omega) = -\frac{\mu}{T}\sigma_o + \frac{\rho s}{(\epsilon+p)}\left(\frac{i}{\omega} + \pi\delta(\omega)\right)$$

$$\bar{\kappa}(\omega) = \frac{\mu^2}{T}\sigma_o + \frac{s^2 T}{(\epsilon+p)}\left(\frac{i}{\omega} + \pi\delta(\omega)\right)$$

Thermoelectric conductivities

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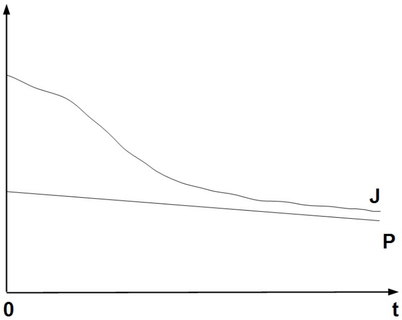


- Their dc limit $\omega \to 0$ is formally infinite. This is due to momentum conservation and the non-zero overlap between the electric and heat currents with momentum:

$$\chi_{JP} = \frac{\delta j^i}{\delta v^i} = \rho$$

$$\chi_{J_Q P} = \frac{\delta j_q^i}{\delta v^i} = sT$$

Finite dc thermal conductivity

- Consider the heat conductivity with open circuit boundary conditions

$$\kappa \equiv T \frac{\delta j_q}{\delta \partial T}\bigg|_{j=0} = \bar{\kappa} - \frac{\alpha^2}{T\sigma}$$

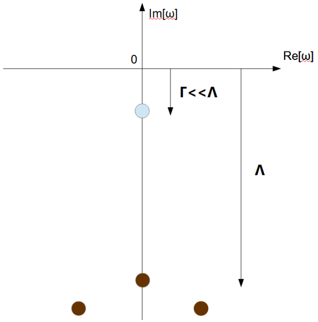
- Finite as $\omega \to 0$

$$\kappa = \frac{(sT + \mu\rho)^2}{T\rho^2}\sigma_o$$

- The open circuit boundary conditions remove the contribution of the sound modes from the thermal conductivity.

- This is the thermal conductivity measured in experiments.

Introducing weak, long wavelength disorder



- Finite dc conductivities? Relax momentum, ie break translations explicitly. The simplest way to treat disorder perturbatively in a 'mean field' way.

$$\partial_t \pi^i + \partial_j \tau^{ij} = -\Gamma \pi^i + \rho \left(E^i + v_k F^{ki} \right) , \quad \Gamma \ll \Lambda \sim 1/\tau_{th}$$

- The conductivity and associated resistivity become

$$\sigma(\omega) = \sigma_o + \frac{\rho^2}{\chi_{PP}} \frac{1}{\Gamma - i\omega} , \quad \rho_{dc} = \frac{1}{\sigma_{dc}} \sim O(\Gamma) \neq 0$$

Disorder is a (dangerously) irrelevant deformation for the resistivity.

Hydrodynamic signatures in electronic flows

- Wiedemann-Franz law for conventional metals

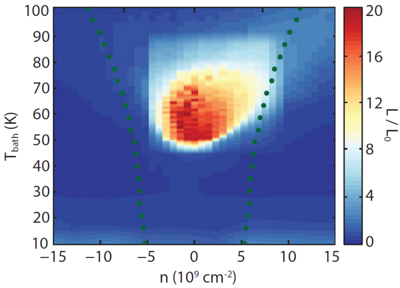
$$\mathcal{L} = \frac{\kappa_e}{\sigma T} = \frac{\pi^2}{3}\left(\frac{k_B^2}{e}\right)^2 \equiv \mathcal{L}_0$$

The WF law holds because both κ_e, $\sigma \sim \tau_{qp}$.

- In very clean Graphene near the charge neutrality point

$$\kappa_e = \frac{s}{\Gamma}, \quad \sigma \sim \sigma_o \quad \Rightarrow \quad \mathcal{L} \sim O\left(\frac{1}{\Gamma}\right) \gg \mathcal{L}_0$$

Independent conduction of heat and charge.



[Crossno et al, Science 351 6277 (2016)]

Hydrodynamic signatures in electronic flows

Other recent experiments

- Backflows and negative resistance in Graphene due to viscous effects [Levitov & Falkovich, Nat. Phys. 12 (2016)], [Bandurin et al, Science 351 (2016)].
- Viscous contributions to the resistance in restricted channels in $PdCoO_2$ [Moll et al, Science 351 2016].
- Viscous contributions to the resistance, violations of WF law and Hall measurements in WP_2 [Gooth et al, arXiv:1706.05925].

Beyond simple hydrodynamics

- How do we compute σ_o, Γ?
- Short-scale disorder?
- Strong disorder?

Need a more microscopic approach!

Inhomogeneous hydrodynamics and memory matrix

- Γ can be computed by perturbing around an equilibrium state with long wavelength disorder [Lucas, 1506.02662]

$$T = \bar{T}(x) + \delta T(t,x)\,, \quad \mu = \bar{\mu}(x) + \delta\mu(t,x)$$

- or using the 'memory matrix' formalism [Forster], [Hartnoll & Hofman, 1201.3917], [Davison, Schalm & Zaanen, 1311.2451]

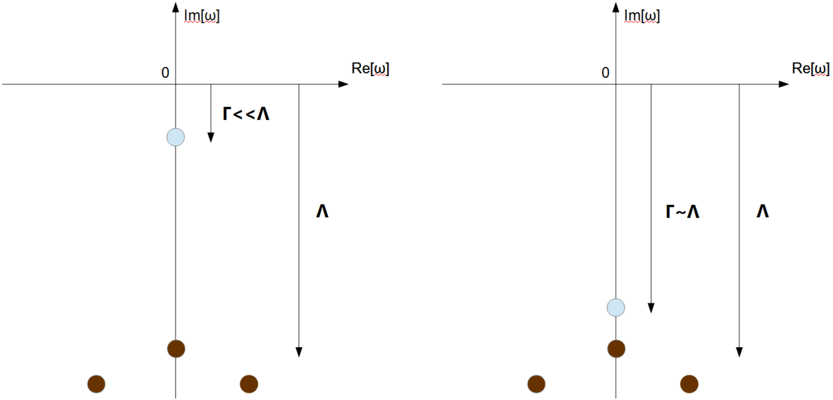
$$H \to H_0 + \epsilon\Delta H + O(\epsilon^2)\,, \quad \dot{P} = i[H,P] = i\epsilon[\Delta H, P] = -\epsilon \int d^2x\, h(x)\, (\partial O)\,(x)$$

$$\sigma_{JJ} = \frac{\chi_{JP}^2}{\chi_{PP}} \frac{1}{M_{PP} - i\omega}$$

$$M_{PP} \equiv \Gamma = \frac{\epsilon^2}{\chi_{PP}} \int \frac{d^2k}{(2\pi)^2} \left|h(k)^2\right| k_x^2 \lim_{\omega\to 0} \frac{\mathrm{Im}\left(G^R_{OO}(\omega,k)\right)}{\omega} + O(\epsilon^3)$$

- When $O = (T^{tt}, J^t)$, Γ determined by η, σ_o.

Coherent to incoherent crossover



1 Hydrodynamics

2 AdS/CFT

A holographic toy-model of momentum relaxation

[Andrade & Withers, 1311.5157]

$$S = \int d^4x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{2}\left(\partial\psi_x^2 + \partial\psi_y^2\right)\right]$$

$$\psi_x = mx\,, \qquad \psi_y = my$$

$$ds^2 = -r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2(dx^2 + dy^2)\,, \quad f(r) = 1 - \frac{m^2}{2r^2} - \frac{r_h^3}{r^3}\left(1 - \frac{m^2}{2r_h^2}\right)$$

The bulk background explicitly breaks translations in the boundary theory

$$\nabla_\mu \langle T^{\mu\nu}\rangle = \sum_{x,y} \partial^i \psi_i \langle O_{\psi_i}\rangle$$

Computing the dc heat conductivity (1)

[Donos & Gauntlett, 1406.4742]

- To compute the dc heat conductivity $\bar{\kappa}_{xx} = -\delta j_q^x / \partial_x \delta T$, we need to turn on perturbations $(\delta g_{tx}(t,r), \delta g_r^x(r), \delta\psi_x(r))$:

  $$\delta g_{tx} = -\zeta(r^2 f)t + \delta g(r)$$

  For this choice of perturbation, all time dependence drops out of the linearized equations.

- Perform boundary change of coords $t = \bar{t} - \zeta \bar{t} x$

  $$ds^2 = -r^2 f(1 - 2\zeta x) d\bar{t}^2 + \ldots \quad \Rightarrow -\zeta = \partial_x T / T$$

Computing the dc heat conductivity (2)

- We can find a radially conserved quantity

$$\partial_r Q(r) = 0\,, \quad Q(r) = (r^2 f)^2 \partial_r \left(\frac{\delta g}{(r^2 f)^2} \right)$$

- Using holographic renormalization, prove

$$Q = T^{tx}$$

  so Q is the heat current at the boundary.

- Regularity at the horizon in Eddington-Finkelstein coordinates

$$t = v - \frac{1}{4\pi T} \ln(r - r_h) \quad \Rightarrow \quad \delta g = -\,\delta g_r^x|_{r=r_h} + \frac{\zeta}{4\pi T} \ln(r - r_h)$$

- Evaluate now Q at the horizon

$$Q(r_h) = \zeta \frac{r_h^2 \left((r^2 f)' \right)^2}{m^2}$$

- Finally,

$$\bar{\kappa} = \frac{1}{T} \frac{\partial Q}{\partial \zeta} = \frac{4\pi s T}{m^2}$$

  Non-perturbative result in m: valid both when $m \ll T$ (coherent) and $m \ll T$ (incoherent).

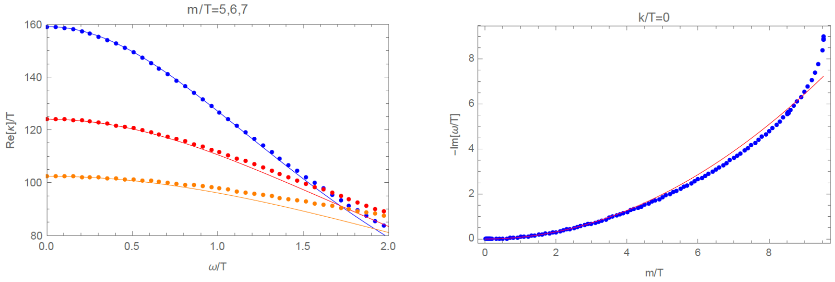
Thermal conductivity in the coherent regime

[Davison & B.G., 1411.1062] By identifying to the hydro formula or by using the memory matrix formula for Γ with $O = \psi$, in the coherent regime $\Gamma \ll \Lambda$

$$\bar{\kappa} = \frac{s^2 T}{\chi_{PP}\Gamma} = \frac{s}{\Gamma} = \frac{4\pi s T}{m^2} \quad \Rightarrow \quad \Gamma = \frac{m^2}{4\pi T}$$

and

$$\bar{\kappa}(\omega) = \frac{s}{\Gamma - i\omega}$$



Computing the momentum relaxation rate

- In the previous slide, we **assumed** that for $m \ll T$, hydrodynamic formulæ should be applicable, and verified it was so. Can we do better?

- Option 1: Using a hydrodynamic expansion around the background, derive that for small $m \ll T$ [Davison & B.G., 1505.05092], [Blake, 1505.06992]

$$\bar{\kappa} = \frac{s}{\Gamma - i\omega}\,, \qquad \Gamma = \frac{m^2}{4\pi T}$$

  Disadvantage: need to know the background analytically.

- Option 2: Other methods based on Einstein equations give the location of the pole analytically by direct computation [Lucas, 1501.05656], assuming there is a low lying pole at $\omega = -i\Gamma$, $\Gamma \ll T$.

- Option 3: Memory matrix prediction (does not rely on either ads/cft or hydrodynamics) [Hartnoll & Hofman, 1201.3917]

$$H \to H_0 + \Delta H\,, \quad \dot{P} = i[H,P] = i[\Delta H, P] = -\int d^2x\, h(x)\,(\partial O)\,(x)$$

$$\Gamma = \frac{1}{\chi_{PP}} \int \frac{d^2k}{(2\pi)^2} \left|h(k)^2\right| k_x^2 \lim_{\omega\to 0} \frac{\mathrm{Im}\left(G^R_{OO}(\omega,k)\right)}{\omega}\Bigg|_{h=0}$$

Computing the momentum relaxation rate (2)

- Applied to our toy model, the previous formula reads

$$\Gamma = \frac{m^2}{sT} \lim_{\omega \to 0} \frac{\mathrm{Im}\left(G^R_{\psi\psi}(\omega, k=0)\right)}{\omega}\Bigg|_{m=0}$$

- We need to compute the spectral weight of the operator dual to ψ in the translation invariant theory.

- This is done by solving the $\delta\psi$ fluctuation equation of motion and taking the ratio

$$G^R_{\psi\psi}(\omega, k=0) = \frac{3\delta\psi_3}{\delta\psi_0}\,, \quad \delta\psi \underset{r\to+\infty}{\to} \delta\psi_0 + \frac{\delta\psi_3}{r^3} + \dots$$

Computing the momentum relaxation rate (3)

- Decompose $\delta\psi = \delta\psi(r)e^{-i\omega t}$. The structure of the equation is

$$\left(r^4 f(r)\delta\psi'(r)\right)' + \#\omega^2\delta\psi = 0$$

The ω dependence only appears at quadratic order.

- We need to impose ingoing boundary conditions

$$\delta\psi(r\sim r_h) = \psi_h(r - r_h)^{-i\omega t/(4\pi T)}(1+\ldots) \sim \psi_h\left(1 - \frac{i\omega}{4\pi T}\ln(r-r_h) + O(\omega^2)\right)$$

- Since we are interested in the $\omega \to 0$ limit, we only need to solve the $\omega = 0$ equation:

$$\delta\psi = \psi_h + \psi_{sing}\int_r^{+\infty}\frac{dr}{r^4 f} \underset{r\sim r_h}{\sim} \psi_h - \frac{\psi_{sing}}{4\pi T r_h^2}\ln(r-r_h)$$

- But then it must be that $\psi_{sing} = i\omega r_h^2\psi_h$ and:

$$\delta\psi(r) = \psi_h\left(1 + i\omega r_h^2\int_r^{+\infty}\frac{dr}{r^4 f} + O(\omega^2)\right)$$

Computing the momentum relaxation rate (4)

$$\delta\psi(r) = \psi_h \left(1 + i\omega r_h^2 \int_r^{+\infty} \frac{dr}{r^4 f} + O(\omega^2) \right)$$

- I can now expand the previous equation close to the boundary $r \to +\infty$ and read off

$$\delta\psi \underset{r\to+\infty}{\to} \delta\psi_0 + \frac{\delta\psi_3}{r^3} + \dots$$

- Finally, we find that

$$\lim_{\omega\to 0} \left. \frac{\mathrm{Im}\left(G^R_{\psi\psi}(\omega, k=0)\right)}{\omega} \right|_{m=0} = r_h^2$$

which we can plug into the expression for Γ

$$\Gamma = \frac{m^2}{sT} \lim_{\omega\to 0} \left. \frac{\mathrm{Im}\left(G^R_{\psi\psi}(\omega, k=0)\right)}{\omega} \right|_{m=0}$$

Recalling that $s = 4\pi r_h^2$, our final result is

$$\Gamma = \frac{m^2 r_h^2}{sT} = \frac{m^2}{4\pi T}$$

Thermal conductivity away from the coherent regime

[Davison & B.G., 1505.05092]

- We can compute the first deviations away from the coherent regime analytically

$$\bar{\kappa}(\omega) = \frac{\chi_{PP}}{T\Gamma}\,, \quad \chi_{PP} = sT\left(1 + 4\pi T\lambda\Gamma + O(\Gamma^2)\right)\,,$$

$$\Gamma = \frac{m^2}{4\pi T}\left(1 + m^2\lambda + O(m^4)\right)\,, \quad \lambda = \frac{\sqrt{3}\pi - 9\log 3}{96\pi^2 T^2}$$

- Corrections both to the momentum relaxation rate and to the static susceptibilities.

- The corrections combine in a non-trivial way such that the dc limit is still

$$\bar{\kappa} = \frac{4\pi s T}{m^2}$$

Thermal conductivity in the incoherent regime

[Davison & B.G., 1411.1062]

- In the incoherent regime, momentum is short-lived and not part of the late time effective theory:

$$\partial_t \epsilon + \nabla \cdot j_\epsilon = 0\,, \quad j_\epsilon = -\bar{\kappa}_o \nabla T$$

- Thermal transport occurs by diffusion of energy rather than sound waves

$$\bar{\kappa} = \bar{\kappa}_o = T\frac{\partial s}{\partial T} D\,, \qquad \bar{\kappa}(\omega) = \frac{i\omega\bar{\kappa}_o}{i\omega - Dk^2}$$