Weibull type speckle distributions as a result of saturation in stimulated scattering processes

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Abstract

During the propagation of an optically smoothed laser beam through a warm plasma the speckle field pattern and the corresponding speckle intensity distribution is modified in time and along the laser propagation direction. It is shown here that the laser plasma interaction can change the character of speckle statistics from an initially exponential-type limit law to a Weibull type law. The Weibull distribution is characterized by a power-law type behaviour in a limited interval of the random variable, which is, in the present case, the speckle intensity. The properties of the speckle distributions are studied using methods of extremal and order statistics. The scattering instability process (here stimulated Brillouin forward scattering) causing the change in speckle statistics has an onset behaviour associated with a 'critical gain' value, as pointed out in work by Rose & DuBois (1993 b). The saturation of the instability process as a function of intensity explains the limited interval of the Weibull type speckle distribution. The differences in the type of the speckle statistics are analyzed by using 'excess over threshold' methods relying on the Generalized Pareto Distribution (GPD), which clearly brings to evidence the transition from an exponential type distribution to the Weibull type distribution as a function of the instability gain value, i. e. from the regime below critical gain to values above the critical gain.

Key words: Laser-Plasma Interaction, Extreme value Theory, Statistical Optics, Non-Gaussian Statistics, Speckles, Stimulated Scattering
Short title: "Weibull distributions in laser plasmas"

1 Introduction

So-called optically "smoothed" laser beams with speckle structure are characterized by well-known Gaussian statistics for the speckle intensity. Propagation of such beams through an active medium like a warm plasma causes modifications to the speckle statistics, both as a function of time and of space. The extremal statistics of laser speckle fields is of crucial importance to understand the onset of non linear phenomena in optics and laser-plasma interaction. Speckle patterns of laser fields generated by optical smoothing techniques contain very intense speckles with peak intensities having many times (typically at the order of ten) the mean intensity of the field pattern. Since these smoothing techniques aim to generate Gaussian statistics for the field values (Cecotti et al., 1995), the abundance of intense speckles decreases following an exponentially decreasing law (Rose & DuBois, 1993a; Garnier, 1999).

The spatial and temporal evolution of laser fields in laser plasma interaction, or more generally in active media, can change the initial distribution of laser speckles along the propagation through the plasma. The notion of non-Gaussian statistics has been evoked for the distribution functions derived from speckle field patterns due to the influence of non linear processes like self-focusing (Lushnikov & Vladimirova 2010) and/or stimulated scattering (mostly stimulated Brillouin forward scattering) (Grech et al. 2006 and 2009, Depierreux et al. 2009, Malka et al. 2003, Schmitt & Afeyan 1997).

By exploring the functional behaviour of the tail of the distribution function, one can distinguish between Gaussian and non-Gaussian statistics. From experimental or numerically determined data, it is however not always evident to conclude which of the three possible types of law of extremal statistics is relevant (Fisher & Tippett, 1928): the 'Gumbel' (or double exponential) law for Gaussian statistics, or the 'Weibull' or the 'Fréchet' law for non Gaussian statistics. The extreme value distribution of Fréchet or Weibull type both decrease slowlier (as a function of the random variable, here speckle intensity) than a law of exponential-type ("heavy tails" versus "light", exponential tails). So-called "heavy tail" behaviour from Lévy statistics, for instance, with a power law dependence in the tail, belongs to the basin of the Fréchet law, and has been investigated in the context of amplifying random media (Montina et al. 2009, Barthélemy et al. 2008, Wiersma 2008).

In laser plasma interaction, however, the potential non linear processes generally saturate with increasing laser intensity. This applies in particular to self focusing and to stimulated scattering. As we shall see later, due to saturation, heavy tail behaviour may be limited to an interval of speckle intensity, and not – like for the above-mentioned cases of Lévy statistics – for the entire tail of the distribution up to the limit of resolution.

Non linear phenomena, like stimulated scattering usually increase with intensity and can therefore be extremely sensitive to even small changes in the speckle intensity, leading eventually to a critical behaviour. Heavy tail behaviour due to modified speckle distributions is therefore of particular interest to estimate the risk of increasing deviations (variance) between different realizations, i. e. from "shot to shot" in experiments.

For the case of Stimulated Raman Scattering, Rose and Mounaix (2011) have recently shown evidence for explosive behaviour in spite of diffraction. Berger et al. (1993) have observed in numerical simulations 'hard tail' behaviour in the intensity distribution which they attribute to filamentation. Mounaix and Divol (2002) have shown that the cumulative response of Stimulated Brillouin Scattering, based on the independence of speckles ("independent hot spot model"), can spread enourmously around a critical gain value associated with the scattering process.

Extremal statistics allows to determine the reliability in the speckle distribution, mostly pertinent in the high-intensity tail of the distribution. From order statistics one can determine the probability with which the n-th most intense speckle of a pattern with $n_{\rm sp}$ speckles can be found at the intensity I_n (hence, I_1, standing for the intensity of the most intense speckle, see e. g. our previous work Hüller & Porzio (2010)). As mentioned before, the speckle distribution laws of standard laser smoothing techniques follow Gaussian statistics; the law describing extremal statistics is consequently the Gumbel (double exponential) law. Non linear phenomena can provoke non-Gaussian statistics due to strong enhancement of the role of speckles at elevated intensity for certain observables, like the total response of the scattered light (Mounaix and Divol, 2002; Hüller, Porzio & Robiche, 2013), as will be discussed later on. The distribution function associated with this processes appears hence to be "stretched" in the tail, with respect to the linear distribution $\bar{F}(I)$, so that the probability to encounter speckles at higher intensities is significantly increased in the interval around I_1, where the intensity I_1 is defined by $n_{\rm sp}\bar{F}(I_1) \equiv 1$ (i.e. $I \simeq \ln n_{\rm sp}$ where $\bar{F} \sim \exp(-I/\langle I\rangle)$).

In this article we show that smoothed laser beams originating from a standard, optical smoothing technique can exhibit a Weibull-type distribution in a limited interval of the speckle peak intensity I, for values of I that are indeed high enough to produce undesirable effects.

The article is organized as follows: In section 2 we recall model equations and distributions arising from order statistics. Approximations are deduced to analyze the behaviour of densities following the value of the gain G of stimulated scattering inside speckles. A particular analysis is devoted to the Variance in subsection 2.2. In section 3.1 we recall the theoretical basis and the 'Excess over Thresholds' methods, which we apply to our model to deduce the nature of the distribution function and its departure from Gaussian statistics. In section 4 we analyze to which type of limit law belong the speckle distribution functions obtained from our model and from numerical simulations, as a function of the gain parameter of stimulated scattering. Section 5 concludes the article. Technical computations are presented in an Appendix.

2 Model Equations

In Refs. Hüller and Porzio (2010) and Porzio and Hüller (2010) we have derived the probability density in intensity for speckles from order statistics. For the k-th speckle in order of its peak intensity I, where k =1 is the most intense of – in total – n_{sp} speckles, the probability density function (*pdf*) is given by

$$f_k(I) = \frac{n_{sp}^k}{\Gamma(k)} e^{-n_{sp}e^{-I}-kI} \ ,$$

where the intensity I is already normalized to the space averaged intensity of the beam $\langle I\rangle$ (here hence $\langle I\rangle \equiv 1$). In a subsequent publication (Hüller, Porzio & Robiche, 2013) we have shown that in a warm plasma, the speckle statistics is modified by plasma induced smoothing due to the stimulated Brillouin forward scattering (SBFS) process.

2.1 The non linear speckle distribution and its *pdf*

The SBFS process induces a non linear modification of the distribution of speckles which can be expressed by taking into account the amplification in the peak intensity of each

$k-$th order speckle. In contrast to the linear initial *pdf* $f_k(I)$, we denote it here as $f_k^{(G)}(I)$; it is given by

$$f_k^{(G)}(I) = \frac{1}{N_k^{(G)}} \frac{n_{sp}^k}{\Gamma(k)} e^{G\min(I,I^*)} e^{-n_{sp}e^{-I}-kI}$$

where G is the standard gain factor for the spatial amplification of stimulated scattering, here SBFS (see Hüller, Porzio and Robiche, 2013; Pesme, 1993); I^* stands for the intensity value from which on the stimuated scattering process is saturated. It means that the scattering seed level $\varepsilon_{\rm seed}$ cannot be amplified beyond the level of the 'pump' wave intensity, i. e. $\varepsilon_{\rm seed}\exp(GI) \leq \varepsilon_{\rm seed}\exp(GI^*) \equiv 1$. The normalisation $N_k^{(G)}$ of the *pdf* is given by $N_k^{(G)} = \int_0^\infty (n_{sp}^k/\Gamma(k)) e^{G\min(I,I^*)} e^{-n_{sp}\exp(-I)-kI} dI$. Numerical simulations of plasma induced smoothing clearly confirm the features of this distribution, so that we use the distribution for further evaluation and to analyse it with respect to its maximum domain of attraction (MDA). (see paragraph 4)

In the following we work out simplified expressions for the probability density $f^{(G)}(I)$ corresponding to the complementary distribution (also named tail distribution) function $\bar{F}^{(G)}(I)$, defined as

$$\bar{F}^{(G)}(I) = 1 - F^{(G)}(I) = n_{sp}^{-1} \sum_k^{n_{sp}} \int_I^\infty f_k^{(G)}(I')dI' \ . \tag{1}$$

This function respresents the complementary distribution sufficiently well at least for $I \gg 1$ (say $I > 4$). In order to derive the *pdf* of the non linear distribution we will make use of the well-known identities for the diverse types of gamma functions, which will be needed for the following expressions, $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, $\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}dt$, $\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt = \sum_{n=0}^\infty (-1)^n x^{a+n}/[n!(a+n)]$, and $\Gamma(a,x) = \Gamma(a,0) - \gamma(a,x) = \Gamma(a) - \gamma(a,x)$ and for k integer $\gamma(k,x) = (k-1)!\left(1 - e^{-x}\sum_{n=0}^{k-1} x^n/n!\right)$.

The probability density $f^{(G)}(I)$ as well as the distribution $F^{(G)}(I)$, following Eq. (1), can then be expressed as follows:

$$f^{(G)}(I) = e^{-n_{sp}e^{-I}} e^{G\min(I,I^*)} \sum_{k=1}^{n_{sp}} \frac{e^{-kI} n_{sp}^k}{D_k(G,0)} \ , \tag{2}$$

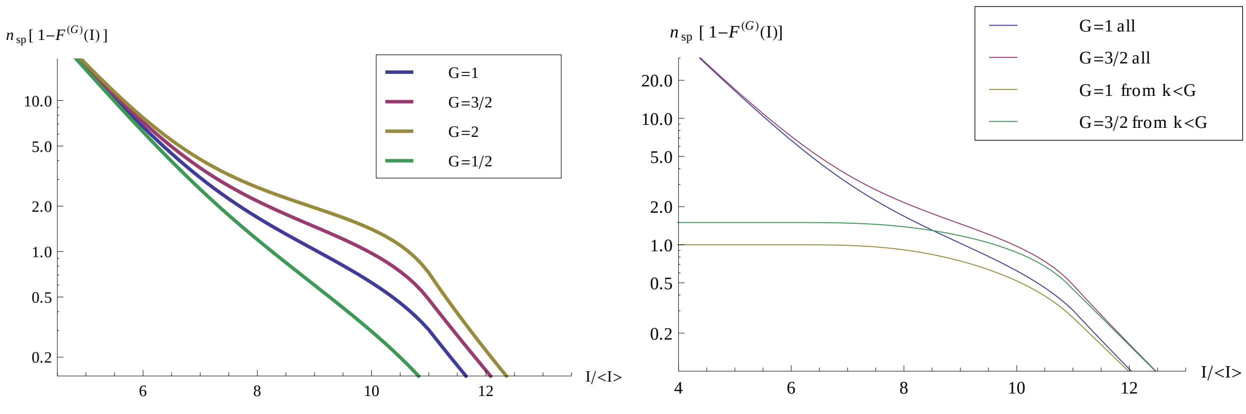
$$\bar{F}^{(G)}(I) = \sum_{k=1}^{n_{sp}} \frac{D_k(G,I)}{D_k(G,0)} \equiv \sum_{k=1}^{n_{sp}} \frac{n_{sp}^k}{D_k(G,0)} \int_I^\infty dI' e^{-n_{sp}e^{-I'}} e^{-kI'+G\min(I',I^*)} \ , \tag{3}$$

with

$$D_k(G,I) \equiv \begin{cases} n_{sp}^G \ \left[\Gamma(k-G, n_{sp}e^{-I^*}) - \Gamma(k-G, n_{sp}e^{-I})\right] + e^{GI^*}\gamma(k, n_{sp}e^{-I^*}) \text{ for } I < I^* \\ \\ e^{GI^*} \ \gamma(k, n_{sp}e^{-I}) \quad \text{for } I \geq I^* \ . \end{cases} \tag{4}$$

Note that the normalizing denominator $D_k(G,0)$ yields $D_k(G=0,0) = \gamma(k, n_{sp}e^{-I})$ for the linear case $G = 0$. In order to understand the non linear modification of the *pdf* $f^{(G)}$ and the distribution $F^{(G)}$, it is instructive to develop Eqs. (2-3) for the particular cases $G = 1/2$ and $G = 3/2$, see Appendix, and for cases with integer values of the gain G, namely

$$\begin{aligned} f^{(G)}(I) \ &\equiv f_a^{(G)}(I) + f_b^{(G)}(I) \qquad \text{for } G \in \mathbb{N} & (5) \\ &\simeq e^{-n_{sp}e^{-I}} e^{G\min(I,I^*)} \left(\sum_{k=1}^{G\in\mathbb{N}} \frac{e^{-kI} n_{sp}^k}{D_k(G,0)} + n_{sp}e^{-(G+1)I} e_{n_{sp}-G-1}(n_{sp}e^{-I}) \right) \ , \end{aligned}$$



(a) (b)

Figure 1: Complementary speckle distribution $\bar{F}^{(G)}(I)$ in the high intensity tail. The total number of speckles considered here is $n_{\rm sp} = 2300$, so that the most intense speckles has its expectation value at $I_1/\langle I \rangle = \log(2300) + \gamma_E \simeq 8.3$. The onset of saturation is at $I^* = 11$ here. Subplot (a) shows the cases $G = 1/2$ (green line), 1 (blue), 3/2 (red), and 2 (yellow-green). Subplot (b) shows only the cases $G = 1$ and 3/2, but distinguishes contribution from all speckles (blue and red line, respectively), and from the most intense speckles $\bar{F}_a^{(G)}$ (yellow and green line, respectively).

with $e_n(x) = \exp(x)\Gamma(n+1,x)/n!$ which approaches $e_n(x) \sim \exp(x)$ for $n \gg x$. In the intensity interval $I < I^*$, i.e. before saturation of amplification, the term $f_b^{(G)}(I) = n_{sp}e^{-I-n_{sp}\exp -I}e_{n_{sp}-G-1}(n_{sp}e^{-I})$ is practically independent of G, except for the missing terms of the first (i.e. the most intense) maxima. For the complementary distribution $\bar{F}^{(G)}$, the same holds for the part $\bar{F}_b^{(G)}(I) = \int_I^\infty f_b^{(G)}(u)du$, being well approximated by $\bar{F}_b^{(G)}(I) \simeq \exp(-I)$ for $I^* > I > 1$,

$$\bar{F}^{(G)}(I) \equiv \bar{F}_a^{(G)}(I) + \bar{F}_b^{(G)}(I) \simeq \sum_{k=1}^{G\in\mathbb{N}} \frac{D_k(G,I)}{D_k(G,0)} + e^{-I} \ , \ \text{for } G \in \mathbb{N} \ . \qquad (6)$$

For $I > I^*$, the contribution $\bar{F}_b^{(G)}(I)$ has a faster, but still exponential decrease, and can be neglected with respect to $\bar{F}_a^{(G)}(I)$. The terms $f_a^{(G)}(I) = \sum_{k=1}^{G}(\ldots)$ and $\bar{F}_a^{(G)}(I) = \sum_{k=1}^{G} D_k(G,I)/D_k(G,0)$, for which the speckle order k (intensity decrease with order) is less or equal G, $k \leq G$, consitute hence the non linear part of the *pdf* and the distribution, respectively, due to an enhancement of the probability to find the most intense speckles at still higher intensity. The complementary distribution $F^{(G)}(I)$ is 'stretched' in the high-intensity tail, i.e. the probability to find intense speckles at intensities higher than in absence of the plasma is considerably increased for $I^* > \log(n_{sp})$, and $F^{(G)}(I)$ therefore exhibits clearly a non exponential decrease, as illustrated in Figures 1. As can be observed in the Figures, there is a drastic change in the functional behaviour of distributions between the domains $G < 1 \leq k$ and $G \geq k \geq 1$.

In the Appendix we develop also approximate expressions for $f^{(G)}(I)$, for cases of the gain coefficient G with half-integer values, such as $G = 1/2$, $3/2$.

2.2 Variance

Although the number of speckles in the high-intensity tail of the distribution is small, the impact of a modified distribution function on physical observables in the regime around and above the critical gain G =1 is important. Moreover, the fluctuations between different realizations of speckle patterns will be particularly pronounced. For this reason we also calculate here moments to the *pdf* of the modified speckle distribution, and compute the variance of the speckle intensity which is an important measure to estimate the differences arising in different realisations of speckle patterns gouverned by the distribution $F^G(I)$. Our results in Ref. (Hüller, Porzio & Robiche 2013) show that in particular for gain values G close to the critical value G =1 the variance can be surprisingly high.

In order to evaluate moments of the *pdf* it is easier to perform a change of variables by replacing the variables n_{sp}, I, and I^*, by $x = n_{sp}\ e^{-I}$, and $x^* = n_{sp}\ e^{-I^*}$, as we have done in the Appendix. The approximate expression of the *pdf* for $f^{(G)}(I) \to \tilde{f}^{(G)}(x)$ then reads

$$\begin{aligned} \tilde{f}^{(G)}(x) &= e^{-x}\left(n^G \sum_{k=1}^{G-1} \frac{x^{k-G}}{D_k(G,0)} + \frac{n^G}{D_{k=G}(G,0)} + x e_{n-G-1}(x) \right) \text{ for } x > x^* \text{ i.e. } I^* > I\ , \\ &= e^{-x}\left((\frac{n}{x^*})^G \sum_{k=1}^{G-1} \frac{x^k}{D_k(G,0)} + \frac{n^G (x/x^*)^G}{D_{k=G}(G,0)} + x(\frac{x}{x^*})^G e_{n-G-1}(x) \right) \text{ for } x < x^* \end{aligned} \quad (7)$$

The expression written in this form distinguishes between three contributions: the sum term, over $k = 1 \dots G-1$, of the first contribution contains the peak term(s) (only if $G > 1$), the second one is the contribution for $G = 1$ only, and corresponds to a plateau-like behaviour as a function of intensity I, while the third contribution is essentially of linear nature and decreases in intensity I. This 3rd contribution as expressed here, since it was derived from order statistics, is only valid for large enough speckles intensities, say $I > 3$. It over-estimates the *pdf* in the low-intensity part.

Consequently, only the two first contributions alter the momenta of the distribution for different values of the gain G. They primarily contribute to the change in the variance as a function of G. For the interval $x < x^*$ (or $I > I^*$) all the terms have powers of $x/x^* < 1$, and therefore are rapidly decreasing.

From the *pdf* we determine the variance, $\mathrm{Var}(I)$ in I to the expectation value $\langle I \rangle \equiv E(I)$ of the speckles, $\mathrm{Var}(I) = E(I^2) - E(I)^2$, via an approximate calculation, keeping only the dominating (i.e. peak and plateau) term(s). By definition, the second moment of a probability density function (*pdf*) of I is given, in our notation, by

$$E(I^2) = \int_0^\infty I^2 f^{(G)}(I) dI = \int_0^n (\log\frac{n}{x})^2 \tilde{f}^{(G)}(x) \frac{dx}{x}\ .$$

Taking for the *pdf* $\tilde{f}^{(G)}(x)$ for even integer values of G, the peak term in the above sum is reached for $k = G/2$, and for odd integer G, two equal peak terms, for $k = [G/2]$ and $k = [G/2] + 1$ have to be considered.

Hence, for most interesting case $G = 1$, one obtains with the leading term,

$$E(I^2)|_{G=1} \simeq \int_{ne^{-I^*}}^n \left(\log\frac{n}{x}\right)^2 \frac{ne^{-x}}{D_{k=1}(G=1,0)} \frac{dx}{x}$$

An approximate expression for the variance with $G = 1$ is hence given by

$$\mathrm{Var}|_{G=1}(I) = \int_{ne^{-I^*}}^n (\log\frac{n}{x})^2 \frac{ne^{-x}}{D_{k=1}(G=1,0)} \frac{dx}{x} - \left(\int_{ne^{-I^*}}^n \log\frac{n}{x}\ \frac{ne^{-x}}{D_{k=1}(G=1,0)} \frac{dx}{x} \right)^2\ ,$$

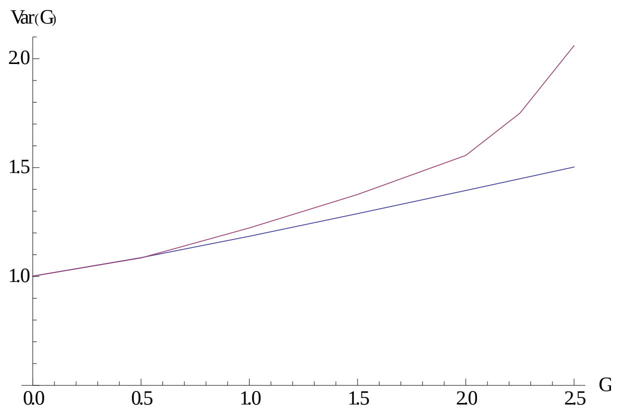


Figure 2: Variance computed from the model *pdf* given by Eqs. (2) and (7) as a function of the gain G for the cases: (blue line) $n_{\rm sp}$ =2300, I^* =11 and (red) $n_{\rm sp}$ =3000, I^* =15.

involving three integrals $\int x^{-1}e^{-x}dx$, $\int x^{-1}(\log x)\ e^{-x}dx$, and $\int x^{-1}(\log\ x)^2\ e^{-x}dx$, which can be expressed in terms of the $\Gamma(a,x)$ function and its derivatives: $\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}\,dt$, $\partial_a\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}\log\ t\,dt$, $\partial_a^2\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}(\log\ t)^2\,dt$.

Based on our model, we have determined the variance $\mathrm{Var}(G)$ for two parameters sets, namely with $n_{\rm sp}$ =2300 and I^* =11 as well as $n_{\rm sp}$ =3000 and I^* =15 showing the increase of the variance as a function of the gain G in the interval 0$\leq G\leq$2.5 due to the increasing peak value of the most intense speckles. The increase in the variance with G is due to the contributions of the leading terms, as discussed above, i. e. due to the most intense speckles. The expectation value (in peak intensity) of the most intense speckles grows rapidly with G, so that already few intense speckles can rise the variance significantly.

3 How to determine the type of statistics

The "non-linear" speckle distribution and its density derived in Eq. (2-3) on the basis of order statistics, for a laser speckle pattern under the influence of stimulated scattering, follow our preceding work Ref. (Hüller, Porzio & Robiche 2013),Eqs. (31) and (33), in which we also have presented simulation results supporting the model results. Both the distribution from the model and from the simulations show a clear departure from an exponential decrease in the high-intensity tail, for for $G\geq$1. This behaviour can be observed in a limited interval in the vicinity of the expectation value of the speckle peak intensity $I_1\simeq\log(n_{sp})+\gamma_{\rm Euler}$ ($\gamma_{\rm Euler}=.5772$).

While a power-law-like behaviour in the tail of complementary distributions is a clear signature of non-Gaussian statistics, it has still to be distinguished between the two types of non-Gaussian limit laws of the extremal behaviour, namely the Weibull or Fréchet laws (Montina et al., 2009; Wiersma, 2008).

The distinction between the different limit laws needs careful analysis of the data for which, however, mathematical methods have been derived allowing to determine the correct type.

For distributions for which a power law behaviour fits in the tail up to the highest measurable value of the variable (here, the intensity) it is plausible to identify the Fréchet law as limit law (Montina et al., 2009; Wiersma, 2008). In cases, however, when the distribution exhibits non-exponential behaviour only in a limited interval, like the case shown in Figs. 1,computed from our model, the distribution has to be carefully determined, in particular the lower bound of the relevant interval.

Fot this reason we examine here the character of the distribution functions on the basis of well-established theory (Pickands 1975, Embrechts et al. 1997).

A good method to plausibly demonstrate the type of statistics is to determine the behaviour by fitting "excesses over a threshold" (Embrechts et al., 1997). This is done on the basis of the generalized Pareto distribution (GPD) for the conditioned distribution function $\bar{F}_{th}(y)$ evaluated at the 'threshold' ('th') speckle intensity I_{th} that has to be suitably chosen. Then, once the type of Pareto distribution for the conditioned F_{th} is known, it is possible to deduce the nature of the underlying distribution F.

3.1 Excess over threshold and Limit theorems

For the distribution function $F(I)$, with $\bar{F} \equiv 1-F$ denoting the complementary distribution function, we determine the 'excess over threshold' probability $\bar{F}_{th}(y)$. This conditional probability is the probability $P(I > I_{th}+y)$ that the speckle intensity I is larger than the value $I_{th}+y$ knowing that I is larger than the threshold intensity I_{th}, i. e. $P(I > I_{th})$. As a function of the distribution $\bar{F}$ and for $y \in \mathbb{R}^+$, it can be expressed as

$$\bar{F}_{th}(y) \equiv \frac{P(I > I_{th}+y)}{P(I > I_{th})} = \frac{\bar{F}(I_{th}+y)}{\bar{F}(I_{th})} \ . \tag{8}$$

It has been shown in Refs. Pickands (1975), Balkema and DeHaan (1974), and in the review of Embrechts et al. (1997)), that for a suitably chosen large threshold value I_{th}, the conditional distribution F_{th} tends toward $h_{a,c}$ the generalized Pareto distribution('GPD', see below) if and only if the (unconditioned) distribution $F(I)$ is in the maximal domain of attraction of 'the generalized extreme value distribution' H_c, with the same parameter c :

$$H_c(y) = \begin{cases} \exp\{-(1+cy)^{-1/c}\} \text{ for } c \neq 0 \\ \exp\{-e^{-y}\} \quad \text{otherwise, i.e. } c = 0 \end{cases} \ , \tag{9}$$

requiring $y > 1/c$ if $c > 0$, and $y < -1/c$ if $c < 0$; for the case $c = 0$ with $y \in \mathbb{R}$ this yields the Gumbel (limit) law, which corresponds to Gaussian statistics.

The three corresponding cases for the generalized Pareto distribution $h_{a,c}$ with $a > 0$ following the value of $c \in \mathbb{R}$, are :

$$h_{c,a}(y) = \begin{cases} \left(1+c\frac{y}{a}\right)^{-1/c} & \text{for } c > 0 \text{ and } y < \infty \ , \\ e^{-y/a} & \text{for } c = 0 \text{ and } y < \infty \ , \\ \left(1-|c|\frac{y}{a}\right)^{1/|c|} & \text{for } c < 0 \text{ and } 0 < y \leq \frac{a}{|c|}. \end{cases} \tag{10}$$

In our case, following Pickands (1975), we find that $\bar{F}_{th}(y)$ can be given in very good approximation by $h_{c,a}$ with $c < 0$, so that F belongs to the extreme value attraction basin of a Weibull law with the same value $c < 0$. It is important to remark in this case that for $y > a/|c|$, $h_{c,a}$ is no longer defined, and the conditioned probability $\bar{F}_{th}(y > a/|c|) = 0$. Therefore the interval for a Weibull-type distribution is limited. Physically this is due to the fact that the non linear process that leads to our distribution $F(I)$ saturates in I, and therefore the interval is limited in I where the Weibull distribution applies.

For distributions that can be described with such a type of functions for the case c <0, a particular property is that, approaching a certain finite value $y_F = |a/c| < \infty$ of the variable y (in our case the speckle intensity above I_{th}), the (unconditioned) distribution has a power-law dependence in the vicinity of $I \sim I_F \equiv I_{th} + y_F$. This feature can be expressed, via the distribution function $F(I)$ and its corresponding probability density $f(I) = F'(I)$, as

$$\lim_{\substack{I \to I_F \\ I < I_F}} \frac{(I_F - I) f(I)}{\bar{F}(I)} \to \alpha \ , \tag{11}$$

with $I_F = I_{th} + y_F$, and with $f(I)$ being the probability density belonging to $\bar{F}(I)$, hence $f^{(G)}(I) \equiv dF^{(G)}(I)/dI$ in our particular case. The upper bound of the interval at $y = y_F$ marks the end point of the theoretical distribution, $y_F \equiv \sup_{y \in \mathbb{R}}\{F(y) < 1\}$, namely the value in y where the conditioned distribution F_{th} $(= 1 - \bar{F}_{th})$ reaches $F_{th}(y = y_F) \equiv 1$ (or $\bar{F}_{th}(y_F)$ =0).

Distributions resulting from the GPD method with negative c are hence associated with Weibull distributions. In a neighborhood of $I = I_{th} + y_F$, for $I \to I_F$, $I < I_F$, (see Embrechts et al. (1997) pp. 152-154) the unconditioned distribution F belongs to the corresponding maximum domain of attraction of the Weibull law Ψ_α,

$$\Psi_\alpha(I) = e^{-(-I)^\alpha} \ , \tag{12}$$

which is qualitatively different from the domain of attraction of the Gumbel law $\sim \exp\{-e^{-I}\}$, to which exponential type distributions from Gaussian statistics belong. From the limit theory, it follows that the power law with an exponent α can be established in the vicinity of $I = I_F$, but only in the branch $I \to I_F$, $I < I_F$, namely

$$\bar{F}(I) \simeq K(I_F - I)^\alpha \text{ for } I_F > I > I_F - K^{-1/\alpha} \ , \tag{13}$$

in which the coefficient K is related the lower bound of the interval of validity of this approximation, namely $I > I_F - K^{-1/\alpha}$. The exponents of the generalized Pareto distribution in Eq. (10), for the case $c < 0$, and of the limit law Eq.(12) are linked (Embrechts et al., 1997); in the ideal limit case, this would yield $\alpha = -1/c$.

In the following we will evaluate the coefficients c, a in Eq. (10) and K and α in Eq. (13) from the distribution of Eq. (3).

4 Results

We have hence examined the distributions following our model, Eqs. (3) for different values of the gain parameter G, namely for $G = 1$, 1.5, 2, and for G =0.5 < 1. Following what has been explained in the previous paragraph, two approaches can be used to find fits to power laws according to the observed features, using
(i) the GPD from 'excess over threshold' theory, Eq. (10), or
(ii) to directly seek for a fit to Eq. (13).

Examining Fig. 1 graphically, one can observe exponential decrease of the distribution in the low-intensity and the very high-intensity tail as a function of I. In between both exponential regimes the distribution is obviously non exponential, but it is not evident to clearly identify where the transition toward the high-intensity tail takes place. Therefore approach (ii) is not the first choice, but can be used once the coefficient found from approach (i) have been determined. One would use I_{th} and y_F (if existing) from (i) and seek for the power exponent α in Eq. (13), and eventually verify the equivalence between the respective coefficients as discussed in section 3.1.

For two sets of the parameters $n_{\rm sp}$ and I^* we have computed the distribution function and determined the coefficients using approach (i). The results are displayed in Table 1, 2, 3, and 4.

gain G	1/2	1	3/2	2
c	.036 $\pm$.04	-0.80 $\pm$.01	-1.83 $\pm$.02	-2.32 $\pm$.05
a	1.6 $\pm$.05	4.3 $\pm$.15	6.4 $\pm$.25	9.0 $\pm$.5
I_{th}	8.0 $\pm$.2	11.3 $\pm$.2	12.0 $\pm$.2	11.5 $\pm$.2
$y_F = \|a/c\|$	-	5.1 $\pm$.2	3.5 $\pm$.7	3.9 $\pm$.3

Table 1: Summary of the parameters for the GPD fit as a function of gain values G for n_{sp} =3000 and I^* =15.

gain G	1/2	1	3/2	2
c	.05$\pm$ .02	-.41 $\pm$.03	-.69 $\pm$.11	-1.1 $\pm$.05
a	1.28 $\pm$.03	2.3 $\pm$.03	3.1 $\pm$.03	4.03 $\pm$.03
I_{th}	6.85 $\pm$.15	8.25 $\pm$.12	8.25 $\pm$.25	8.2 $\pm$.25
$y_F = \|a/c\|$	-	6.0 $\pm$.4	5.1 $\pm$.3	3.5 $\pm$.4

Table 2: Summary of the parameters for the GPD fit as a function of gain values G for n_{sp} =2300 and I^* =11.

We have first determined the conditioned distribution $\bar{F}_{th}$ by varying the threshold intensity I_{th} and seeking for convergence in the values of c and a of the fit to the function $h_{c,a}(y)$ of Eq. (10). The value of I_{th} can be assumed to be in the vicinity of $\log n_{\rm sp}$ and has to be chosen as $I_{th} < I^*$ (since otherwise non linear effects will already be saturated). Pickands in Ref. Pickands (1975) suggests a method to determine the value of c based on specific quantile values of $\bar{F}_{th}(y)$, namely by determining the values y_1 and y_2 where $\bar{F}_{th}(y_1)$ =50% and $\bar{F}_{th}(y_2)$ =25% such that $c = \log(y_2/y_1 - 1)/\log(2)$.

Numerical algorithms allow, however, to more reliably find the values of c and a by searching the best fit of $h_{c,a}(y)$ to $\bar{F}_{th}(y)$. This is what we have done to obtain the values in Tables 1 and 2. The fit to $\bar{F}_{th}(y) = 1 - F_{th}(y)$ is shown in Fig. 3 for the particular case G =3/2, $n_{\rm sp}$ =3000 (or $\log n_{\rm sp} \simeq 8$), and I^* =15. The values for c and a are stable with respect to the choice of the threshold value in the vicinity of $I_{th} \sim 10 \pm 1$ at least, being close to the expectation value $I_1 = \log n_{\rm sp} + .5772$ of the peak intensity of the most intense speckle (determined from order statistics, Ref. Hüller and Porzio (2010).

The width of the intensity interval corresponding to a Weibull law is given by $y_F = |a/c|$ if $c < 0$, so that a Weibull-type distribution is found in $I_{th} \lesssim I < I_F = I_{th} + y_F$. The values found here ($I \gtrsim I_{th}$) are great enough to reasonably guarantee convergence toward the Weibull law.

The results show a clear distinction between the case G =0.5 < 1 and the other cases with $G \geq$1 : for G =0.5, the value of c is very small so that the behaviour, as a function of the excess parameter y, is not significantly departing from an exponential decrease, while there is a clear non-exponential behaviour for $G \geq$1 in the interval $y_F > y >$0. For the function $h_{c,a}$ of the GPD a finite upper limit of this interval exists due to negative value of c, simply due to the fact that otherwise the probability for $y > y_F$ would exceed 100%. The tail of the conditioned distribution $\bar{F}_{th}$ shows a smooth decrease for $y \sim y_F$, instead of an abrupt behaviour of $h_{c,a}$ at y_F, corresponding to the end of validity of the Weibull law. Beyond y_F (or $I \gtrsim I_F$) the distribution is again of exponential type because of the onset of saturation.

We observe a clear and systematic decrease of the coefficient c with the gain value G, starting from a small value $|c| \ll 1$ for $G < 1$. As exposed in the previous paragraph, it follows that the exponent of the power law behaviour corresponding to Eq. (13), namely $\alpha = -1/c$, hence decreases with increasing G, explaining a flattening of the distribution $\bar{F}(I)$ in the interval $I_{th} \lesssim I < I_F$. Both the flattening and the great enough intensity values imply the onset of a heavy tail distribution function which has to be seen as a 'caveat' for extremal statistics, eventually causing undesirable effects.

In addition to the method with approach (i), where we determined the coefficients to the function Eq. (10) via fitting to the conditioned distribution $\bar{F}_{th}$, we have also directly analyzed the distributions $\bar{F}(I)$ – computed from Eq. (3) for the different G-values – with respect to the power law of Eq. (13). As input to get obtain a best fit, we have used the values of $I_F = I_{th} + y_F$ from Tables 1 and 2. The results of the fit for the coefficients α and K are listed listed in the Tables 3 and 4.

gain G	1/2	1	3/2	2
$\alpha = 1/\lvert c\rvert$	$\gg 1$	1.24 ±.02	0.54 ±.05	0.43 ±.02
K		0.12 ±.02	0.45 ±.05	0.55 ±.15
I_{th}	8.0 ±.2	11.3 ±.2	12.0 ±.2	11.5 ±.2
$I_{th} + y_F$	no limit	16.3 ±.5	15.5 ±.5	15.5 ±.5

Table 3: Summary of parameters of the Weibull type power law valid in the limit $I_{th} \lesssim I < I_F = I_{th} + y_F$. Here n_{sp}=3000 and I^* =15, corresponding to Table 1.

gain G	1/2	1	3/2	2
$\alpha = 1/\lvert c\rvert$	$\gg 1$	2.5 ±.2	1.45 ±.2	0.9 ±.15
K		.015 ±.005	.1 ±.07	.25 ±.07
I_{th}	6.85 ±.15	8.25 ±.12	8.25 ±.25	8.2 ±.25
$I_{th} + y_F$	no limit	14.3 ±.6	13.3 ±.5	11.7 ±.6

Table 4: Summary of parameters of the Weibull type power law valid in the limit $I_{th} \lesssim I < I_F = I_{th} + y_F$. Here n_{sp}=2300 and I^* =11 corresponding to Table 2.

The values determined via both approaches – i. e. approach (i) with the GPD method with Eq. (10) and via approach (ii) using the power law expression Eq. (13) – prove to be consistent, in particular what concerns the interval where the Weibull type distribution has to be considered.

For the interval $I < I_F - K^{-1/\alpha}$ the distribution keeps the characteristics of an exponential distribution, meaning that smaller intensity speckles still obey Gaussian statistics. The consistency with the power-law dependence in a limited interval is also illustrated graphically in Figures 3, 4, 5, and 6. In Fig. 3 the power-law according to the function Eq. (10) is directly compared with the conditioned complementary distribution $\bar{F}_{th}$ for the example case with $G = 3/2$, in Fig. 4, for the same case, we show the comparison between the complementary speckle distribution $\bar{F}(I)$ and the power law deduced via the GPD in a linear scale, while in Figures 5 and 6 the comparison between the computed complementary distributions and the power law according to Eq. (13) is depicted in log scale for G =1, 3/2, and 2. One can see that the values for each $\bar{F}_{th}$ fit very well to the power law curves $\sim K(I_F - I)^\alpha$ for the values and intervals indicated in Tables. 3 and 4.

As in Ref. Hüller, Porzio, and Robiche (2013) we have performed 2-dimensional (2D) simulations with the code HARMONY2D (Hüller et al., 2006 and 2013) for the same parameters as in the computation of $\bar{F}_{th}$ from Eq. (3) for $n_{\rm sp}$ =2300, I^* =11, and for G =1 and 3/2. The distributions are determined from snapshots from the speckle field pattern taken from a plasma layer of approximately 2.5 times a speckle length, as explained in Ref. Hüller, Porzio, and Robiche (2013). The simulation results, see Figures 7 and 8, show very good agreement both with the computation using $\bar{F}_{th}$ from Eq. (3). Evidently, also the Weibull-type power-law behaviour can be observed in the same speckle intensity interval.

The good agreement between simulations and our model calculations also confirms that the upper limit of the power-law distribution cannot be associated with the way how we model the saturation of the stimulated scattering process in Eq. (3), namely via the saturation of the exponential growth $\sim \exp\{G \min(I, I^*)\}$ in I for $I \geq I^*$. In the simulations the saturation also arises around $I \sim I^*$ naturally, but less abruptly. Therefore, applying the abrupt onset of saturation in a non linear model Eq. (3), instead of a softer transition avoiding the cusp in the exponential growth at $I = I^*$, has no qualitative influence on the result, i. e. on the upper limit in the validity interval of the Weibull-type power-law behaviour in the distribution.

5 Discussion and Conclusions

In numerous articles based on theoretical and experimental work, the onset of non-Gaussian statistics in the observed speckle distribution function has been mentioned (Schmitt & Afeyan, 1997, Grech et al., 2006 and 2009, Lushnikov & Vladimirova, 2010, Depierreux et al. 2009, Montina et al. 2009). In particular laser beams produced with optical smoothing techniques and propagating through plasmas show a departure from the exponential-type complementary distribution of the incident beam even behind relatively short plasma layers.

We have shown here that for laser beams undergoing laser-induced smoothing via the process of stimulated Brillouin forward scattering (SBFS), this process is responsible for the modifications in the peak intensity of intense speckles. The resulting speckle distribution functions are henced “stretched” in their tail, yielding a functional behaviour different from an exponential-type decrease.

Distribution functions with exponential-type behaviour in the tail and distributions with power law behaviour, even in a limited interval, have essentially different properties what concerns extremal statistics due to the strictly different limit laws (Embrechts et al., 1997).Heavy tail distributions, as those with power law behaviour, may cause strong deviations and an important variance in physical observables from one to another realisation, leading to desastrous "shot to shot" behaviour.

The speckle distributions analyzed in this work have been computed on the basis of our model (Eq. (3)) , and with the help of numerical simulations. The model is limited to the first layer, in the laser propagation direction, where speckles of equal size (generated by optical smoothing) undergo stimulated scattering. Multiple scattering, occuring further inside in longer plasmas (Schmitt & Afeyan, 1997) is not taken into account here. From the simulation results of a plasma layer, snapshots, corresponding to few realizations of speckle patterns have been taken to determine the statistics. The simulations results confirm the model results.

We have shown in the frame of this work that speckle distributions, modified in a plasma as a consequence of stimulated forward scattering, may deviate from Gaussian statistics. By making use of generalized Pareto distributions (GPD), we have identified power law type distributions under certain conditions: precisely, the distributions discussed here belong to the Weibull type power law that exhibits heavy tail behaviour. The inspection of the distribution from our model established here and from previous work, as well as from numerical simulations, show a clear qualitative change between distributions found for gain values, G, of SBFS around the critical value G =1. The distributions found for values G <1 still show an exponential-type decrease, so that speckle distributions for this regime (G <1) are still governed by Gaussian statistics. For values G ≥1, however, our analysis clearly shows that a power-law behaviour of the speckle (complementary) distribution governs a limited interval in the tail, but with an upper limit, beyond of which the distribution is again of exponential type. The limited interval is associated with Weibull distributions, not following Gaussian statistics. The upper limit of the interval is due to a saturation of the underlying non linear process, namely stimulated scattering. We should remind that for different realizations of speckle patterns physical observables like the scattered field strength can strongly vary with the gain value for $G \geq 1$, as indicated by the computation of the variance of the speckle statistics as a function of G.

In presence of temporal smoothing, either introduced via optical techniques ('Smoothing by Spectral Dispersion' (SSD), 'Induced Spatial Incoherence' (ISI), etc.) or induced by the plasma itself, speckle distributions will evolve, and particularly the tail of the distribution will be altered. Sufficiently fast smoothing, such that the correlation time (Mounaix et al., 2000) is short compared to the instability process (here forward SBS), may avoid a vigorous onset of the instability and hence the formation of heavy tail distributions. The plasma can take itself the function of an active smoothing medium in altering the effective speckle size further inside the plasma (Schmitt & Afeyan, 1997) and with a succesive decrease of the correlation time (Depierreux et al. 2009).

In absence of temporal smoothing or in case of relatively long correlation times of the smoothing method, heavy tail speckle distributions may occur for sufficiently high laser intensity (see above condition for the SBFS gain).

While we have used 'excess over threshold' methods to analyze speckle distributions modified by stimulated scattering in laser plasmas, those methods can be more generally applied to determine the nature of speckle statistics from experimental and simulation data.

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Appendix: simplified expressions for the *pdf* and its moments

In this Appendix we drop, for simplicity, the subscript in n_{sp}, standing for the total number of speckles: we write hence n instead of n_{sp}. The probability density (*pdf*) in Eq. (2)

$$f^{(G)}(I) = e^{-ne^{-I}} e^{G\min(I,I^*)} \left(\sum_{k=1}^{n} \frac{e^{-kI} n^k}{n^G(\Gamma(k-G, ne^{-I^*}) - \Gamma(k-G, n)) + e^{GI^*}\gamma(k, ne^{-I^*})} \right)$$

can be approximated for entire values of G, $G \in \mathbb{N}^*$, by

$$f^{(G)}(I) = e^{-ne^{-I}} e^{G\min(I,I^*)} \left(\sum_{k=1}^{G} \frac{e^{-kI} n^k}{D_k(G,0)} + ne^{-(G+1)M} e_{n-(G+1)}(ne^{-I}) \right) \qquad (14)$$

where $ne^{-(G+1)I} e_{n-(G+1)}(ne^{-I})$ is the approximation kept for the part of the sum in $f^{(G)}(I)$ of Eq. (2) starting with $n = G+1$, namely

$$\begin{aligned} \textstyle\sum_{k=G+1}^{n} \frac{e^{-kI} n^k}{D_k(G,0)} &= \sum_{k=G+1}^{n} \frac{e^{-kI} n^k}{n^G(\Gamma(k-G, ne^{-I^*}) - \Gamma(k-G, n)) + e^{GI^*}\gamma(k, ne^{-I^*})} \\ &\sim \sum_{k=G+1}^{n} \frac{e^{-kI} n^k}{n^G \gamma(k-G, n)} \sim \sum_{k=G+1}^{n} \frac{e^{-kI} n^k}{n^G \Gamma(k-G)} \end{aligned}$$

where $e_m(x) = \sum_{j=0}^{m} x^j/j!$ is the truncated exponential series.

For the denominator in these expressions, as defined in Eq. (4), $D_k(G,0)$, one can find, by using the approximation of the exponential integral $\int_{ne^{-I^*}}^{n} (e^{-t}/t)dt \sim (I^* - \log n)\exp(-ne^{-I^*})$,

$$\begin{aligned} D_k(G,0) &\simeq \tfrac{n^G(-1)^{G-k}}{(G-k)!} \left([I^* - \log n] e^{-ne^{-I^*}} + e^{-n} \textstyle\sum_{m=0}^{G-k-1} \frac{(-1)^m m!}{n^{m+1}} - e^{-ne^{-I^*}} \sum_{m=0}^{G-k-1} \frac{(-1)^m m!}{(ne^{-I^*})^{m+1}} \right) \\ &\quad + e^{GI^*}(k-1)! \left(1 - e^{-ne^{-I^*}} \textstyle\sum_{m=0}^{k-1} \frac{(ne^{-I^*})^m}{m!} \right) \\ &\simeq \tfrac{n^G(-1)^{G-k}}{(G-k)!} \left((I^* - \log n) e^{-ne^{-I^*}} + e^{-ne^{-I^*}} \textstyle\sum_{m=0}^{G-k-1} \frac{(-1)^{m+1} m!}{(ne^{-I^*})^{m+1}} \right) \\ &\quad + e^{GI^*}(k-1)! \left(1 - e^{-ne^{-I^*}} \textstyle\sum_{m=0}^{k-1} \frac{(ne^{-I^*})^m}{m!} \right) \ , \end{aligned} \qquad (15)$$

or in the still more simplified form

$$\tilde{D}_k(G,0) \simeq \frac{n^G(-1)^{G-k}}{(G-k)!}(I^* - \log n)e^{-ne^{-I^*}} + e^{GI^*}(k-1)! \left(1 - e^{-ne^{-I^*}} \sum_{m=0}^{k-1} \frac{(ne^{-I^*})^m}{m!} \right) \ , \qquad (16)$$

keeping only the dominating terms. Taking $ne^{-I} = x$ and $ne^{-I^*} = x^*$, the approximate expressions for the *pdf* in the limits $I^* \geq I$, equivalently to $x^* \leq x$, and $I^* < I$, equivalently to $x^* < x$, for $f^{(G)}(I) \to \tilde{f}^{(G)}(x)$ read

$$\tilde{f}^{(G)}(x) = \begin{cases} e^{-x}(\frac{n}{x})^G \sum_{k=1}^{G} \frac{x^k}{D_K(G,0)} + xe^{-x} e_{n-G-1}(x) \text{ for } x^* \leq x \quad (I^* \geq I) \ , \\ e^{-x}(\frac{n}{x^*})^G \sum_{k=1}^{G} \frac{x^k}{D_k(G,0)} + xe^{-x}(\frac{x}{x^*})^G e_{n-G-1}(x) \text{ for } x^* > x \quad (I^* < I) \end{cases} \qquad (17)$$

For the case that the gain G assumes the value of an entire number, the sum in the expression for the *pdf* simplifies, such that for $x^* < x$ we obtain

$$\tilde{f}^{(G)}(x) = e^{-x} n^G \sum_{k=1}^{G-1} \frac{x^{k-G}}{D_k(G,0)} + \frac{e^{-x} n^G}{D_{k=G}(G,0)} + x e^{-x} e_{n-G-1}(x) \ .$$

The first part of the sum contains, for odd integer G, two neighbouring peak terms of the same value, for $k = [G/2]$ and $[G/2]+1$, and for even integer G, a single peak term for $k = G/2$. The second part of the sum yields a 'plateau'-like term, and the last part is exponentially decreasing.

In terms of x and x^* the denominators $D_k(G,0)$ and $\tilde{D}_k(G,0)$ are given by

$$\begin{aligned} D_k(G,0) \simeq\ & \tfrac{n^G(-1)^{G-k}}{(G-k)!} \Big((-\log x^*) e^{-x^*} + e^{-n} \textstyle\sum_{m=0}^{G-k-1} \tfrac{(-1)^m m!}{n^{m+1}} - e^{-x^*} \textstyle\sum_{m=0}^{G-k-1} \tfrac{(-1)^m m!}{(x^*)^{m+1}} \Big) \\ & + (\tfrac{n}{x^*})^G (k-1)! \Big(1 - e^{-x^*} \textstyle\sum_{m=0}^{k-1} \tfrac{(x^*)^m}{m!} \Big) \end{aligned} \tag{18}$$

and

$$\tilde{D}_k(G,0) \simeq \frac{n^G(-1)^{G-k}}{(G-k)!} (-\log x^*) e^{-x^*} + (\frac{n}{x^*})^G (k-1)! \left(1 - e^{-x^*} \sum_{m=0}^{k-1} \frac{(x^*)^m}{m!} \right) \ . \tag{19}$$

We express in the following the *pdf* from the general expression Eq. (2) for selected values of G.

Approximations of the *pdf* for selected cases

For $G = 1/2$ one obtains

$$f^{(1/2)}(I) \simeq e^{-ne^{-I}} e^{(1/2)\min(I,I^*)} e^{-I} \frac{\sqrt{n}}{\sqrt{\pi}} \left(1 + \sum_{k=1}^{n-1} \frac{(2ne^{-I})^k}{(2k-1)!!} \right) \ , \tag{20}$$

and for $G = 3/2$

$$f^{(3/2)}(I) \simeq \frac{e^{-ne^{-I}} e^{(3/2)\min(I,I^*)}\ ne^{-I}}{n^{3/2} \int_{ne^{-I}}^{n} \frac{e^{-t}}{t^{3/2}} dt (1 - e^{-ne^{-I^*}}) e^{(3/2)I^*}} + \frac{e^{-ne^{-I}} e^{(3/2)\min(I,I^*)} n^2 e^{-2I}}{n^{3/2}\sqrt{\pi}} \left(1 + \sum_{k=1}^{n-2} \frac{(2ne^{-I})^k}{(2k-1)!!} \right) \tag{21}$$

For the most interesting case $G = 1$, we obtain

$$f^{(1)}(I) \simeq \frac{e^{-ne^{-I}} e^{\min(I,I^*)} ne^{-I}}{n \int_{ne^{-I}}^{n} \frac{e^{-t}}{t} dt + (1 - e^{-ne^{-I^*}}) e^{I^*}} + e^{-ne^{-I}} e^{\min(I,I^*)} ne^{-2I} e_{n-2}(ne^{-I}) \ . \tag{22}$$

Finally, we give an approximate expression for non-integer G-values :

$$f^{(G)}(I) \simeq e^{-ne^{-I}} e^{G\min(I,I^*)} \left(\sum_{k=1}^{[G]} \frac{(ne^{-I})^k}{n^G \int_{ne^{-I}}^{n} \frac{e^{-t}}{t^{1+G-k}} dt + e^{GI^*} \int_0^{ne^{-I}} \frac{e^{-t}}{t^{1-k}} dt} + \sum_{k=[G]+1}^{n} \frac{(ne^{-I})^k}{n^G \Gamma(k-G)} \right) \ . \tag{23}$$

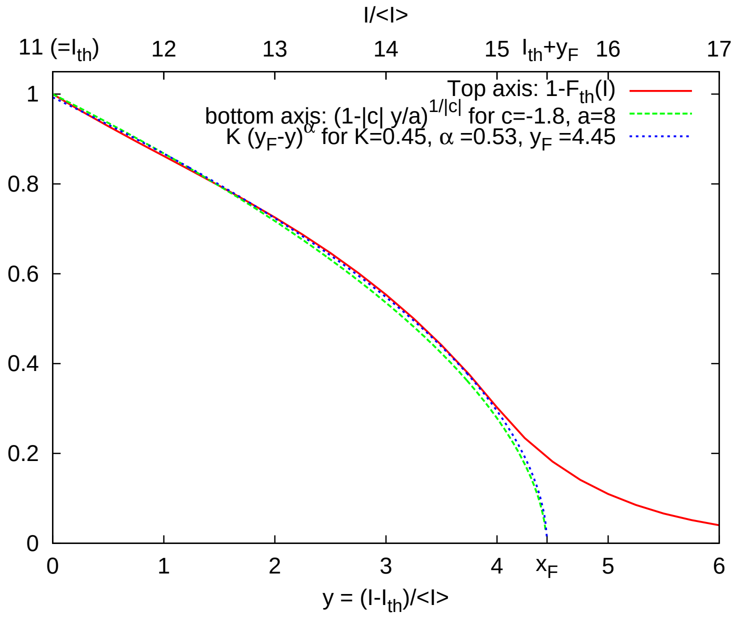

Figure 3: Comparison between the complementary 'excess over threshold' distribution function $\bar{F}_{th} = 1 - F_{th}$ deduced from our model, Eq. (1), the generalized Pareto distribution, following Eq. (10), $(1 - |c|y/a)^{1/|c|}$, and the power law expression $K(y_F - y)^{\alpha}$ with $y_F = a/|c|$, for the case G =3/2, with c =≃ -1.8, a =8., y_F ≃4.4, K =0.45, and α =0.53.

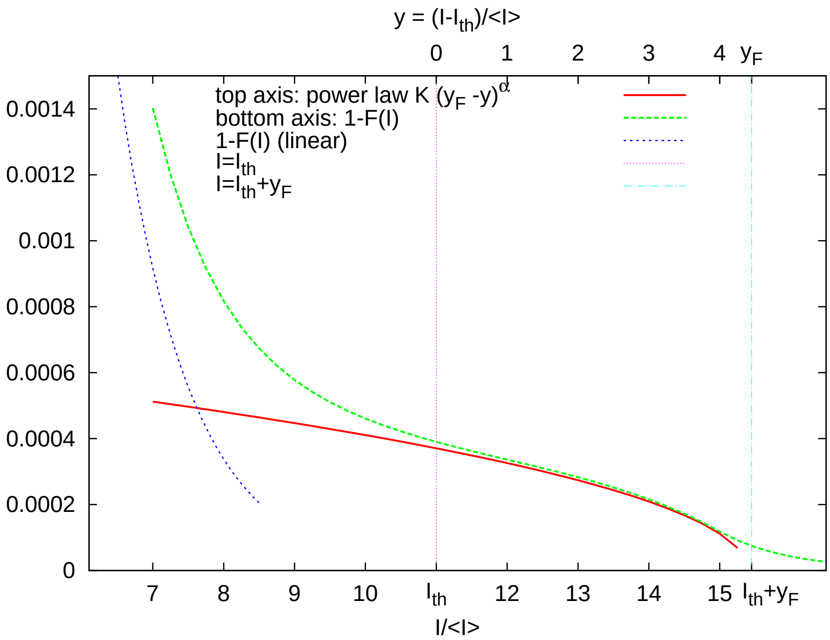

Figure 4: Illustration of non-exponential behaviour of the complementary distribution $\bar{F}(I) = 1 - F(I)$ (from our model Eq.(1)) in the interval between $I = I_{th}$ ($y = 0$, top axis) and $I \equiv I_F = I_{th} + y_F$ ($y = y_F$, top axis) where the behaviour can be approximated by a power-law type function $K(y_F - y)^{\alpha}$ deduced from a generalized Pareto distribution (GPD). Shown is the case of G =3/2 with y_F ≃4.4, α ≃0.53, and K ≃0.45, consistent with the parameters c ≃-1.8, a ≃8 from the GPD.

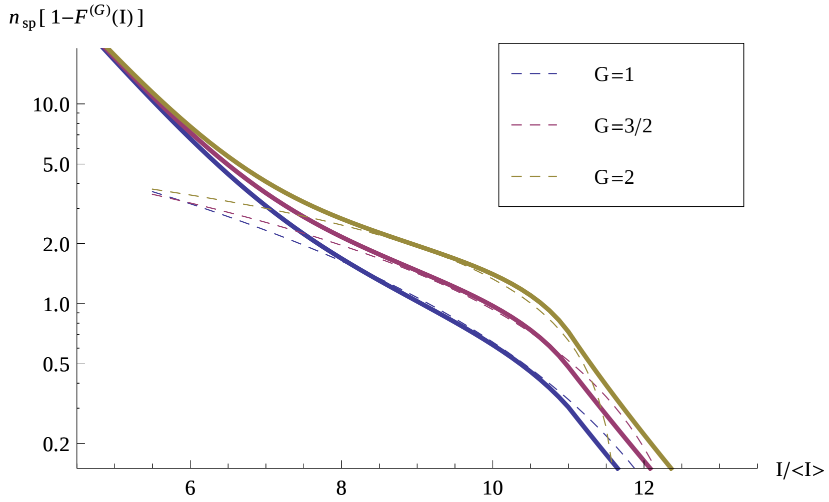

Figure 5: Complementary speckle distribution $\bar{F}^{(G)}(I)$ (multiplied by $n_{\rm sp}$ in the high intensity tail for the gain values G =1 (blue line), 3/2 (red), and 2 (yellow-green) and the power laws deduced with the help of the GPD, and taken from Table 4, i.e. G =1 : α =2.3, I_F =13.7; G =3/2 : α =1.45 and I_F =13; and G =2 : α =0.85 and I_F =11.6. Solid lines: model; dashed lines: power law following Eq. (13). The total number of speckles considered here is $n_{\rm sp}$ = 2300. The onset of saturation is set to I^* =11 here.

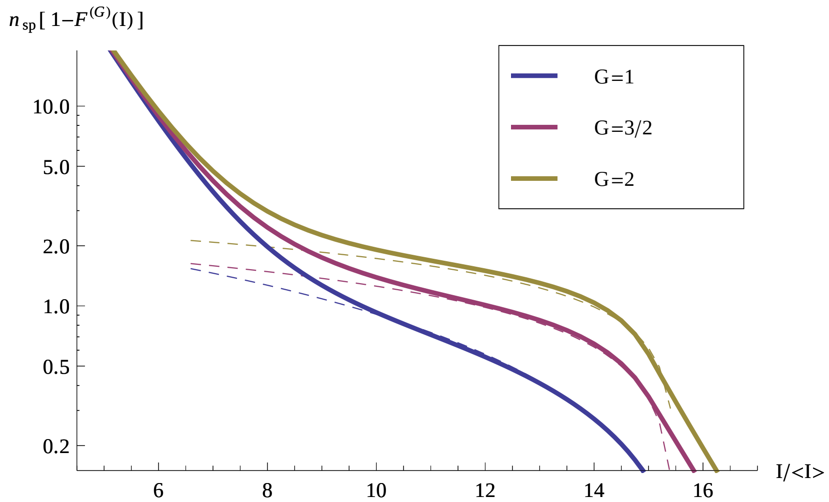

Figure 6: Complementary speckle distribution $\bar{F}^{(G)}(I)$ (multiplied by $n_{\rm sp}$ in the high intensity tail for the gain values G =1 (blue line), 3/2 (red), and 2 (yellow-green) and the power laws deduced with the help of the GPD, and taken from Table 3, i.e. G =1 : α =0.8, I_F =16.4; solid lines: model; dashed lines; power law following Eq. (13); G =3/2: α =0.54 and I_F =15.5; and G =2: α =0.43 and I_F =15.4. Parameters $n_{\rm sp}$ = 3000 and I^* =15 here.

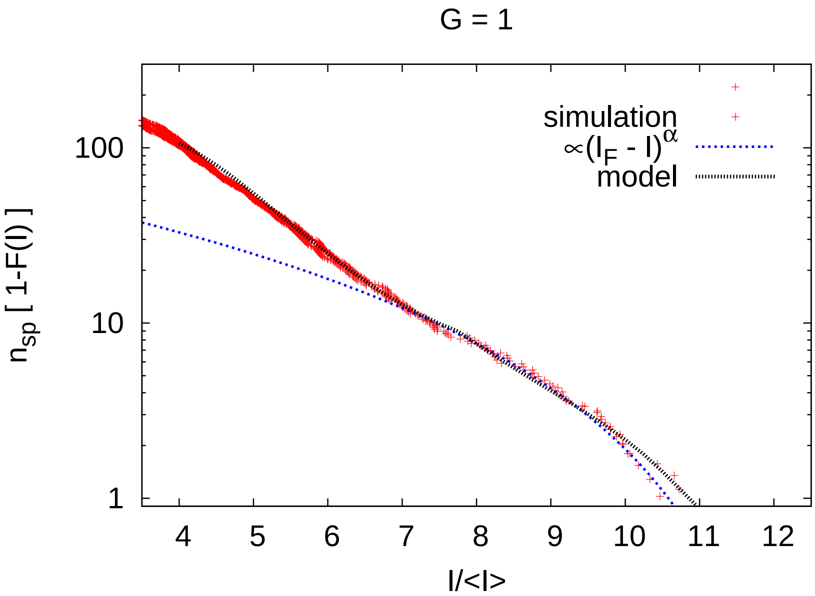

Figure 7: For the case G =1: comparison between the complementary distribution of speckles obtained from simulations of forward SBS (as in Hüller, Porzio, and Robiche (2013)),the power law dependence $\bar{F} \propto (I_F - I)^{\alpha}$, and the model following Eq. (3). The power law dependence determined via the GPD is valid in the limited interval $I_{th} \lesssim I < I_F$, with $I_F \simeq 12.2$, $I_{th} \simeq 8$, and $\alpha \simeq 2.25 \ldots 2.45$. The total number of speckles in the simulations was $n_{\rm sp} = 2300$.

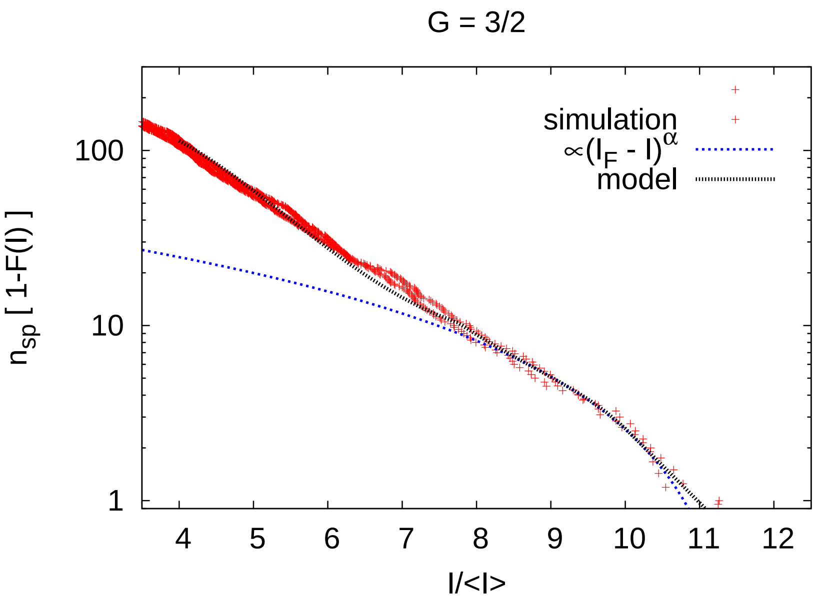

Figure 8: As in Fig. 7, but for G =1.5: comparison between the complementary distribution of speckles obtained from simulations of forward SBS, the power law dependence $\bar{F} \propto (I_F - I)^{\alpha}$, and the model following Eq. (3). The power law dependence is valid in the limited interval $I_{th} \lesssim I < I_F$, with $I_F \simeq 11.7$, $I_{th} \simeq 8$, $\alpha \simeq 1.45 \ldots 1.5$.